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# Spinor Types in Infinite Dimensions

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**Abstract.** The Cartan - Dirac classification of spinors into types is generalized to infinite dimensions. The main conclusion is that, in the statistical interpretation where such spinors are functions on $\mathbb{Z}_2^\infty$, any real or quaternionic structure involves switching zeroes and ones. There results a maze of equivalence classes of each type. Some examples are shown in $L^2(\mathbb{T})$. The classification of spinors leads to a parametrization of certain non-associative algebras introduced speculatively by Kaplansky.
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## Contents



## 1. Introduction

Let $H$ be a separable real prehilbert space and $\mathfrak{C} = C(H)$ the Clifford algebra of $H$, i.e., the quotient of the tensor algebra $\mathcal{T}_\mathbb{R}(H)$ of $H$ by the ideal generated by the elements of the form

$$h \otimes h' + h' \otimes h + 2 < h, h' >$$

with $h, h' \in H$.

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In two little known papers from 1954, Gårding and Wightman parametrized (up to equivalence) the unitary representations of the so-called Canonical Commutation and Anticommutation Relations. The first essentially amounts to parametrizing the unitary representations of the infinite-dimensional Heisenberg group $\mathcal{H}$, while the second amounts to doing the same for $\mathfrak{C}$. Their work, based on original examples by von Neumann [15], show that both have "a true maze" of equivalence classes of irreducibles, in striking contrast to the finite case. Abusing language, one says that *Stone-von Neumann fails in infinite dimensions* in both cases. The standard representations appearing in QFT constitute a special class characterizable by the existence of vacua – vectors annihilated by all the annihilation operators. One calls these Bose-Fock in the case of $\mathcal{H}$, or, abusing again, Fermi-Fock, in the case of $\mathfrak{C}$, or simply *Fock* representations. According to ordinary use in finite dimensions, the unitary representations of $\mathfrak{C}$ will be called here *complex spinor* structures or simply *spinors*, and the particular realization derived from the construction of Gårding and Wightman, *GW spinors.*

In this article we determine the *type* of these spinors and deduce some conclusions. Recall that a real (resp., quaternionic) structure on a complex Hilbert space is an antilinear, norm-preserving operator $S$ (resp., $Q$) such that $S^2 = I$ (resp., $Q^2 = -I$). As in the finite dimensional case, a complex representation of $\mathfrak{C}$ is said to be of *real, quaternionic* or *complex type*, according to whether it commutes with an $S$, a $Q$, or neither, conditions that are mutually exclusive when the representation is irreducible.

The question of type is basic in finite dimensions, where its solution was found apparently first by Cartan and rediscovered later by Jordan, Wigner and Dirac. The fact is that every (complex) representation of $C(\mathbb{R}^n)$ is a multiple of a unique irreducible one (for $n \not\equiv 3, 7 \mod(8)$), or a sum of multiples of two unique irreducible ones (for $n \equiv 3, 7 \mod(8)$). The irreducible ones are of real type for $n \equiv 0, 6$, of complex type for $n \equiv 1, 5$ and of quaternionic type for $n \equiv 2, 3, 4$ [5][6][14]. In the physics literature $S$ and $Q$ are called *charge-conjugation operators* and the irreducible spinors of real type *Majorana spinors.*

In infinite dimensions we find mazes of inequivalent irreducible spinors of each of the three types. The key condition for a spin-invariant real or quaternionic structure to exist is that in their dyadic representation (cf. §2), changing all 0's to 1's and all 1's to 0's must be a meaningful operation among spinors. This rules out all representations common in physics: Fock, anti-Fock, Canonical.

Because the questions of reducibility and equivalence of the GW representations are not completely resolved -indeed, they may be essentially unsolvable in general, the GW parametrization works better in practice as a source of examples than as an instrument of proof. Our results are an exception to this rule: the GW parametrization is well fit to describe the breakdown into types and yields a neat answer. We now mention some specific consequences.

The spinors of real type yield the orthogonal representations of $\mathfrak{C}$ in real Hilbert spaces. If $S$ is a spin-invariant real structure then $\{v : Sv = v\}$ is an invariant real form which, by restriction, provides a real representation of $\mathfrak{C}$ and every real representation must arise in this way.

When $\dim_{\mathbb{R}} H = 1, 3, 7$, the real irreducible representations of $\mathfrak{C}$ have dimensions $2, 4, 8$, respectively, and are in correspondence with the classical division algebras [2]. For this, the property of $\mathfrak{C}$ having a module of dimension equal to one plus the number of generators, is crucial. Of course, this property holds when $\dim H = \infty$ too, so it is natural to search for infinite-dimensional analogs of quaternions and octonions. This possibility was considered by Kaplansky in the fifties [13], who ruled out strict analogs and proposed weaker alternatives. Although he seemed doubtful of their existence as well, examples were found in the nineties [7][17]. We give here a parametrization of all such algebras up to equivalence, concluding that there are mazes of inequivalent ones.

