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On Embedding of Lie Conformal Algebras into Associative Conformal Algebras

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Abstract. We prove that a Lie conformal algebra $\mathfrak{L}$ with bounded locality function is embeddable into an associative conformal algebra $\mathfrak{A}$ with the same bound on the locality function. If $\mathfrak{L}$ is nilpotent, then so is $\mathfrak{A}$, and the nilpotency index remains the same. We also give a list of open questions concerning the embedding of Lie conformal algebras into associative conformal.
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Introduction

Conformal algebras. A conformal algebra is, roughly speaking, a linear space $\mathfrak{A}$ with infinitely many bilinear products $(n) : \mathfrak{A} \times \mathfrak{A} \to \mathfrak{A}$, parameterized by an non-negative integer n, and a derivation $D : \mathfrak{A} \to \mathfrak{A}$. An important property of these products is that for any fixed $a, b \in \mathfrak{A}$ we have $a(n)b = 0$ when n is large enough. See § 1. below for formal definitions.

Conformal algebras where introduced in [7] as the simplification of the vertex algebra structure. Sometimes they are also called "vertex Lie algebras" [4, 9]. Curiously, similar algebraic structures appeared in the Hamiltonian formalism in the theory of non-linear evolution equations [5]. For more information on conformal algebras see e.g. [3, 7, 8, 13] and the references therein.

Formulation of the problem. For any variety of algebras, like associative, Lie, Jordan, etc, there is the corresponding variety of conformal algebras, see § 1.4 for the rigorous statement. Given an associative conformal algebra $\mathfrak{A}$, we can define a different family of products $[n] : \mathfrak{A} \times \mathfrak{A} \to \mathfrak{A}$ by

$$a[n]b = a(n)b - \sum_{s \geqslant 0} (-1)^{n+s} \frac{1}{s!} D^s \big(b(n+s)a\big). \tag{1}$$

With these new products (and the same derivation D), the space $\mathfrak{A}$ becomes a Lie conformal algebra, which we will denote by $\mathfrak{A}^{(-)}$. This is analogous to the

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fact that on any associative algebra A one can define the Lie algebra structure by using commutators, or Jordan algebra structure by using anti-commutators. Like in these classical cases, a natural question is whether any Lie conformal algebra can be obtained as a subalgebra of $\mathfrak{A}^{(-)}$ for some associative conformal algebra $\mathfrak{A}$. In this case, $\mathfrak{A}$ is called *an enveloping* algebra of $\mathfrak{L}$.

As shown in [11], the answer to the above question is negative for the following reason. Let $\mathfrak{L}$ be a Lie conformal algebra, generated by a set $\mathcal{G}$. Consider all words $w = g_1(n_1)\cdots(n_{l-1})g_l \in \mathfrak{L}$ (with arbitrary order of parentheses) for $g_i \in \mathcal{G}$ and $n_i \in \mathbb{Z}_+$. Suppose that there is an integer $S(l) = S_{\mathfrak{L},\mathcal{G}}(l)$ with the following property: any word w as above with $\sum_i n_i \geqslant S(l)$ is zero. We call $S(l)$ *the locality function* of $\mathfrak{L}$. It is easy to show that if $|\mathcal{G}| < \infty$, then $S_{\mathfrak{L},\mathcal{G}}(l)$ always exists and for a different set of generators $\mathcal{H}$, the difference $|S_{\mathfrak{L},\mathcal{G}}(l) - S_{\mathfrak{L},\mathcal{H}}(l)|$ has at most linear growth in l. We have shown in [11] that if a finitely generated Lie conformal algebra $\mathfrak{L}$ is embeddable into an associative conformal algebra, then $S(l)$ must have linear growth. On the other hand, for a free Lie conformal algebra the growth of $S(l)$ is quadratic [11, 12], so it is not embeddable into associative.

Results of this paper. We prove the following theorem:

Theorem 1. *Let $\mathfrak{L}$ be a Lie conformal algebra, generated by a set $\mathcal{G}$. Assume that there is $K \in \mathbb{Z}$ such that $S_{\mathfrak{L},\mathcal{G}}(l) \leqslant K$ for any l. Then $\mathfrak{L}$ is embeddable into an associative conformal algebra $\mathfrak{A}$ such that $S_{\mathfrak{A},\mathcal{G}}(l) \leqslant K$ for all l.*

In particular, the condition of Theorem 1 applies when the algebra $\mathfrak{L}$ is nilpotent. We say that $\mathfrak{L}$ is nilpotent of index k if all words $a_1(m_1)\cdots(m_{l-1})a_l$ are zero when $l \geqslant k$. Then we can prove

Theorem 2. *Any nilpotent Lie conformal algebra has a nilpotent associative conformal enveloping algebra of the same index.*

We remark that not all algebras that satisfy the assumption of Theorem 1 are nilpotent. For example, the loop algebra $\mathfrak{L}$ (see §1.6 below) has locality function 1, but is not nilpotent. See §2.1 for other examples.

Both Theorems 1 and 2 will be derived from Proposition 2 in §2.1. The argument was partially inspired by [1].

Open questions. First of all, it remains unclear whether a linear growth of the locality function is a sufficient condition for the embedding of a Lie conformal algebra into an associative. My guess is that this is false, but I don't have a counterexample.

An important special case is when the Lie conformal algebra is of *finite type*, which means that it is a module of finite rank over the algebra of polynomials in D [3]. It is shown in [11] that such algebras have linear locality functions. It has been conjectured [11] that a finite type Lie conformal algebra has a finite type conformal associative enveloping algebra. This conjecture is closely related to another conjecture that states that a finite type torsion-free Lie conformal algebra always has a faithful module of finite type (see §1.7 for the definitions and further discussion). This is the conformal analogue of classical Ado's theorem [1, 6].

The results of this paper show embeddability when the locality function is uniformly bounded. In §2.1 we conjecture that a central extension of a finite type Lie conformal algebra with bounded locality function also has a bounded locality function. This would imply a general way of generating examples of such algebras.

1. Definitions and Notations

All algebras and spaces are assumed to be over a field $\Bbbk$ of characteristic zero. Throughout the paper we will use the divided powers notation $x^{(n)} = \frac{1}{n!}x^n$, and $\mathbb{Z}_+$ will stand for the set of non-negative integers.

