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Division-ample sets and the Diophantine problem for rings of integers

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Résumé. Nous demontrons que le dixième problème de Hilbert pour un anneau d'entiers dans un corps de nombres K admet une réponse négative si K satisfait à deux conditions arithmétiques (existence d'un ensemble dit *division-ample* et d'une courbe elliptique de rang un sur K). Nous lions les ensembles division-ample à l'arithmétique des variétés abéliennes.

Abstract. We prove that Hilbert's Tenth Problem for a ring of integers in a number field K has a negative answer if K satisfies two arithmetical conditions (existence of a so-called *division-ample* set of integers and of an elliptic curve of rank one over K). We relate division-ample sets to arithmetic of abelian varieties.

1. Introduction

Let K be a number field and let $\mathscr{O}_K$ be its ring of integers. *Hilbert's Tenth Problem* or *the diophantine problem* for $\mathscr{O}_K$ is the following: is there an algorithm (on a Turing machine) that decides whether an arbitary diophantine equation with coefficients in $\mathscr{O}_K$ has a solution in $\mathscr{O}_K$.

The answer to this problem is known to be negative for $K = \mathbf{Q}$ ([5]) and for the following fields K by reduction to the field $K = \mathbf{Q}$: K of complex degree ≤ 2 over a totally real field (Denef and Lipshitz [6], [7], [8]), K with exactly one pair of complex embeddings (Pheidas [9] and Shlapentokh [14]) and subfields of all those (including cyclotomic fields, and hence all abelian number fields; Shapiro and Shlapentokh [12]). This reduction consists in finding a *diophantine model* (cf. [3]) for integer arithmetic over $\mathscr{O}_K$. The problem is open for general number fields (for a survey see [10] and [13]), but has been solved conditionally, e.g. by Poonen [11] (who shows that the set of rational integers is diophantine over $\mathscr{O}_K$ if there exists an elliptic

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curve over $\mathbf{Q}$ that has rank one over both $\mathbf{Q}$ and K). In this paper, we give a more general condition as follows:

Theorem 1.1. *The diophantine problem for the ring of integers $\mathscr{O}_K$ of a number field K has a negative answer if the following exist:*

(1) *an elliptic curve defined over K of rank one over K;*
(2) *a* division-ample *set $A \subseteq \mathscr{O}_K$.*

Definition. A set $A \subseteq \mathscr{O}_K$ is called *division-ample* if the following three conditions are satisfied:

- (diophantineness) A is a diophantine subset of $\mathscr{O}_K$;
- (divisibility-density) Any $x \in \mathscr{O}_K$ divides an element of A;
- (norm-boundedness) There exists an integer $\ell > 0$, such that for any $a \in A$, there is an integer $\widetilde{a} \in \mathbf{Z}$ with $\widetilde{a}$ dividing a and $|N(a)| \leq |\widetilde{a}|^{\ell}$.

Proposition 1.1. *A division ample set exists if either*

(1) *there exists an abelian variety G over $\mathbf{Q}$ such that*

$$\operatorname{rk} G(\mathbf{Q}) = \operatorname{rk} G(K) > 0; \text{ or}$$

(2) *there exists a commutative (not necessarily complete) group variety G over $\mathbf{Z}$ such that $G(\mathscr{O}_K)$ is finitely generated and such that* $\operatorname{rk} G(\mathbf{Z}) = \operatorname{rk} G(\mathscr{O}_K) > 0$.

From (1) in this proposition, it follows that our theorem includes that of Poonen, but it isolates the notion of "division-ampleness" and shows it can be satisfied in a broader context. It would for example be interesting to construct, for a given number field K, a curve over $\mathbf{Q}$ such that its Jacobian satisfies this condition.

As we will prove below, part (2) of this proposition is satisfied for the relative norm one torus $G = \ker(N_K^{KL})$ for a number field L linearly disjoint from K, if K is quadratic imaginary (choosing L totally real).

It would be interesting to know other division-ample sets, in particular, such that are not subsets of the integers.

The proof of theorem 1.1 will use divisibility on elliptic curves and a Lemma from algebraic number theory of Denef and Lipshitz. Some of our arguments are similar to ones in [11], but we have avoided continuous reference both for reasons of completeness and because our results have been obtained independently.

2. Lemmas on number fields

In this Section we collect a few facts about general number fields which will play a rôle in subsequent proofs. Fix K to be a number field, let $\mathscr{O} = \mathscr{O}_K$ be its ring of integers, and let h denote the class number of $\mathscr{O}$.