There are families of representations of $\mathfrak{C}$ on $L^2(\mathbb{T})$ or $L^2(\mathbb{R})$, of real or quaternionic type which seem to have analytic content. We discuss two operators,

$$D = \sum_{k=1}^{\infty} a_k \partial_k, \qquad D' = \sum_{k=1}^{\infty} a_k^* \partial_k$$

where $a_k$, $a_k^*$, are the creation and annihilation operators associated to the spin structure and the $\partial_k$ are certain dyadic difference operators. Notably, for the standard Fermi-Fock representations they diverge off the vacuum. But for the spinor structures in $L^2(\mathbb{T})$ they have a dense domain and relate neatly with the real and quaternionic structures.

In the statistical interpretation of the creation and annihilation operators, a real or quaternionic structure necessarily empties all occupied states and fills all non-occupied ones. This may be an unlikely feature for particles or fields, but not necessarily for other systems modelled with 0's and 1's.

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## 2. Gårding-Wightman spinors

Let

$$X = \mathbb{Z}_2^\infty$$

be the set of sequences $x = (x_1, x_2, \dots)$ of 0's and 1's, and $\Delta \subset X$ the subset consisting of sequences with only finitely many 1's. Then $X$ is an abelian group under componentwise addition modulo 2 and $\Delta$ is the subgroup generated by the sequences $\delta^k$, where $\delta_j^k$ is the Kronecker symbol. The product topology on $X$ is compact and is generated by the sets

$$X_k = \{x : \ x_k = 1\}, \qquad X_k' = \{x : \ x_k = 0\},$$

which, therefore, also generate the canonical $\sigma$-algebra of Borel sets in $X$. Let

$$\chi_k, \ \chi_k',$$

denote the characteristic functions of the sets $X_k, X_k'$, respectively.

We will realize all the complex spinor structures on $L^2$ spaces of $\mathbb{C}$-valued functions on $X$ or direct integrals thereof. As a motivation, let us realize the standard finite even-dimensional spinors in this manner. For each positive integer $N$ consider the vector space

$$V_N = \{f : \mathbb{Z}_2^N \to \mathbb{C}\}.$$

Then, clearly, $\dim V_N = 2^N$ and the operators

$$\begin{aligned} J_k f(x) &= -i(-1)^{x_1+\ldots+x_{k-1}}\ f(x+\delta^k) \\ J'_k f(x) &= \ (-1)^{x_1+\ldots+x_k}\ f(x+\delta^k) \end{aligned} \tag{2.1}$$

where $1 \leq k \leq N$, $x \in \mathbb{Z}_2^N$, addition is modulo 2 and the $\delta^k$ is the standard basis of $\mathbb{Z}_2^N$, define an irreducible complex representation of the Clifford algebra $C(\mathbb{R}^{2N})$ -the unique one modulo equivalence. The unitarity is relative to the natural $L^2$ inner product in $V_N$, which in turn is associated to the measure on $\mathbb{Z}_2^N$ where each point has measure $1$.

When $N = \infty$, in order to reach all equivalence classes one must allow for more general measures on the group $X = \mathbb{Z}_2^\infty$ and replace $\mathbb{C}$-valued functions for sections of appropriate fiber spaces over $X$. Three natural but very different measures on $X$ that generalize the finite case are:

The *Haar* measure of $X$, $\mu_X$.

The *Fermi-Fock* measure on $X$, $\mu_\Delta$, supported on the discrete set $\Delta$ with each point having measure $1$. More generally,

The *Canonical* measures, $\mu_{x_o+\Delta}$, supported on translates of $\Delta$.

The first is invariant under all translations in $X$ while the second is invariant only under those from $\Delta$. It is $\mu_\Delta$ that leads to the representations that appear most in QFT, however implicitly. It ignores all the points $x$ with infinitely many $x_i = 1$, or "occupied states", on the basis that the total number of fermions must be finite. In any case, (2.1) define irreducible representations of $\mathfrak{C}$ on $L^2(X, \mu_X)$ and on $L^2(X, \mu_{x_o+\Delta})$ of very different nature.

The next theorem is Gårding and Wightman's main result in [9], rephrased to fit our setting. A sketch of its proof is included in an appendix.

Recall that two measures $\lambda, \mu$ on the same Borel algebra of sets are said to be *equivalent* if they have the same sets of measure zero. Equivalently, if there exists locally integrable functions, denoted by $d\lambda/d\mu$ and $d\mu/d\lambda$, such that for any measurable set $A$, these Radon-Nikodym derivatives satisfy

$$\lambda(A) = \int_A \frac{d\lambda}{d\mu}\, d\mu, \qquad \mu(A) = \int_A \frac{d\mu}{d\lambda}\, d\lambda.$$

$\mu$ is said to be *quasi-invariant by* $\Delta$ if $\mu$ is invariant under translations by elements of $\Delta$.