1.1. The definition.

Definition 1. [7] *A* conformal algebra *is a space* $\mathfrak{A}$ *equipped by a collection of bilinear products* $A \otimes A \to A$, $a \otimes b \mapsto a(n)b$, *indexed by* $n \in \mathbb{Z}_+$, *and a linear map* $D : \mathfrak{A} \to \mathfrak{A}$ *such that*

(C1) $a(n)b = 0$ *for* $n \gg 0$*;*

(C2) $D\big(a(n)b\big) = (Da)(n)b + a(n)(Db) = -n\, a(n-1)b + a(n)(Db)$*.*

If only the condition (C2) *is satisfied, then we will call* $\mathfrak{A}$ a preconformal algebra.

Iterating (C2), we get

$$\begin{aligned} \big(D^{(k)}a\big)(n)b &= (-1)^n \binom{n}{k} a(n-k)b, \\ a(n)\big(D^{(k)}b\big) &= \sum_{s \geqslant 0} \binom{n}{s} D^{(k-s)}\big(a(n-s)b\big). \end{aligned} \tag{2}$$

1.2. Formal series. A typical way of constructing a conformal algebra is as follows. Take a "usual" algebra A. Consider the space of formal series $A[[z, z^{-1}]]$. We will write a series $\alpha \in A[[z, z^{-1}]]$ as

$$\alpha(z) = \sum_{n \in \mathbb{Z}} \alpha(n)\, z^{-n-1}, \quad \alpha(n) \in A.$$

For $\alpha, \beta \in A[[z, z^{-1}]]$ define their n-th product

$$(\alpha(n)\beta)(z) = \sum_{m \in \mathbb{Z}} \big(\alpha(n)\beta\big)(m)\, z^{-m-1}$$

by

$$(\alpha(n)\beta)(z) = \operatorname{Res}_w \alpha(w)\beta(z)(z-w)^n,$$

so that

$$\big(\alpha(n)\beta\big)(m) = \sum_{s=0}^{n} (-1)^s \binom{n}{s} \alpha(n-s)\beta(m+s). \tag{3}$$

Series $\alpha, \beta \in A[[z, z^{-1}]]$ are *local* of order $N \in \mathbb{Z}_+$, if

$$\alpha(w)\beta(z)(w-z)^N = 0.$$

In terms of coefficients this means

$$\sum_{s=0}^{N}(-1)^s\binom{N}{s}\alpha(n-s)\beta(m+s)=0$$

for any $m,n\in\mathbb{Z}$. It is easy to see that if α and β are local of order N, then $\alpha(n)\beta=0$ for $n\geqslant N$.

Proposition 1. [7, 8] *If $\mathfrak{A}\subset A[[z,z^{-1}]]$ is a space of series such that*

(i) *any two series $\alpha,\beta\in\mathfrak{A}$ are local;*

(ii) *$\alpha(n)\beta\in\mathfrak{A}$ for any $\alpha,\beta\in\mathfrak{A}$;*

(iii) *$\partial_z\alpha\in\mathfrak{A}$ for any $\alpha\in\mathfrak{A}$,*

then $\mathfrak{A}$ is a conformal algebra with $D=\partial_z$. If only conditions (ii) *and* (iii) *hold, then $\mathfrak{A}$ is a preconformal algebra.*

Remark 1. *If A is either associative or Lie algebra, then condition* (i) *of Proposition 1 can be weakened: it is enough to assume the locality only for a set of generators $\mathcal{G}$ of $\mathfrak{A}$. This fact is know as* the Dong's lemma.

1.3. Coefficient algebra. Conversely, any conformal algebra can be obtained as in Proposition 1. Moreover, to any conformal algebra $\mathfrak{A}$ there corresponds a "usual" algebra $A=\operatorname{Coeff}\mathfrak{A}$, called the *coefficient algebra* of $\mathfrak{A}$, and the inclusion $\pi:\mathfrak{A}\hookrightarrow A[[z,z^{-1}]]$ with the following universal property. For any other homomorphism $\varphi:\mathfrak{A}\to B[[z,z^{-1}]]$ of $\mathfrak{A}$ to the space of formal series, such that $\varphi(\mathfrak{A})$ satisfies the conditions of Proposition 1, there is a unique algebra homomorphism $\rho:A\to B$ such that $\rho(\pi(a))=\varphi(a)$ for any $a\in\mathfrak{A}$.

The coefficient algebra $A=\operatorname{Coeff}\mathfrak{A}$ is constructed in the following way. Consider the space of Laurent series $\mathfrak{A}[t,t^{-1}]$ in an independent variable t with coefficients in $\mathfrak{A}$. For $a\in\mathfrak{A}$, denote $a(n)=at^n$. As a linear space A is isomorphic to the quotient of $\mathfrak{A}[t,t^{-1}]$ over the subspace generated by the vectors $(Da)+n\,a(n-1)$ for $a\in\mathfrak{A}$. The formula for the product in A is derived from (3):

$$a(m)b(n)=\sum_{s\geqslant 0}\binom{m}{s}\big(a(s)b\big)(m+n-s).$$

Note that the sum here is finite due to (C1). The canonical inclusion $\mathfrak{A}\to A[[z,z^{-1}]]$ is given by $a\mapsto\sum_{n\in\mathbb{Z}}a(n)\,z^{-n-1}$.

In §2.2 below we are going to need to following fact:

Lemma 1. [8, 10] *Assume that $\mathfrak{A}$ is a free $\Bbbk[D]$-module, and let $\mathcal{B}\subset\mathfrak{A}$ be its basis over $\Bbbk[D]$. Then the set $\{\,b(n)\mid b\in\mathcal{B},\,n\in\mathbb{Z}\,\}$ is a $\Bbbk$-linear basis of* $\operatorname{Coeff}\mathfrak{A}$.