Let $N = N_{\mathbf{Q}}^K$ be the norm from K to $\mathbf{Q}$, and let $n = [K : \mathbf{Q}]$ denote the degree of K. Let $|$ denote "divides" in $\mathscr{O}$.

First of all, we will say a subset $S \subseteq K^n$ is "diophantine over $\mathscr{O}$" if its set of representatives $\widetilde{S} \subseteq (\mathscr{O} \times (\mathscr{O} - \{0\}))^n$ given by

$$\widetilde{S} := \{(a_i, b_i)_{i=1}^n \in (\mathscr{O} \times (\mathscr{O} - \{0\}))^n \mid (a_i/b_i)_{i=1}^n \in S\}$$

is diophantine over $\mathscr{O}$. Recall that "$x \neq 0$" is diophantine over $\mathscr{O}$ ([8] Prop. 1(b)), hence S is diophantine over $\mathscr{O}$ if and only if it is diophantine over K.

Recall that there is no unique factorisation in general number fields, but we can use the following valuation-theoretic remedy:

Definition. Let $x \in K$. If $x^h = \frac{a}{b}$ for $a, b \in \mathscr{O}$ with $(a, b) = 1$ (the ideal generated by a and b), we say that $a = \mathrm{wn}(x)$ is *a weak numerator* and $b = \mathrm{wd}(x)$ is *a weak denominator* for x.

Lemma 2.1. (1) *For any $x \in K$ a weak numerator and a weak denominator exists and is unique up to units.*

(2) *for any valuation v,*

- $v(x) > 0 \iff v(\mathrm{wn}(x)) > 0$, *and then* $v(\mathrm{wn}(x)) = hv(x)$;
- $v(x) < 0 \iff v(\mathrm{wd}(x)) > 0$, *and then* $v(\mathrm{wd}(x)) = -hv(x)$.

(3) *For $a \in \mathscr{O}, x \in K$, "$a = \mathrm{wn}(x)$" and "$a = \mathrm{wd}(x)$" are diophantine over $\mathscr{O}$.*

Proof. Since $\mathscr{O}$ is a Dedekind ring, (x) has a unique factorisation in fractional ideals

$$(x) = \mathfrak{p}_1 \cdots \mathfrak{p}_r \cdot \mathfrak{q}_1^{-1} \cdots \mathfrak{q}_s^{-1}.$$

We let a be a generator for the principal ideal $(\mathfrak{p}_1 \cdots \mathfrak{p}_r)^h$ and b a generator for $(\mathfrak{q}_1 \cdots \mathfrak{q}_s)^h$; these are obviously a weak numerator/denominator for x. Uniqueness, (2) and (3) are obvious. □

Lemma 2.2 (Denef-Lipshitz [8])**.** (1) *If $u \in \mathbf{Z} - \{0\}$ and $\xi \in \mathscr{O}$ satisfy the divisibility condition*

$$2^{n!+1} \prod_{i=0}^{n!-1} (\xi + i)^{n!} \mid u$$

then for any embedding $\sigma : K \hookrightarrow \mathbf{C}$

$$(*)_u \qquad |\sigma(\xi)| \leq \frac{1}{2} \sqrt[n!]{|N(u)|}.$$

(2) *If $\widetilde{u} \in \mathbf{Z} - \{0\}, q \in \mathbf{Z}$ and $\xi \in \mathscr{O}$ satisfy $(*)_{\widetilde{u}}$ for any embedding $\sigma : K \hookrightarrow \mathbf{C}$ and $\xi \equiv q \bmod \widetilde{u}$, then $\xi \in \mathbf{Z}$.*

Proof. Easy to extract from the proof of Lemma 1 in [8]. □

3. Lemmas on elliptic curves

Let E denote an elliptic curve of rank one over K, written in Weierstrass form as

$$E \ : \ y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6,$$

let T be the order of the torsion group of $E(K)$, and let P be a generator for the free part of $E(K)$. Define $x_n, y_n \in K$ by $nP = (x_n, y_n)$.

Lemma 3.1. *For any integer r the set $rE(K)$ is diophantine over K and, if r is divisible by T, then $rE(K) = \langle rP \rangle \cong (\mathbf{Z}, +)$.*

Proof. A point $(x, y) \in K \times K$ belongs to $rE(K) - \{0\}$ if and only if $\exists\ (x_0, y_0) \in E(K) \ : \ (x, y) = r(x_0, y_0)$. As the addition formulæ on E are algebraic with coefficients from K, this is a diophantine relation. The last statement is obvious. □

Lemma 3.2 ([2] for $K = \mathbf{Q}$, [11])**.** *There exists an integer $r > 0$ such that for any non-zero integers $m, n \in \mathbf{Z}$,*

$$m|n \iff \mathrm{wd}(x_{rm})|\,\mathrm{wd}(x_{rn}).$$

Proof. We reduce the claim to a statement about valuations using Lemma 2.1(ii). The theory of the formal group associated to E implies that if $n = mt$ and v is a finite valuation of K such that $v(x_{rm}) < 0$, then $v(x_{rmt}) \leq v(x_{rm}) - 2v(t) \leq v(x_{rm})$ ([15] VII.2.2).