Now, consider triples

$$(\mu, \mathcal{V}, \mathcal{C})$$

where

- $\mu$ is a positive Borel measure on $X$, quasi-invariant under translations by $\Delta$.
- $\mathcal{V} = \{V_x\}_{x\in X}$ is a family of complex Hilbert spaces, invariant under translations by $\Delta$ and such that the function $x \mapsto \nu(x) = \dim V_x$ is measurable.

• $\mathcal{C} = \{c_k : \; k \in \mathbb{N}\}$ is a family of unitary operators $c_k(x) : V_x \to V_{x+\delta^k} = V_x$ depending measurably on $x$ and satisfying

$$\begin{aligned} c_k^*(x) &= c_k(x+\delta^k) \\ c_k(x)c_l(x+\delta^k) &= c_l(x)c_k(x+\delta^l) \end{aligned} \tag{2.2}$$

for all $\delta \in \Delta$ and almost all $x \in X$.

We will often write $(\mu, \nu, \mathcal{C})$ instead of $(\mu, \mathcal{V}, \mathcal{C})$, in view of the fact that changing $\mathcal{V}$ unitarily will yield equivalent representations. Given such triple, consider the Hilbert space

$$V = V_{(\mu,\nu,\mathcal{C})} = \int_X^{\oplus} V_x \; d\mu(x)$$

and define operators on $V$ by

$$\begin{aligned} J_k f(x) &= -i(-1)^{x_1+\ldots+x_{k-1}} \sqrt{\frac{d\mu(x+\delta^k)}{d\mu(x)}} \; c_k(x) \;\; f(x+\delta^k) \\ J_k' f(x) &= \quad (-1)^{x_1+\ldots+x_k} \sqrt{\frac{d\mu(x+\delta^k)}{d\mu(x)}} \; c_k(x) \;\; f(x+\delta^k) \end{aligned} \tag{2.3}$$

where an $f \in V$ is regarded as an assignment $x \mapsto f(x) \in V_x$ and all sums are modulo 2.

In the real Hilbert space $H$, we fix an orthogonal basis with a given pairing, $\{h_k, h_k'\}$, and define an $\mathbb{R}$-linear

$$\pi = \pi_{(\mu,\nu,\mathcal{C})} : H \to \mathrm{End}_{\mathbb{C}}(V)$$

by

$$\pi(h_k) = J_k, \qquad \pi(h_k') = J_k'.$$

**Theorem 2.4.** *The operators $J_1, J_1', J_2, J_2', \ldots$ are mutually anticommuting orthogonal complex structures and, therefore, $\pi = \pi_{(\mu,\nu,\mathcal{C})}$ extends to a unitary representation of $\mathfrak{C}$ on $V$. Conversely, every spinor structure on a separable complex Hilbert space is unitarily equivalent to some $\pi_{(\mu,\nu,\mathcal{C})}$.*

The proof of the theorem is in the appendix.

*Remarks.* (a) Gårding and Wightman give a recursive formula for all possible systems of $\mathcal{C}$'s, hence Theorem (2.4) gives an effective parametrization of all separable Clifford modules. Although the matters of equivalence and irreducibility are not resolved, a lot is known in interesting special cases [4],[9],[8].

(b) The relation between the operators $J_k, J_k'$ and the operators $a_k, a_k^*$ of the Canonical Conmutation Relations:

$$a_k = \frac{1}{2}\,(J_k' + iJ_k) \qquad\qquad a_k^* = \frac{1}{2}\,(-J_k' + iJ_k)$$

(c) When $\nu(x) = 1$, $V_x$ can be identified with $\mathbb{C}$, the direct integral becomes

$$V = L^2(X, \mu)$$

and the $c_k(x)$'s are just complex numbers of modulus one depending measurably in $x$. The Fermi-Fock representation corresponds to the triple $(\mu_\Delta, 1, \{1\})$. Von Neumann's first examples of non-Fock representations, were special cases of infinite tensor products, which in our notation are the $V_{(\mu,1,\mathcal{C}^\otimes)}$, with

$$c_k^\otimes(x) = \omega_k^{(-1)^{x_k}},$$

the $\omega_k$ being fixed complex numbers of absolute value $1$. In particular,

$$V_{(\mu_X,1,\{1\})}$$

with $\mu_X$ the Haar measure, is one such. As we shall see, this has a natural realization on $L^2$ of the circle.

(d) While $V_{(\mu_X,1,\{1\})}$ and $V_{(\mu_\Delta,1,\{1\})}$ are given by the same formulae as those of the finite-dimensional case, namely (2.1), they are inequivalent: in the first, the characteristic function of the point $\mathbf{0} = (0, 0, \dots)$ gives a non-zero vector annihilated by all the operators $a_k^*$, while the second has no such "vacuum" vector.

(e) Although the GW representations can be discussed more intrinsically in terms of the "Clifford-Weyl systems" of [3], we prefer to keep $\{h_k, h'_k\}$ as an implicit parameter, to be in tune with previous publications. One must keep in mind that this is not just a notational issue: different basis may yield inequivalent representations (cf. Berezin's notion of $G$-equivalence [4]). We will return to this issue in §5.