Remark 2. *The requirement that $\mathfrak{A}$ is a free $\Bbbk[D]$-module is not as restrictive as it might appear. In any conformal algebra $\mathfrak{A}$ one can define the so-called* torsion ideal $\mathfrak{t} = D\text{-tor}\,\mathfrak{A} + \bigcap_{n\geqslant 0} D^n\mathfrak{A}$, *where* $D\text{-tor}\,\mathfrak{A} = \{\, a \in \mathfrak{A} \mid \exists\, p(D) \in \Bbbk[D],\ p(D) \neq 0,\ p(D)a = 0 \,\}$ *is the D-torsion of $\mathfrak{A}$, so that $\mathfrak{A}/\mathfrak{t}$ is a free $\Bbbk[D]$-module. It is easy to show [3] that $\mathfrak{t}$ belongs to the left annihilator of $\mathfrak{A}$, i.e. $a(n)b = 0$ for any $a \in \mathfrak{t}$, $b \in \mathfrak{A}$ and $n \in \mathbb{Z}_+$.*

1.4. Varieties of conformal algebras. In the case when the coefficient algebra $A = \operatorname{Coeff}\mathfrak{A}$ belongs to a certain variety of algebras, the conformal algebra $\mathfrak{A}$ is said to belong to the corresponding conformal variety. For example, if A is an associative (respectively, a Lie or a Jordan) algebra, then $\mathfrak{A}$ is called an associative conformal (respectively, a Lie conformal or a Jordan conformal) algebra. Moreover, if A belongs to a certain variety of algebras, then any conformal subalgebra of $A[[z, z^{-1}]]$ as in Proposition 1 belongs to the corresponding conformal variety. In this paper we deal only with Lie or associative conformal algebras. To distinguish between them, we will denote the products in a conformal algebra by $[n]$ whenever the product in the coefficient algebra is denoted by the brackets $[\,\cdot,\,\cdot\,]$.

There is a correspondence between the identities in a conformal algebra and the identities in its coefficient algebra. An identity R holds in $A = \operatorname{Coeff}\mathfrak{A}$ if and only if a certain identity (or family of identities) $\operatorname{Conf} R$ holds in $\mathfrak{A}$. For example, the associativity $(ab)c = a(bc)$, the Jacoby identity $[[a,\,b],\,c] = [a,\,[b,\,c]] - [b,\,[a,\,c]]$ and the (skew-)symmetry $ab = \pm\, ba$ correspond to the following conformal identities respectively:

conformal associativity:

$$\big(a(m)b\big)(n)c = \sum_{s\geqslant 0} (-1)^s \binom{m}{s} a(m-s)\big(b(n+s)c\big) \tag{4}$$

conformal Jacoby identity:

$$\big(a[m]b\big)[n]c = \sum_{s\geqslant 0} (-1)^s \binom{m}{s} \Big(a[m-s]\big(b[n+s]c\big) - b[n+s]\big(a[m-s]c\big)\Big)$$

quasi-symmetry:

$$a(n)b = \pm \sum_{s\geqslant 0} (-1)^{s+n} D^{(s)}\big(b(n+s)a\big) \tag{5}$$

We will need in §2.2 the following strengthening of the above correspondence.

Lemma 2. *Let A be an associative algebra, and $\mathfrak{A} \subset A[[z, z^{-1}]]$ a preconformal algebra of formal series with coefficients in A. Then the identity* (4) *holds in $\mathfrak{A}$.*

Proof. Take $k \in \mathbb{Z}$. The k-th coefficient of the left- and right-hand sides of (4) are, respectively

$$\sum_{i,j\geqslant 0} (-1)^{i+j} \binom{m}{i}\binom{n}{j} a(m-i)b(n+i-j)c(k+j)$$

and

$$\sum_{i,j,s\geqslant 0}(-1)^{i+j+s}\binom{m}{s}\binom{m-s}{i}\binom{n+s}{j}a(m-s-i)b(n+s-j)c(k+i+j).$$

Replace the indices in the second formula by the rule $i\to i-s,\ \ j\to j-i+s$, and then we are done by the combinatorial identity

$$\sum_{s=0}^{i}(-1)^{s+j}\binom{m}{s}\binom{m-s}{i-s}\binom{n+s}{j-i+s}=(-1)^{i+j}\binom{m}{i}\binom{n}{j}.$$

■

For a Lie conformal algebra $\mathfrak{L}$ denote by $Z(\mathfrak{L})=\{\,a\in\mathfrak{L}\,|\,a(n)b=0\ \forall\, b\in\mathfrak{L},\ n\in\mathbb{Z}_+\,\}$ the center of $\mathfrak{L}$. Due to the quasi-symmetry (5), we have $b(n)a=0$ for any $a\in Z(\mathfrak{L}),\ \ b\in\mathfrak{L}$ and $n\in\mathbb{Z}_+$.

1.5. The relation between associative and Lie conformal algebras. Let $\mathfrak{A}$ be an associative conformal algebra. Then we can define another family of products $[n],\ \ n\in\mathbb{Z}_+$, on $\mathfrak{A}$ by the formula (1). This will define a Lie conformal algebra structure on $\mathfrak{A}$, which we will denote by $\mathfrak{A}^{(-)}$. Recall that any associative algebra A can be turned into a Lie algebra $A^{(-)}$ by taking the commutator $[a,\,b\,]=ab-ba$ for the product. It is easy to check that $(\operatorname{Coeff}\mathfrak{A})^{(-)}=\operatorname{Coeff}(\mathfrak{A}^{(-)})$.

Here is another useful formula that holds in $\mathfrak{A}$:

$$a(m)b(n)c-b(n)a(m)c=\sum_{s\geqslant 0}\binom{m}{s}\big(a[s]b\big)(m+n-s)c \qquad (6)$$

In §2.2 we will deal with the following situation. Let $\mathfrak{L}$ be a Lie conformal algebra and $L=\operatorname{Coeff}\mathfrak{L}$ be its coefficient Lie algebra. Let $A\supset L$ be an associative enveloping algebra of L. Then we get $\mathfrak{L}\subset A[[z,z^{-1}]]$. Let $\mathfrak{A}\subset A[[z,z^{-1}]]$ be the associative preconformal algebra generated by $\mathfrak{L}$. By Lemma 2, the conformal associativity (4) holds in $\mathfrak{A}$. The following statement is checked in a similar way.