For the converse, we start by choosing r_0 in such a way that r_0P is non-singular modulo all valuations v on K. By the theorem of Kodaira-Néron ([15], VII.6.1), such r_0 exists and it actually suffices to take $r_0 = 4\prod_v v(\Delta_E)$, where Δ_E is the minimal discriminant of E, and the product runs over all finite valuations on K for which $v(\Delta_E) \neq 0$. Note that then, $v(x_{r_0n}) < 0 \iff r_0nP = 0$ in the group E_v of non-singular points of E modulo v.

We claim that for an arbitrary finite valuation v on K, if $v(x_{r_0n}) < 0$ and $v(x_{r_0m}) < 0$, then $v(x_{(r_0m, r_0n)}) < 0$, where $(\cdot, \cdot)$ denotes the gcd in $\mathbf{Z}$. Indeed, the hypothesis means $r_0mP = r_0nP = 0$ in E_v. Since there exist integers $a, b \in \mathbf{Z}$ with $(r_0m, r_0n) = ar_0m + br_0n$, we find $(r_0m, r_0n)P = 0$ in E_v, and hence the claim.

The main theorem of [1] states that for any sufficiently large $M(\geq M_0)$, there exists a finite valuation v such that $v(x_M) < 0$ but $v(x_i) \geq 0$ for all $i < M$. We choose $r = r_0M_0$. Pick such a valuation v for $M = rm$. The hypothesis implies that $v(x_{rn}) < 0$ and hence $v(x_{r(m,n)}) < 0$. But $r(m, n) \leq rm$ and $v(x_i) \geq 0$ for any $i < rm$. Hence $r(m, n) = rm$ so m divides n. □

Lemma 3.3. *Any $\xi \in \mathscr{O} - \{0\}$ divides the weak denominator of some x_n.*

Proof. The set $E(\mathscr{O}/\xi)$ is finite but contains $\{nP \bmod \xi\}_{n\in\mathbf{Z}}$. Hence there are $a \neq b \in \mathbf{Z}$ with $aP = bP \bmod \xi$, so $NP = 0 \bmod \xi$ for $N = a - b \neq 0$. Therefore, ξ divides $\operatorname{wd}(x_N)$. □

Lemma 3.4. *Let m, n, q be integers with $n = mq$. Then*

$$\operatorname{wd}(x_m) | \operatorname{wn}(\frac{x_n y_m}{y_n x_m} - q).$$

Proof. The formal power series expansion for addition on E around 0 ([15], IV.2.3) implies that $\frac{x_n}{y_n} = q\frac{x_m}{y_m} + O((\frac{x_m}{y_m})^2)$, whence the result. □

4. Proof of the main theorem

Let $\xi \in \mathscr{O}$. Given an elliptic curve E of rank one over K as in the main theorem, we use the notation from Section 3 for this E — in particular, choose a suitable r such that Lemma 3.2 applies; we also choose ℓ which comes with the definition of A. We claim that the following formulæ give a diophantine definition of $\mathbf{Z}$ in $\mathscr{O}$:

$$\xi \in \mathbf{Z} \iff \exists\, m, n \in rT\mathbf{Z},\, \exists\, u \in A - \{0\} \begin{cases} (1) & m | n \\ (2) & 2^{n!+1} \prod_{i=0}^{n!-1} (\xi^{\ell n!} + i)^{n!} | u \\ (3) & u^h | \operatorname{wd}(x_m) \\ (4) & \operatorname{wd}(x_m) | \operatorname{wn}(\frac{x_n y_m}{x_m y_n} - \xi) \end{cases}$$

4.1. Any $\xi \in \mathbf{Z}$ satisfies the relations. If $\xi \in \mathbf{Z}$, then a u satisfying (2) exists because A is division-dense. By Lemma 3.3, there exists an m satisfying (3) for this u. Define $n = m\xi$ for this m. Then (1) is automatic and (4) is the contents of Lemma 3.4.