For further results on the Gårding-Wightman parametrization, see [4][8].

## 3. Real and Quaternionic structures

If $U$ is a real module over $\mathfrak{C}$, then $\mathbb{C}\otimes U$ is a complex module over $\mathbb{C}\otimes\mathfrak{C}$, which comes with the $\mathfrak{C}$-invariant decomposition

$$\mathbb{C}\otimes U = U \oplus_{\mathbb{R}} iU.$$

$U$ is an *invariant real form* of $\mathbb{C}\otimes U$. Conversely, any module over $\mathbb{C}\otimes\mathfrak{C}$ with an invariant real form determines a real module over $\mathfrak{C}$ simply by restriction. Hence, parametrizing the invariant real forms of the Gårding-Wightman modules up to unitary equivalence, is the same as parametrizing the real representations of $\mathfrak{C}$ up to orthogonal equivalence.

The first problem is equivalent to that of determining the $\mathbb{C}$-antilinear operators

$$S : V \to V$$

which commute with the action of $\mathfrak{C}$ and such that

$$S^2 = 1, \qquad ||Sf|| = ||f||. \tag{3.1}$$

The invariant real form associated to $S$ is then $\{v \in V : \ Sv = v\}$ and $S$ becomes complex-conjugation relative to it.

The map

$$x \mapsto \check{x} = x + \mathbf{1}$$

where the sum is modulo 2 and $\mathbf{1}$ is the point with ones in all slots, is an involution of the set $X$, which switches all zeroes to ones and viceversa. There are induced involutions on subsets of $X$ and on functions and measures on $X$:

$$\check{A} = \{\check{x} : \ x \in A\}, \qquad \check{f}(x) = f(\check{x}), \qquad \check{\mu}(A) = \mu(\check{A}).$$

**Theorem 3.2.** $\pi_{(\mu,\nu,\mathcal{C})}$ *admits an invariant real form if and only if the measures* $\mu$ *and* $\check{\mu}$ *are equivalent,* $\check{\nu}(x) = \nu(x)$ *for almost all* $x \in X$ *and there exist a measurable family of antilinear operators*

$$r(x) : \ V_x \to V_{\check{x}} \cong V_x$$

*that preserve norms and satisfy*

$$\begin{aligned} r(x)r(\check{x}) &= 1 \\ r(x)c_k(\check{x}) &= (-1)^k c_k(x) r(x+\delta_k) \end{aligned} \tag{3.3}$$

*for all* $k \in \mathbb{N}$ *and almost all* $x \in X$.

*Proof.* If $\mu$ and $\check{\mu}$ are equivalent, $\nu = \check{\nu}$ a.e. and $r(x) : \ H_x \to H_x$ is as stated, then the operator

$$Sf(x) = \sqrt{\frac{d\check{\mu}(x)}{d\mu(x)}}\, r(x) f(\check{x})$$

is an invariant real structure in $V(\mu, \nu, \mathcal{C})$. Indeed, it is clearly antilinear, it is norm-preserving because both $r(x)$ and

$$Tf(x) = \sqrt{\frac{d\check{\mu}(x)}{d\mu(x)}}\, f(\check{x}) \tag{3.4}$$

are so, and

$$S^2 f(x) = \sqrt{\frac{d\check{\mu}(x)}{d\mu(x)}}\, r(x) Sf(\check{x}) = r(x) r(\check{x}) f(x) = f(x)$$

showing that $S$ is involutive. As for invariance,

$$\begin{aligned}
SJ_kf(x) &= \\
&= \sqrt{\frac{d\check{\mu}(x)}{d\mu(x)}}r(x)(J_kf)(\check{x}) \\
&= i(-1)^{\check{x}_1+\ldots+\check{x}_{k-1}}\sqrt{\frac{d\mu(\check{x})}{d\mu(x)}}\sqrt{\frac{d\mu(\check{x}+\delta^k)}{d\mu(\check{x})}}\ r(x)\ c_k(\check{x})f(\check{x}+\delta^k) \\
&= i(-1)^{x_1+\ldots+x_{k-1}+k-1}\sqrt{\frac{d\mu(\check{x}+\delta^k)}{d\mu(x)}}\ (-1)^k c_k(x)r(x+\delta_k)f(\check{x}+\delta^k) \\
&= -i(-1)^{x_1+\ldots+x_{k-1}}\ c_k(x)r(x+\delta_k)\sqrt{\frac{d\mu(\check{x}+\delta^k)}{d\mu(x)}}f(\check{x}+\delta^k) \\
&= -i(-1)^{x_1+\ldots+x_{k-1}}\ c_k(x)\ \sqrt{\frac{d\mu(x+\delta^k)}{d\mu(x)}}\ r(x+\delta^k)\Big(\sqrt{\frac{d\check{\mu}(x+\delta^k)}{d\mu(x+\delta^k)}}f((x+\delta^k)\check{}\ )\Big) \\
&= -i(-1)^{x_1+\ldots+x_{k-1}}\ c_k(x)\ \sqrt{\frac{d\mu(x+\delta^k)}{d\mu(x)}}\ Sf(x+\delta^k) \\
&= J_kSf(x)
\end{aligned}$$

Finally, since $J_k'f(x) = i(-1)^{x_k}J_kf(x)$ and $S(\rho_kf) = -\rho_kSf$ for $\rho_k(x) = (-1)^{x_k}$, it follows that $SJ_k' = J_k'S$ as well.