Lemma 3. *The formula* (6) *holds for any* $a,b\in\mathfrak{L},\ \ c\in\mathfrak{A}$ *and* $m,n\in\mathbb{Z}_+$.

1.6. Example: loop algebras. Let $\mathfrak{g}$ be an algebra. Let $L=\mathfrak{g}[[t,t^{-1}]]$. Denote $a(m)=at^m$ for $a\in\mathfrak{g}$ and $m\in\mathbb{Z}$, so that $a(m)b(n)=(ab)(m+n)$. For $a\in\mathfrak{g}$ set

$$\widetilde{a}=\sum_{m\in\mathbb{Z}}a(m)\,z^{-m-1}\in L[[z,z^{-1}]].$$

It is easy to see that $\widetilde{a}$ and $\widetilde{b}$ are local of order 1 and $\widetilde{a}(0)\widetilde{b}=\widetilde{ab}$. Let $\mathfrak{L}\subset L[[z,z^{-1}]]$ be the conformal algebra generated by $\widetilde{a}$ for all $a\in\mathfrak{g}$. It is called the *loop algebra* of $\mathfrak{g}$. As a $\Bbbk[D]$-module, $\mathfrak{L}$ is freely generated by $\mathcal{G}=\{\,\widetilde{a}\,|\,a\in\mathfrak{g}\,\}$, so Lemma 1 implies that $L=\operatorname{Coeff}\mathfrak{L}$.

We remark that a monomial $\widetilde{a}_1(m_1)\cdots\widetilde{a}_{l-1}(m_{l-1})\widetilde{a}_l\in\mathfrak{L}$ (with arbitrary order of parentheses) is equal to 0 if $\sum m_i>0$, so the locality function $S_{\mathfrak{L},\mathcal{G}}=1$. It is easy to show that if a conformal algebra has locality function 1, then it must be a loop algebra.

In the case when $\mathfrak{g}$ is a Lie algebra, it often comes with an invariant bilinear form $\langle\cdot\,|\,\cdot\rangle$, and then the corresponding loop algebras L and $\mathfrak{L}$ have central extensions $\widehat{L} = L \oplus \Bbbk\mathsf{c}$ and $\widehat{\mathfrak{L}} = \mathfrak{L} \oplus \Bbbk\mathsf{c}$. The brackets in $\widehat{L}$ are $[a(m), b(n)] = [a, b](m+n) + m\delta_{m,-n}\langle a\,|\,b\rangle$, the locality of $\widetilde{a}$ and $\widetilde{b}$ is 2, and the conformal products are $\widetilde{a}(0)\widetilde{b} = \widetilde{[a, b]}$, $\widetilde{a}(1)\widetilde{b} = \mathsf{c}$. We identify c with $\mathsf{c}(-1) \in \widehat{L}[[z, z^{-1}]]$. This is called the *affine Lie algebra* corresponding to $\mathfrak{g}$. The locality function of $\widehat{\mathfrak{L}}$ corresponding to the generators $\mathcal{G} \cup \{\mathsf{c}\}$ is equal to 2.

1.7. Representations of conformal algebras. Let M be a $\Bbbk[D]$ module of finite rank.

Definition 2. [2] *A* conformal operator α *on* M *is a series*

$$\alpha = \sum_{n\geqslant 0} \alpha(n)\, z^{-n-1} \in \mathrm{End}(M)[[z^{-1}]],$$

such that

(CO1) *for any fixed* $v \in M$ *we have* $\alpha(n)v = 0$ *for* $n \gg 0$,

(CO2) $[D, \alpha(n)] = -n\,\alpha(n-1)$.

Denote the space of all conformal operators by $\mathrm{CEnd}(M) \subset \mathrm{End}(M)[[z^{-1}]]$.

In fact, any conformal operator $\alpha \in \mathrm{CEnd}(M)$ is zero on D-tor M, so we can assume that M is a free $\Bbbk[D]$-module without loss of generality.

We observe that the formula (3) makes sense when $\alpha, \beta \in \mathrm{CEnd}(M)$, and also $\mathrm{CEnd}(M)$ is closed under the derivation $D = \partial_z$, so it can be shown that $\mathrm{CEnd}(M)$ is an associative conformal algebra [2].

Let $\mathfrak{A}$ be an associative (respectively, a Lie) conformal algebra, then by definition, M is a module over $\mathfrak{A}$ if there is a conformal algebra homomorphism $\mathfrak{A} \to \mathrm{CEnd}(M)$ (respectively, $\mathfrak{A} \to \mathrm{CEnd}(M)^{(-)}$). For example, the algebra $\mathfrak{A}$ is a module over itself with the representation map $\mathfrak{A} \to \mathrm{CEnd}(\mathfrak{A})$ given by $a \mapsto \sum_{n\geqslant 0} a(n)\, z^{-n-1}$.

It follows that if a Lie conformal algebra $\mathfrak{L}$ has a faithful finite type module M, then $\mathrm{CEnd}(M)$ is an associative conformal enveloping algebra of $\mathfrak{L}$. In particular, this applies to the case when $Z(\mathfrak{L}) = 0$ so that $\mathfrak{L}$ is a faithful module over itself. Together with Theorems 1 and 2 this provides some grounds to the conjecture that any finite type Lie conformal algebra is embeddable into an associative conformal algebra.

For further information about these and other conformal algebras consult e.g. the reviews [8, 13] and the references therein.

2. Conformal algebras with bounded locality function

Let $\mathfrak{L}$ be a conformal algebra generated by a set $\mathcal{G} \subset \mathfrak{L}$. Recall that the *locality function* $S(l) = S_{\mathfrak{L},\mathcal{G}}(l)$ is an integer such that any word $w = g_1(n_1)\cdots(n_{l-1})g_l \in \mathfrak{L}$, where $g_i \in \mathcal{G}$ and $n_i \in \mathbb{Z}_+$, is zero whenever $\sum_i n_i \geqslant S(l)$. If $|\mathcal{G}| < \infty$, then the existence of $S(l)$ follows from (C1). If $\mathfrak{L}$ is a Lie conformal algebra, then the quantitative version of Dong's lemma [12] implies that if the locality

$N(a,b)$ of any two generators $a, b \in \mathcal{G}$ is uniformly bounded by $N \in \mathbb{Z}_+$, then $S(l) \leqslant \frac{1}{2}Nl(l-1) - l + 1$.