4.2. A ξ satisfying the relations is rational. Let $q \in \mathbf{Z}$ satisfy $n = qm$ (which exists by (1)). Then Lemma 3.4 implies that

$$\operatorname{wd}(x_m) | \operatorname{wn}(\frac{x_n y_m}{x_m y_n} - q),$$

which can be combined with (4) using the non-archimedean triangle inequality to give

$$\operatorname{wd}(x_m) | \operatorname{wn}(\xi - q) = (\xi - q)^h.$$

By (3), then also $u | \xi - q$.

By norm-boundedness of A we can find $\widetilde{u} \in \mathbf{Z}$ such that $\widetilde{u} | u$ and $|N(u)| \leq \widetilde{u}^\ell$. We still have

$$(*) \quad \xi \equiv q \bmod \widetilde{u}; \quad \widetilde{u}, q \in \mathbf{Z}.$$

Condition (2) implies that Lemma 2.2(1) can be applied with $\xi^{\ell n!}$ in place of ξ, so for any complex embedding σ of K we find

$$(**) \quad |\sigma(\xi)| \leq \frac{1}{2}|N(u)|^{\frac{1}{\ell n!}} \leq \frac{1}{2}N(\widetilde{u})^{\frac{1}{n!}}.$$

Because of $(*)$ and $(**)$, we can apply Lemma 2.2(2) to conclude $\xi \in \mathbf{Z}$.

4.3. The relations (1)-(4) are diophantine over $\mathscr{O}$. By 2.1 and 3.1, for $a \in \mathscr{O}$, the relations $\exists n \in rT\mathbf{Z} : a = \mathrm{wn}(x_n)$ and $\exists n \in rT\mathbf{Z} : a = \mathrm{wd}(x_n)$ are diophantine. By the diophantineness of A, the membership $u \in A$ is diophantine, and $u \neq 0$ is diophantine ([8], Prop. 1(b)). Condition (1) is diophantine because of Lemma 3.2. Conditions (2)-(4) are obviously diophantine using 2.1. □

5. Proof of the proposition and discussion of division-ample sets

5.1. Rank-preservation over Q. Suppose there exists an abelian variety G of dimension d over $\mathbf{Q}$ such that $\mathrm{rk}\, G(\mathbf{Q}) = \mathrm{rk}\, G(K) > 0$ (note that $G(K)$ is finitely generated by the Mordell-Weil theorem). Let T denote the (finite) order of the torsion of $G(K)$ and consider the free group $TG(K) \cong \mathbf{Z}^r$. The assumption implies that $G(\mathbf{Q})$ is of finite index $[G(K) : G(\mathbf{Q})]$ in $G(K)$. The choice of an ample line bundle on G gives rise to a projective embedding of G in some projective space with coordinates $\langle x_i \rangle_{i=1}^N$, where G is cut out by finitely many polynomial equations and the addition on G is algebraic in those coordinates. Suppose $\{t_i\}$ are algebraic function of the coordinates, and local uniformizers at the unit $\mathbf{0} = (1 : 0 : \cdots : 0)$ of G (i.e., $\hat{\mathscr{O}}_{G,\mathbf{0}} = \mathbf{Q}[[t_1, \ldots, t_d]]$). Define

$$A_G := \{\mathrm{wd}(t_2(P)) \ : \ P \in T[G(K) : G(\mathbf{Q})] \cdot G(K) \text{ and } t_1(P) = 1\}.$$

We claim that A_G is division-ample. Indeed, the three conditions are satisfied:

(a) A_G is obviously diophantine over $\mathscr{O}$ (the diophantine definition comes from the chosen embedding of G).

(b) The analogue of Lemma 3.3 remains valid, so A_G is divisibility-dense. Indeed, it suffices to prove that a given non-zero integer ξ divides $t_2(NP)$ for some N (where $t_1(NP) = 1$). Since $G(\mathbf{Z}/\xi)$ is finite, there is a non-zero N for wich $NP = \mathbf{0} \bmod \xi$, and then $t_2(NP) = 0 \bmod \xi$.

(c) Since by assumption, all elements of A_G are in $\mathbf{Z}$, we can set $\widetilde{a} = a$, $\ell = n$ for any $a \in A_G$ to get the required norm-boundedness.

Remark. From available computer algebra, the construction of elliptic curves which fit the above can be automated. One can compute ranks of elliptic curves over $\mathbf{Q}$ quite fast using `mwrank` by J. Cremona [4], and over number fields using the `gp`-package of D. Simon [16]. Michael Stoll

has written a MAGMA-package that computes the rank of Jacobians of genus two curves over $\mathbf{Q}$ ([17]). Unfortunately, the current state of affairs in computational arithmetical geometry doesn't include an algorithm for the rank of abelian varieties of dimension ≥ 2 over arbitrary number fields (although the necessary descent theory exists). We will therefore restrict to examples involving elliptic curves.