Conversely, let $S$ be an arbitrary $\mathfrak{C}$-invariant, antilinear operator on $V = \int_X V_x\ d\mu(x)$. Let $N_k$, $N_k'$ be the operators on $V$ defined by

$$N_k = a_k^*a_k \qquad\qquad N_k' = a_ka_k^*$$

As it can be seen in the proof of Theorem 2.4 (see the Appendix the details) $N_k$ and $N_k'$ are projections on $V$, moreover they act as multiplication by the characteristic functions of the sets $X_k$ and $X_k' = \check{X}_k$, respectively. Since $2a_k = J_k' + iJ_k$ and $2a_k^* = -J_k' + iJ_k$, one obtains the relations

$$Sa_k^* = -a_kS, \qquad Sa_k = -a_k^*S, \qquad SN_k = N_k'S. \tag{3.5}$$

If $L_\phi$ denotes the operator of multiplication by the $\mathbb{C}$-valued bounded measurable function $\phi$, the third equation in (3.5) implies that

$$SL_\phi = L_{\check{\phi}}S \tag{3.6}$$

for $\phi = \chi_k$ or $\phi = \chi_k'$. Since the $X_k$ generate the $\sigma$-algebra of Borel sets of $X$, (3.6) must hold for any measurable characteristic function and, *a fortiori*, for any essentially bounded function $\phi$. As a consequence,

$$\operatorname{Supp}(Sf) = (\operatorname{Supp}(f))\check{} \tag{3.7}$$

for all $f \in V$. Indeed, if $F = \operatorname{Supp}(f)$, then $\operatorname{Supp}(Sf) = \operatorname{Supp}(S(\chi_Ff)) = \operatorname{Supp}(\chi_{\check{F}}S(f)) \subset \check{F}$; since $S$ is an involution, the equality follows.

In order to see that $\mu$ and $\check{\mu}$ are equivalent, let $E \subset X$ be a measurable set contained in some $E_n = \{x : \ \nu(x) = n\}$. We can identify all $V_x$, $x \in E_n$, with a fixed $V_{(n)}$. Let $u$ be a unit vector of $V_{(n)}$ and define $f \in \int_X^{\oplus} V_x \, d\mu(x)$ by

$$f(x) = \begin{cases} \chi_E(x)u & x \in E_n \\ 0 & x \notin E_n, \end{cases}$$

On one hand,

$$\|f\|^2 = \int_X (f(x), f(x))d\mu(x) = \int_E (f(x), f(x))d\mu(x) = \int_E (u, u)d\mu(x) = \mu(E).$$

On the other, because $S$ preserves norms and $Sf(x)$ is supported in $\check{E}$,

$$\|f\|^2 = \|Sf\|^2 = \int_X (Sf(x), Sf(x))d\mu(x) = \int_{\check{E}} (Sf(x), Sf(x))d\mu(x).$$

Therefore $\mu(\check{E}) = 0 \ \Rightarrow \ \mu(E) = 0$ for any $E$ contained in some $E_n$. The last restriction can now be dropped and the implication be reversed, so $\mu$ and $\check{\mu}$ are indeed equivalent.

To show that $\nu(x) = \nu(\check{x})$ for almost all $x$, suppose the contrary: $\exists n \leq \infty$, $m < n$ and $E \subset E_n$ such that $\mu(E) > 0$ and $\check{E} \subset E_m$. Since $\mu$ and $\check{\mu}$ are equivalent, $\mu(\check{E}) > 0$. As before, identify all $V_x$, $x \in E_n$, with a fixed $V_{(n)}$. Let $\{v_i\}$ be an orthonormal basis of $V_{(n)}$ and $F \subset E$ a measurable subset. Then

$$f_i(x) = \chi_F(x)v_i$$

are elements mutually orthogonal in $V$. If we let ${}^rV$ be $V$ regarded as a real Hilbert space with the inner product $Re(u, v)$, the $f_i$ remain orthogonal in ${}^rV$. Since $S$ is antilinear and preserves norm,

$$(Sf_i, Sf_j) = \overline{(f_i, f_j)} = 0.$$

for $i \neq j$. Because of (3.7), the $Sf_j$ must vanish off $\check{F}$, and we can conclude that

$$\int_F Re(Sf_i(\check{x}), Sf_j(\check{x}))d\check{\mu}(x) = \int_{\check{F}} Re(Sf_i(x), Sf_j(x))d\mu(x) = 0$$

Since $\mu$ is equivalent to $\check{\mu}$ and $F$ is arbitrary, this implies that

$$Re(Sf_i(x), Sf_j(x)) = 0$$

almost everywhere in $\check{E}$. On the other hand,

$$\begin{aligned} \check{\mu}(F) &= \int_F |f_i(x)|^2 d\mu(x) = \int_X |f_i(x)|^2 d\mu(x) \\ &= \|f_i\|^2 = \|Sf_i\|^2 = \int_{\check{F}} |Sf_i(x)|^2 d\mu(x) \end{aligned}$$

shows that the $|Sf_j(x)|$ cannot vanish identically. We conclude that $\{Sf_i(x)\}_{i=1}^n$ is a linearly independent set in $V_x$ for almost all $x$ in $\check{E} \subset E_m$, which is a contradiction since $m < n$.