2.1. The main Proposition. To every generator $g \in \mathcal{G} \subset \mathfrak{L}$ we assign a weight $\operatorname{wt} g \in \mathbb{Z}_+$. (The word "degree" will be used later for different purpose). For a monomial $w = g_1[n_1]\cdots[n_{l-1}]g_l$ or $w = g_1(n_1)\cdots(n_{l-1})g_l$ (with arbitrary order of parentheses), where $g_i \in \mathcal{G}$ and $n_i \in \mathbb{Z}_+$, we set $\operatorname{wt} w = \sum_i \operatorname{wt} g_i + \sum_i n_i$. We also set $\operatorname{wt} D = -1$.

Both Theorem 1 andTheorem 2 follow from the following statement:

Proposition 2. *Assume that there is an integer $r \geqslant 0$ such that any monomial of weight r or more is equal to zero in $\mathfrak{L}$. Then $\mathfrak{L}$ is embedded into an enveloping conformal associative algebra $\mathfrak{A}$, that has the same property: any conformal monomial w in $\mathcal{G}$ is zero in $\mathfrak{A}$ whenever* $\operatorname{wt} w \geqslant r$.

Theorem 1 (respectively, Theorem 2) is a special case of Proposition 2 obtained by setting $\operatorname{wt} g = 1$ (respectively, $\operatorname{wt} g = 0$) for every $g \in \mathcal{G}$. We will prove Proposition 2 in §2.2.

We have seen one example of a Lie conformal algebra with bounded locality function in §1.6. It is easy to see that any central extension of a loop algebra also has a bounded locality function. We state the following conjecture.

Conjecture 1. *Let $\mathfrak{L}$ be a Lie conformal algebra of finite type generated by a finite set $\mathcal{G}$ so that $S_{\mathfrak{L},\mathcal{G}}(l) < K$. Let M be a $\Bbbk[D]$-module, generated by a set $\mathcal{M}$, on which $\mathfrak{L}$ acts trivially, and let $\widehat{\mathfrak{L}}$ be a central extension of $\mathfrak{L}$. Then $S_{\widehat{\mathfrak{L}},\mathcal{G}\cup\mathcal{M}}$ is also uniformly bounded.*

By "central extension" we mean that there is a short exact sequence of conformal algebra homomorphisms $0 \to M \to \widehat{\mathfrak{L}} \to \mathfrak{L} \to 0$, such that $M \subseteq Z(\widehat{\mathfrak{L}})$.

2.2. Proof of Proposition 2.

2.2.1. *Two filtrations on $\mathfrak{L}$.*
Let $\mathfrak{L}$ be a Lie conformal algebra, satisfying the conditions of Proposition 2. Define a filtration

$$\mathfrak{L} \supseteq \ldots \supseteq \mathfrak{L}'_{i-1} \supseteq \mathfrak{L}'_i \supseteq \mathfrak{L}'_{i+1} \supseteq \ldots \supseteq \mathfrak{L}'_r = 0$$

on $\mathfrak{L}$ by setting

$$\mathfrak{L}'_i = \operatorname{Span}\left\{ w = D^m g_1[n_1]\cdots[n_{l-1}]g_l \;\middle|\; g_j \in \mathcal{G},\ \operatorname{wt} w \geqslant i \right\}.$$

We have $\mathfrak{L}'_i = 0$ for $i \geqslant r$ due to the fact that there are no words of weight l or more in $\mathfrak{L}$. Clearly, we also have $\bigcup_i \mathfrak{L}'_i = \mathfrak{L}$. For an element $a \in \mathfrak{L}$ set $\deg' a = \max\{ i \,|\, a \in \mathfrak{L}'_i \}$. Note that for a Lie conformal monomial w in $\mathcal{G}$ we have $\operatorname{wt} w \leqslant \deg' g$.

Here are some easy properties of this filtration that we are going to need:

Lemma 4. (a). $\mathfrak{L}'_i[n]\mathfrak{L}'_j \subseteq \mathfrak{L}'_{i+j+n}$, $D\mathfrak{L}'_i \subseteq \mathfrak{L}'_{i-1}$.

(b). $\mathfrak{L} = \Bbbk[D]\mathfrak{L}'_0$.

Proof. (a) follows from the fact that the formulas (2) are homogeneous. To prove (b), note that since every generator $g \in \mathcal{G}$ has $\operatorname{wt} g \geqslant 0$, an element $a \in \mathfrak{L}$ of negative degree must belong to $D\mathfrak{L}$. ■

Next we set

$$\mathfrak{L}_i = \left\{ a \in \mathfrak{L} \;\middle|\; \exists n : D^n a \in \mathfrak{L}'_{i-n} \right\}.$$

This defines another filtration on $\mathfrak{L}$ of the form

$$\mathfrak{L} \supseteq \ldots \supseteq \mathfrak{L}_{i-1} \supseteq \mathfrak{L}_i \supseteq \mathfrak{L}_{i+1} \supseteq \ldots .$$

Set $\mathfrak{L}_\infty = \bigcap_i \mathfrak{L}_i$ and $\deg a = \sup\{ i \,|\, a \in \mathfrak{L}_i \}$ for $a \in \mathfrak{L}$. Clearly, we have $\mathfrak{L}'_i \subseteq \mathfrak{L}_i$, therefore $\deg a \geqslant \deg' a$. More precisely, we have

$$\deg a = \sup_n \{ n + \deg' D^n a \}. \tag{7}$$

Here are some properties of the filtration $\{\mathfrak{L}_i\}$:

Lemma 5. (a). $\mathfrak{L}_i[n]\mathfrak{L}_j \subseteq \mathfrak{L}'_{i+j+n}$. *In particular,* $\mathfrak{L}_i[n]\mathfrak{L}_j = 0$ *if* $i+j+n \geqslant r$.

(b). $\mathfrak{L}_r \subseteq Z(\mathfrak{L})$.

(c). $\deg Da = \deg a - 1$ *for any* $a \in \mathfrak{L}$.