Example. In the style of Poonen's result, the elliptic curve $y^2 = x^3 + 8x$ has rank one over $\mathbf{Q}$ and over $\mathbf{Q}(\sqrt{2})$, $\mathbf{Q}(\sqrt[3]{2})$ and $\mathbf{Q}(\sqrt[4]{2})$. However, this curve acquires rank two over $\mathbf{Q}(\sqrt[5]{2})$.

The curve $y^2 = x^3 + 14x$ has rank two over $\mathbf{Q}$ and over $\mathbf{Q}(\sqrt[5]{2})$, and the curve $y^2 = x^3 + \sqrt[5]{2}x^2 + 8x$ has rank one over $\mathbf{Q}(\sqrt[5]{2})$.

We conclude that the diophantine theory of the ring of integers of $\mathbf{Q}(\sqrt[n]{2})$ is undecidable for $n \leq 5$ ($n \leq 3$ also covered by known results).

Remark. We ask: given K, can one construct in some clever way a curve C over $\mathbf{Q}$ such that its Jacobian satisfies the above conditions?

5.2. Rank-preservation over Z. A similar construction (of which we leave out the details) can be performed if there exists a commutative (not necessarily complete) group variety G over $\mathbf{Z}$ such that $G(\mathscr{O})$ is finitely generated and such that $\operatorname{rk} G(\mathbf{Z}) = \operatorname{rk} G(\mathscr{O}) > 0$. We will work out an easy example. Maybe a variation of this example can help one eliminate the second condition in the main theorem.

Example. Let L be another number field, linearly disjoint from K. Let $\langle a_i \rangle$ denote a $\mathbf{Z}$-basis for $L/\mathbf{Q}$ (this is also a basis for $\mathscr{O}_{KL}$ over $\mathscr{O}_K$). Let T_L denote the norm one torus $N_{\mathbf{Q}}^L(\sum a_i x_i) = 1$. Then $T_L(\mathbf{Z}) \cong \mathscr{O}_L^*$ and

$$T_L(\mathscr{O}_K) = \ker(N_K^{KL} : \mathscr{O}_{KL}^* \to \mathscr{O}_K^*),$$

hence (by surjectivity of the relative norm) $\operatorname{rk} T_L(\mathscr{O}_K) = \operatorname{rk} \mathscr{O}_{KL}^* - \operatorname{rk} \mathscr{O}_K^*$. In particular, $T_L(\mathscr{O}_K) = T_L(\mathbf{Z})$ iff

$$r_{KL} + s_{KL} = r_K + s_K + r_L + s_L - 1$$

where r_M, s_M denote the number of real, respectively half the number of complex embeddings of a number field M. If L and K are linearly disjoint, $r_{KL} = r_K r_L$, and the condition simplifies to

$$(r_K + s_K - 1)(r_L - 1) + (r_K + 2s_K - 1)s_L = 0.$$

The only non-trivial solution is $r_K = 0, s_K = 1$ (i.e., K complex quadratic) choosing $r_L > 1, s_L = 0$.

Remark. In all these examples, division-ample sets are actually subsets of the integers. Can one find a division-ample set which does not consists of just ordinary integers?

Remark (December 2005)**.** Mazur and Rubin have shown that there exist infinitely many number fields over which the rank of every elliptic curve *defined over* $\mathbf{Q}$ is even, assuming the Parity Conjecture. More specifically, they show that if $E/\mathbf{Q}$ is an elliptic curve and $K/\mathbf{Q}$ a Galois extension such that $\mathrm{Gal}(K/\mathbf{Q})$ has a non-cyclic 2-Sylow and such that the discriminant of E is coprime to that of K, then the root number of E/K is $+1$ (compare: Rubin, talk at AIM-workshop (2005); Rubin and Mazur in: Kazuya Kato's Birthday volume of Doc. Math. (2003), pp. 585–607).

On the other hand, Poonen and Shlapentokh have remarked that the argument in [11] continues to hold under the weaker assumption that there exists an elliptic curve over $\mathbf{Q}$ retaining its positive rank over the number field K (not necessarily of rank one), see: Poonen, talk at AIM-workshop (2005); Shlapentokh, Elliptic Curves Retaining Their Rank in Finite Extensions and Hilbert's Tenth Problem, preprint (2004).

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