We may now assume that $V_x = V_{\check{x}}$, so the operator

$$Tf(x) = \sqrt{\frac{d\check{\mu}(x)}{d\mu(x)}}f(\check{x})$$

is well defined. It is $\mathbb{C}$-linear, unitary and satisfies the relations

$$T^2 = I, \qquad TN_k = N_k'T \tag{3.8}$$

The first is clear while the second follows from

$$\begin{aligned} TN_kf(x) &= \sqrt{\frac{d\check{\mu}(x)}{d\mu(x)}}N_kf(\check{x}) = \sqrt{\frac{d\check{\mu}(x)}{d\mu(x)}}\chi_k(\check{x})f(\check{x}) = \sqrt{\frac{d\check{\mu}(x)}{d\mu(x)}}\chi_k'(x)f(\check{x}) \\ &= N_k'Tf(x) \end{aligned}$$

The product

$$R = ST$$

is then antilinear, bounded and commutes with all the $N_k$ and $N_k'$. This implies that $R$ acts fiberwise, as an antilinear operator-valued function $r(x)$. In fact, if $R$ would be $\mathbb{C}$-linear, rather than $\mathbb{C}$-antilinear operators, this follows from the Spectral Theorem. In our case we argue as follows: the condition that $R$ commutes with the $N_k$ and $N_k'$ implies

$$RL_\phi = L_\phi R,$$

for any essentially bounded real-valued function $\phi$. On each $E_n$ we can assume, as before, that all ${}^rV_x$ are the same ${}^rV_{(n)}$, so it is enough to define $r(x)v$ for $v \in {}^rV_{(n)}$. Identifying $v$ with $\chi_{E_n}(x)v$, $Rv$ is an element of

$$\int_{E_n} {}^rV_x \; d\mu(x) \subset {}^rV$$

and, therefore representable as a $V_x$-valued function $x \mapsto (Rv)(x)$. Now

$$r(x)v := (Rv)(x)$$

defines our desired operator-valued function. Clearly, $r(x)$ is antilinear, preserves norms, and satisfies

$$r(x)f(x) = (Rf)(x) = (STf)(x)$$

for all $f \in V$. Because $T$ is an involution, this is equivalent to $r(x)(Tf)(x) = Sf(x)$, yielding a pointwise formula for $S$:

$$Sf(x) = \sqrt{\frac{d\check{\mu}(x)}{d\mu(x)}}r(x)f(\check{x}). \tag{3.9}$$

Since

$$f(x) = S^2 f(x) = \sqrt{\frac{d\check{\mu}(x)}{d\mu(x)}} r(x) (\sqrt{\frac{d\mu(x)}{d\mu(\check{x})}} r(\check{x})(f(x))) = r(x) r(\check{x}) f(x)$$

we obtain $r(x)r(\check{x}) = I$ for almost all $x \in X$. Because $S$ commutes with the $J_k, J_k'$, themselves,

$$r(x)(-1)^{x_1+\ldots+x_{k-1}+k-1} c_k(\check{x}) = (-1)^{x_1+\ldots+x_{k-1}+1} c_k(x) r(x+\delta_k),$$

must hold a.e.; the calculation is straightforward. ∎

All this applies to the finite, even case as well. A measure $\mu$ on $\mathbb{Z}_2^N$ is quasi-invariant if and only if every point has non-zero mass and any two such measures are equivalent. Take $\mu(\{x\}) = 1$, $\nu(x) = 1$ and $c_k(x) = I$ for all $x \in \mathbb{Z}_2^N$. From (3.3) one deduces that

$$r(\mathbf{1}) = (-1)^{\frac{N(N+1)}{2}} r(\mathbf{0}).$$

Assuming, as we may, that $r(\mathbf{0})$ is the standard conjugation on $\mathbb{C}$, we see that $V$ splits over $\mathbb{R}$ if and only if $N(N+1)/2$ is an even integer, i.e., for

$$N \equiv 0, 3 \pmod 4$$

as we mentioned earlier.