Proof. (a) Let $a \in \mathfrak{L}_i$ and $b \in \mathfrak{L}_j$. Since $a[n]b = 0$ for $n \gg 0$, we can, using induction, assume that $a[s]b \in \mathfrak{L}'_{i+j+s}$ for any $s > n$.

There are $k, m \in \mathbb{Z}_+$, such that $D^{(k)}a \in \mathfrak{L}'_{i-k}$ and $D^{(m)}b \in \mathfrak{L}'_{j-m}$. Using Lemma 4 (a) and (2), we get

$$\begin{aligned} \mathfrak{L}'_{i+j+n} &\ni \big(D^{(k)}a\big)[k+m+n]\big(D^{(m)}b\big) \\ &= (-1)^k \binom{k+m+n}{k} \sum_{s=0}^{m} \binom{m+n}{m-s} D^{(s)}\big(a[n+s]b\big) \\ &\equiv (-1)^k \binom{k+m+n}{k} \binom{m+n}{m} \; a[n]b \mod \mathfrak{L}'_{i+j+n}, \end{aligned}$$

since by Lemma 4 (a) and induction, $D^{(s)}\big(a[n+s]b\big) \in \mathfrak{L}'_{i+j+n}$ for $s > 0$.

(b) Since $\mathfrak{L}'_0 \subseteq \mathfrak{L}_0$, Lemma 4 (b) implies that $\mathfrak{L} = \Bbbk[D]\mathfrak{L}_0$, and by (a) we have $a[n]b = 0$ for any $a \in \mathfrak{L}_0$ and $b \in \mathfrak{L}_r$.

(c) By Lemma 4 (a) we have $\deg' Da \geqslant \deg' a - 1$. Using this and (7), we get

$$\deg Da = \sup_{n \geqslant 0} \left\{ n + \deg' D^{n+1} a \right\} = \sup_{n \geqslant 0} \left\{ n - 1 + \deg' D^n a \right\} = \deg a - 1.$$

■

It follows from (b) and (c) that $\mathfrak{L}_\infty$ is a central ideal of $\mathfrak{L}$.

2.2.2. *The basis* $\mathcal{B}$. Let $\mathcal{B}_i \subset \mathfrak{L}_i$ be a $\Bbbk$-linear basis of $\mathfrak{L}_i$ modulo $\mathfrak{L}_{i+1} + D\mathfrak{L}_{i+1}$. By Lemma 4 (b), if $i < 0$, then $\mathcal{B}_i = \varnothing$. Denote $\mathcal{B} = \bigcup_{i=0}^{r-1} \mathcal{B}_i$. Let $\mathfrak{T} = \Bbbk[D]\mathfrak{L}_r$. By Lemma 5 (a) and (b), this is a central ideal of $\mathfrak{L}$.

Lemma 6. *The set $\mathcal{B}$ is a $\Bbbk[D]$-linear basis of $\mathfrak{L} \mod \mathfrak{T}$. The expansion of an element $a \in \mathfrak{L}$ of $\deg a = i$ in this basis is*

$$a = \sum_{n\in\mathbb{Z}_+,\, b\in\mathcal{B}} k_{n,b}\, D^n b + a_0, \qquad k_{n,b} \in \Bbbk,\ a_0 \in \mathfrak{T}, \tag{8}$$

such that $\deg D^n b = \deg b - n \geqslant i$ whenever $k_{n,b} \neq 0$.

Proof. First we show that any element $a \in \mathfrak{L}_i$ has expansion (8). Indeed, if $i \geqslant r$, this is obvious, so by induction we can assume that any $a \in \mathfrak{L}_{i+1}$ has such an expansion. Now, we can decompose $a = a_1 + a_2$ so that $a_1 \in \operatorname{Span}_{\Bbbk} \mathcal{B}_i$ and $a_2 \in \mathfrak{L}_{i+1} + D\mathfrak{L}_{i+1}$; by induction, a_2 has an expansion (8), therefore, so does a.

This shows that $\mathcal{B}$ spans $\mathfrak{L}$ modulo $\mathfrak{T}$ over $\Bbbk[D]$. Let us prove that the set $\mathcal{B}$ is linearly independent over $\Bbbk[D]$ modulo $\mathfrak{T}$.

We observe that in the $\Bbbk[D]$-module $\mathfrak{L}/\mathfrak{T}$ we have $\operatorname{Ker} D = 0$. Indeed, every element $t \in \mathfrak{T}$ can be written as $t = Dt_0 + t_1$ for $t_0 \in \mathfrak{T}$ and $t_1 \in \mathfrak{L}_r$. So if $Da \in \mathfrak{T}$, then write $Da = Dt_0 + t_1$, and get $D(a - t_0) = t_1 \in \mathfrak{L}_r$, hence by Lemma 5 (c) we get $\deg(a - t_0) \geqslant r + 1$. Therefore, $a - t_0 \in \mathfrak{T}$, hence also $a \in \mathfrak{T}$.

Now assume that we have a linear relation

$$\sum_{n\in\mathbb{Z}_+,\, b\in\mathcal{B}} k_{n,b}\, D^n b \in \mathfrak{T}, \qquad k_{n,b} \in \Bbbk.$$

By the previous paragraph, we can assume that the set $\mathcal{B}' = \{\, b \in \mathcal{B} \mid k_{0,b} \neq 0 \,\}$ in non-empty. Let i be the minimal degree of the elements in this set, and let $\mathcal{B}'_i = \mathcal{B}_i \cap \mathcal{B}'$ be the elements degree i in $\mathcal{B}'$. The linear relation above implies that $\mathcal{B}'$ is linear dependent modulo $D\mathfrak{L} + \mathfrak{L}_{i+1}$, which contradicts the definition of $\mathcal{B}_i$, since $D\mathfrak{L} \cap \mathfrak{L}_i = D\mathfrak{L}_{i+1}$ due to Lemma 5 (c). ∎

2.2.3. *The algebra* $\mathfrak{U}$. Let $L = \operatorname{Coeff} \mathfrak{L}$ be the coefficient Lie algebra of $\mathfrak{L}$. Consider its universal enveloping algebra $U(L)$ and let U be the augmentation ideal of $U(L)$.

Consider the space $U[[z, z^{-1}]]$ of formal series with coefficients in U. Let $\mathfrak{U} \subset U[[z, z^{-1}]]$ be the associative preconformal algebra generated by the series $\sum_{m\in\mathbb{Z}} a(m)\, z^{-m-1}$ for $a \in \mathfrak{L}$. By Lemma 2 and Lemma 3 the identity (4) holds for any $a, b, c \in \mathfrak{U}$ and (6) holds for any $a, b \in \mathfrak{L}$ and $c \in \mathfrak{U}$.

Define a filtration $\mathfrak{U} \supseteq \ldots \supseteq \mathfrak{U}_{i-1} \supseteq \mathfrak{U}_i \supseteq \mathfrak{U}_{i+1} \supseteq \ldots$ on $\mathfrak{U}$ by setting

$$\mathfrak{U}_i = \operatorname{Span}\Big\{\, D^m a_1(m_1)\cdots a_{l-1}(m_{l-1}) a_l \;\Big|\; a_j \in \mathfrak{L},\ m_j \in \mathbb{Z}_+,\ \sum_{j=1}^{l-1} m_j + \sum_{j=1}^{l} \deg a_j - m \geqslant i \,\Big\}.$$

Here the order of parentheses is arbitrary, but note that using the formula (4), it is enough to take only right-normed words.

The filtration $\{\mathfrak{U}_i\}$ is defined in a similar way to the filtration $\{\mathfrak{L}'_i\}$ on $\mathfrak{L}$, constructed in §2.2.1, so it satisfies the properties, analogous to Lemma 4 (a), which are proved in the same way:

Lemma 7. $\mathfrak{U}_i(n)\mathfrak{U}_j \subseteq \mathfrak{U}_{i+j+n}, \ \ D\mathfrak{U}_i \subseteq \mathfrak{U}_{i-1}$.

2.2.4. Let $T = \operatorname{Coeff}\mathfrak{T} \subset L$ be the coefficient algebra of $\mathfrak{T}$. Then T is a central ideal of L and we have $L/T = \operatorname{Coeff}(\mathfrak{L}/\mathfrak{T})$. By Lemma 1, since $\mathcal{B}$ is a $\Bbbk[D]$-linear basis of $\mathfrak{L} \mod \mathfrak{T}$, the set $\{\, b(n) \,|\, b \in \mathcal{B},\, n \in \mathbb{Z} \,\} \subset L$ is a $\Bbbk$-linear basis of L mod T.

Let $N \subset U$ be the ideal of U generated by $T = \operatorname{Coeff}\mathfrak{T}$. Then the algebra U/N is equal to the augmentation ideal of the universal enveloping algebra $U(L/T)$. Choose a linear order on $\mathcal{B}$. For $m, n \in \mathbb{Z}$ and $a, b \in \mathcal{B}$ we will write $a(m) < b(n)$ if either $m < n$ or $m = n$ and $a < b$. The PBW theorem states that the set

$$\mathcal{S} = \Big\{\, b_1(n_1)\cdots b_l(n_l) \ \Big|\ b_i \in \mathcal{B},\, n_i \in \mathbb{Z},\, b_i(n_i) \geqslant b_{i+1}(n_{i+1}) \,\Big\} \subset U \tag{9}$$

is a $\Bbbk$-linear basis of $U \mod N$. We will order the words from $\mathcal{S}$ first by length and then alphabetically from right to left, so that for $u = b_1(n_1)\cdots b_l(n_l)$, $u' = b'_1(n'_1)\cdots b'_l(n'_l) \in \mathcal{S}$ we write $u \leqslant u'$ if $b_l(n_l) = b'_l(n'_l), \ldots, b_{i+1}(n_{i+1}) = b'_{i+1}(n'_{i+1})$, but $b_i(n_i) \leqslant b'_i(n'_i)$ for some $1 \leqslant i \leqslant l$.

Since T is a central ideal of L, the set $N_2 = TN \subset N$ is a proper subideal of N spanned by all words $a_1(m_1)\cdots a_l(m_l)$ for $a_i \in \mathfrak{L}, \ \ m_i \in \mathbb{Z}, \ \ l \geqslant 2$, such that $a_i \in \mathfrak{T}$ for some $1 \leqslant i \leqslant l$. Clearly we have $N_2 \cap L = 0$.

Similarly, we define $\mathfrak{N} \subset \mathfrak{U}$ to be the span of all words

$$D^n a_1(n_1)\cdots a_{l-1}(n_{l-1})a_l, \qquad a_i \in \mathfrak{L}$$

(for arbitrary order of parentheses), such that $a_i \in \mathfrak{T}$ for some $1 \leqslant i \leqslant l$, and $\mathfrak{N}_2 \subset \mathfrak{N}$ to be the span of the same words of length at least 2. Clearly, $\mathfrak{N}$ and $\mathfrak{N}_2$ are ideals of $\mathfrak{U}$ such that $u(n) \in N$ for $u \in \mathfrak{N}$ and $u(n) \in N_2$ for $u \in \mathfrak{N}_2$, and we have $\mathfrak{N}_2 \cap \mathfrak{L} = 0$.

2.2.5. *Basis in* $\mathfrak{U}$. Recall from §2.2.2 that we have a set $\mathcal{B} \subset \mathfrak{L}$ which is a $\Bbbk[D]$-linear basis of $\mathfrak{L}/\mathfrak{T}$. Define

$$\mathcal{W} = \left\{ b_1(n_1)\big(b_1(n_2)\cdots\big(b_{l-1}(n_{l-1})b_l\big)\cdots\big) \in \mathfrak{U} \ \left|\ \begin{array}{c} b_i \in \mathcal{B},\ n_i \in \mathbb{Z}_+ \\ b_i(n_i) \geqslant b_{i+1}(n_{i+1}) \\ \text{for } 1 \leqslant i \leqslant l-2 \end{array} \right.\right\}.$$

For $w = b_1(n_1)\big(b_1(n_2)\cdots\big(b_{l-1}(n_{l-1})b_l\big)\cdots\big) \in \mathcal{W}$ set $\deg w = \sum_j \deg b_j + \sum_j n_j$, so that $w \in \mathfrak{U}_{\deg w}$.