Assume now that $V$ is infinite dimensional and separable. The axiom of choice implies that there are always plenty of solutions $r(x)$ to the equations (3.3), whatever the data. Indeed, let $\tilde{X} = X/\sim$, where $x \sim y$ if and only if $y = \check{x}$ or $x - y \in \Delta$. Choose an element $x_p \in p$ from each class $p \in \tilde{X}$ and define $r(x_p)$ in an arbitrary manner. Then

$$r(\check{x}_p) = r(x_p)^{-1}, \qquad r(x_p+\delta) = (-1)^k c_k(x_p)^* r(x_p) c_k(\check{x}_p)$$

defines $r(x)$ for all $x$. However, most of these solutions -and often all those associated to a given $\mathcal{C}$, will be non-measurable.

**Corollary 3.10.** *If $\mu$ is discrete and $V$ is irreducible over $\mathbb{C}$, then it is irreducible over $\mathbb{R}$. In particular, this is the case for the Fermi-Fock representations.*

*Proof.* If $\mu$ is discrete and $V_{(\mu,\nu,\mathcal{C})}$ is irreducible, then $\mu$ is supported in some set of the form $x_o+\Delta$ [8]. Then $\check{\mu}$ is supported in $(x_o+\Delta)\check{}$, which is disjoint from $x_o+\Delta$ and, therefore, cannot be equivalent to $\mu$. ∎

The proof above is based on results from [8],[9], involving relations among the ergodicity of the measure $\mu$, the nature of its support and the irreducibility of $\pi_{(\mu,\nu,\mathcal{C})}$. In the next result ergodicity is used in the statement, so we recall that $\mu$ is *ergodic* under translations by $\Delta$ if any $\Delta$-invariant set has measure zero or its complement has measure zero. This is equivalent to asking that every essentially bounded measurable function invariant under translations by $\Delta$ (in the sense that $f(x+\delta) = f(x)\ \forall \delta \in \Delta$ and a.a. $x \in X$) is constant (i.e., $f(x) = c$ for some $c$ and a.a. $x \in X$). In our case, both the Haar measure $\mu_X$ and the discrete measures $\mu_{x_o+\Delta}$ are ergodic for elementary reasons. Worth mentioning here is the fact that *if $\mu$ is quasi-invariant, discrete and ergodic, then*

$$\mu \cong \mu_{x_o+\Delta}$$

*for some $x_o \in X$* [8]. This is used in the last proof.

**Corollary 3.11.** *Suppose that $\mu$ is ergodic and that $d\mu(x+\delta^k)/d\mu(x)$ is bounded away from zero and infinity as a function of $x$ and $k$. Then $V(\mu, 1, \{1\})$ is irreducible over $\mathbb{R}$. More generally, this is true of all the tensor product representations $V(\mu, 1, \mathcal{C}^{\otimes})$.*

*Proof.* When $\nu = 1$, the operator-valued function $r(x)$ of the Theorem is complex valued and (3.3) implies $r(x+\delta^k) = (-1)^k r(x)$. By hypothesis, $\exists C > 0$ such that

$$\frac{1}{C} < \Big|\frac{d\check{\mu}(x+\delta^k)}{d\mu(x)}\Big| < C$$

for all $k$ and almost all $X$. For any measurable essentially bounded function like $r(x)$, the difference $r(x+\delta^k) - r(x)$ must go to zero as $k \to \infty$, at least in measure (see e.g., Theorem 4 in [8]). This is incompatible with that identity and $r$ being invertible.

We conclude that $V_{(\mu,1,\{1\})}$ has no invariant real forms. That the same is true for tensor product representations follows by a similar argument, using that for $c_k(x) = \omega_k^{(-1)^{x_k}}$,

$$c_k(\check{x}) = \omega_k^{(-1)^{x_k+1}} = c_k(x)^{-1}$$

so that (3.3) becomes

$$r(x+\delta^k) = (-1)^k \omega_k^{2(-1)^{x_k}} r(x).$$

The irreducibility over $\mathbb{R}$ follows from the irreducibility over $\mathbb{C}$, which in turn is implied by the ergodicity of $\mu$. Indeed, any complex linear operator commuting with $\mathfrak{C}$ must commute with the projection operators $N_k$ and, therefore, consist of multiplication by a function $f(x)$. That the operator commutes with the $J$'s themselves implies, as in the proof of the Theorem, that $f(x)$ is invariant under translation by all elements of the subgroup $\Delta$. By ergodicity, $f$ must be constant. ∎

We next give a "normal form" for spinors of real type, in the case when the multiplicities $\nu(x)$ are $1$. In this case one may set

$$V_x = \mathbb{C}$$

for all $x$ and the direct integral defining $V$ is an ordinary space of complex-valued square-integrable functions:

$$V = V_{(\mu,\mathcal{V},\mathcal{C})} = L^2(X, \mu).$$

Like any space of complex-valued functions, this has a canonical real structure, namely

$$V_{\mathbb{R}} = L^2(X, \mu)_{\mathbb{R}} = \{f \in V : \ f(x) \in \mathbb{R} \ a.e.\}$$

for which the corresponding $S$-operator is

$$(Rf)(x) = \overline{f(x)}.$$

As we will see, this cannot remain invariant under a non-trivial spin structure.