Lemma 8. *The set $\mathcal{W}$ is a $\Bbbk[D]$-linear basis of $\mathfrak{U}/\mathfrak{N}$, such that the expansion of an element $u \in \mathfrak{U}_i$ in this basis has form*

$$u = \sum_{n \in \mathbb{Z}_+,\, w \in \mathcal{W}} k_{n,w}\, D^{(n)} w + u_0, \quad k_{n,w} \in \Bbbk,\ u_0 \in \mathfrak{N}, \tag{10}$$

where $\deg D^{(n)} w = \deg w - n \geqslant i$ whenever $k_{n,w} \neq 0$.

Proof. Let us show first that $\mathcal{W}$ is linearly independent over $\Bbbk[D]$ modulo $\mathfrak{N}$. Suppose

$$u = \sum_{n,w} k_{n,w}\, D^{(n)} w \in \mathfrak{N}.$$

Then $u(-1) = \sum_{n,w} k_{n,w}\, w(-n-1) \in N$. For

$$w = b_1(n_1)\big(b_1(n_2)\cdots\big(b_{l-1}(n_{l-1})b_l\big)\cdots\big) \in \mathcal{W}$$

we compute, iterating (4),

$$w(-n-1) = \sum_{i_1=0}^{n_1}\cdots\sum_{i_{l-1}=0}^{n_{l-1}} (-1)^{i_1+\ldots+i_{l-1}} \binom{n_1}{i_1}\cdots\binom{n_{l-1}}{i_{l-1}} \\ \times b_1(n_1-i_1)\cdots b_{l-1}(n_{l-1}-i_{l-1})b_l(i_1+\ldots+i_{l-1}-n-1).$$

If we expand $w(-n-1)$ into a linear combination of the elements of $\mathcal{S}$ (see (9)) modulo N, then the minimal term among the terms of maximal length in this expansion will be

$$b_1(n_1)\cdots b_{l-1}(n_{l-1})b_l(-n-1).$$

We observe that these terms are different for every pair $n \in \mathbb{Z}_+$, $w \in \mathcal{W}$. Therefore, the set $\{\, w(-n-1) \,|\, n \in \mathbb{Z}_+,\ w \in \mathcal{W}\,\}$ is linearly independent modulo N, hence all $k_{n,w} = 0$.

Now we show that any element $u \in \mathfrak{U}_i$ has expansion (10). By definition, $\mathfrak{U}_i$ is spanned by words $D^m a_1(m_1)\cdots a_{l-1}(m_{l-1})a_l$ for $a_j \in \mathfrak{L}$ and $m_j \in \mathbb{Z}_+$, such that $\sum_j m_j + \sum_j \deg a_j - m \geqslant i$. Expand every a_j in such a word into a linear combination (8), and then use (4) and (6) to write this word in the form (10). By Lemma 6, every term in the expansion (8) for a_j will have degree at least $\deg a_j$, so the condition on degrees in (10) follows from the fact that the relations (4) and (6) are homogeneous. ■

2.2.6. *The ideal* $\mathfrak{I}$. Set

$$\mathfrak{I} = \operatorname{Span}_{\Bbbk[D]} \left\{ a_1(m_1)\cdots a_{l-1}(m_{l-1})a_l \in \mathfrak{U} \;\middle|\; \begin{array}{c} a_j \in \mathfrak{L},\ m_j \in \mathbb{Z}_+,\ l \geqslant 2 \\ \sum_j m_j + \sum_j \deg a_j \geqslant r \end{array} \right\}.$$

Lemma 9. (a). *$\mathfrak{I}$ is an ideal of $\mathfrak{U}$.*

(b). *For any $a, b \in \mathfrak{L}$ we have $a(n)b \in \mathfrak{I}$ for $n \gg 0$.*

(c). $\mathfrak{I} \cap \mathfrak{L} = 0$.

Proof. (a) Let $u = a_1(m_1)\cdots a_{l-1}(m_{l-1})a_l$ be a generator of $\mathfrak{I}$. Then $a(n)u \in \mathfrak{I}$ for any $a \in \mathfrak{L}_0$ and $n \in \mathbb{Z}_+$. Since $\mathfrak{L} = \Bbbk[D]\mathfrak{L}_0$ by Lemma 4 (b), we get $\mathfrak{L}(n)\mathfrak{I} \subseteq \mathfrak{I}$ for any $n \in \mathbb{Z}_+$. But $\mathfrak{U}$ is generated by $\mathfrak{L}$ as an algebra, therefore (4) implies that $\mathfrak{U}(n)\mathfrak{I} \subseteq \mathfrak{I}$ for any $n \in \mathbb{Z}_+$.

(b) We have $a(n)b \in \mathfrak{I}$ for $n \geqslant l - \deg a - \deg b$.

(c) Let $u = a_1(m_1)\cdots a_{l-1}(m_{l-1})a_l \in \mathfrak{I}$ as above. It follows from Lemma 8 and §2.2.4 that the expansion (10) of u will have the following two properties:

(i) $\deg w \geqslant r$ whenever $k_{n,w} \neq 0$;

(ii) $u_0 \in \mathfrak{N}_2$.

Clearly, the expansion (10) of any $\Bbbk[D]$-linear combination of such elements u also has properties (i) and (ii). We are left to note, that if a linear combination (10) with properties (i) and (ii) belongs to $\mathfrak{L}$, then it is 0. Indeed, any word $w \in \mathcal{W}$ of degree r or more has length at least 2, since $\deg b \leqslant r-1$ for any $b \in \mathcal{B}$. Therefore, all $k_{n,b} = 0$ and the combination is in $\mathfrak{N}_2$. But we also have $\mathfrak{N}_2 \cap \mathfrak{L} = 0$. ■

Now Proposition 2 easily follows from Lemma 9: Take $\mathfrak{A} = \mathfrak{U}/\mathfrak{I}$. This is an associative conformal algebra, since the conformal associativity holds by Lemma 2 and the locality is due to (b), it contains $\mathfrak{L}$ because of (c) and any word $w = g_1(n_1)\cdots(n_{l-1})g_l$, $g_i \in \mathcal{G}$, of weight r or more belongs to $\mathfrak{I}$, therefore, $w = 0$ in $\mathfrak{A}$.

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