Consider instead the real structure

$$S_o f(x) = \overline{Tf(x)} = \sqrt{\frac{d\mu(\check{x})}{d\mu(x)}}\ \overline{f(\check{x})}. \tag{3.12}$$

whose space of real vectors can be written as

$$V^{\mathbb{R}} = L^2(X,\mu)^{\mathbb{R}} = \{f \in V :\ \overline{f(x)\sqrt{d\mu(x)}} = f(\check{x})\sqrt{d\mu(\check{x})}\}. \tag{3.13}$$

**Proposition 3.14.** *$\pi_{(\mu,1,\mathcal{C})}$ leaves $L^2(X,\mu)^{\mathbb{R}}$ invariant if and only if*

$$c_k(\check{x}) = (-1)^k \overline{c_k(x)}.$$

*In such case, $r(x)f(x) = Rf(x) = \overline{f(x)}$.*

*Proof.* For any $\pi_{(\mu,\nu,\mathcal{C})}$

$$\begin{aligned}
J_k T f(x) &= -i(-1)^{x_1+\ldots+x_{k-1}} \sqrt{\frac{d\mu(x+\delta^k)}{d\mu(x)}}\ c_k(x)\ Tf(x+\delta^k) \\
&= -i(-1)^{x_1+\ldots+x_{k-1}} \sqrt{\frac{d\mu(x+\delta^k)}{d\mu(x)}}\ c_k(x)\ \sqrt{\frac{d\check{\mu}(x+\delta^k)}{d\mu(x+\delta^k)}} f(\check{x}+\delta^k) \\
&= -i(-1)^{x_1+\ldots+x_{k-1}} \sqrt{\frac{d\mu(\check{x}+\delta^k)}{d\mu(x)}}\ c_k(x)\ \ f(\check{x}+\delta^k)
\end{aligned}$$

so

$$\begin{aligned}
TJ_kTf(x) &= \sqrt{\frac{d\mu(\check{x})}{d\mu(x)}} J_kTf(\check{x}) \\
&= \sqrt{\frac{d\mu(\check{x})}{d\mu(x)}} (-i)(-1)^{\check{x}_1+\ldots+\check{x}_{k-1}} \sqrt{\frac{d\mu(\check{\check{x}}+\delta^k)}{d\mu(\check{x})}}\ c_k(\check{x}) f(\check{\check{x}}+\delta^k) \\
&= -i(-1)^{x_1+\ldots+x_{k-1}} \sqrt{\frac{d\mu(x+\delta^k)}{d\mu(x)}}\ (-1)^{k+1} c_k(\check{x}) f(x+\delta^k) \\
&= \tilde{J}_k f(x)
\end{aligned}$$

with

$$\tilde{c}_k(x) = (-1)^{k+1} c_k(\check{x}).$$

Now, $J_k S_o = J_k RT = J_k TR = T\tilde{J}_k R$, since $T$ is real. On the other hand, looking at the formula for $J_k$, it is clear that $RJ_kR = \hat{J}_k$, with $\hat{c}_k(x) = -\overline{c_k(x)}$. Therefore

$$S_o J_k S_o = RTJ_kTR = J_k'$$

with

$$c_k'(x) = \hat{\tilde{c}}_k(x) = -\overline{\tilde{c}_k(x)} = -(-1)^{k+1}\overline{c_k(\check{x})} = (-1)^k \overline{c_k(\check{x})}.$$

In particular, $S_o$ commutes with the $J_k$ iff $c_k' = c_k$, which translates into the condition of the Theorem. ∎

*Remark.* The assumption $\nu = 1$ can be dropped altogether, provided we measurably fix a real structure $\sigma(x)$ on each $V_x$, invariant under translations by $\Delta$ and checking, and replace $R$ and bars for $\sigma(x)$; (3.14) remains true. For the next result, however, the restriction $\nu = 1$, which is a property of the equivalence class of a representation, seems essential.

**Theorem 3.15.** *Every pair $(\pi, S)$ consisting of a unitary representation of $\mathfrak{C}$ with $\nu = 1$, together with an invariant real structure, is unitarily equivalent to a GW representation on $V = L^2(X, \mu)$, having $L^2(X, \mu)^{\mathbb{R}}$ as invariant real form and multipliers satisfying $c_k(\check{x}) = (-1)^k \overline{c_k(x)}$.*

*Proof.* Realize $\pi$ as a GW representation $\pi_{(\mu,1,\mathcal{C})}$. By (3.2), $\mu$ is equivalent to $\check{\mu}$, the derivative $d\mu(\check{x})/d\mu(x)$ exists a.e. and the operator $T$ of (3.4) is a well defined unitary operator on $V$. Let $r(x)$ be the operator-valued function associated to $(\pi, S)$,

$$r(x)(f(x)) = (STf)(x).$$

Since each $r(x)$ is antilinear and norm preserving, $R \circ r(x)$ is a linear, unitary operator on $V_x = \mathbb{C}$ and therefore has the form

$$R \circ r(x) = \omega(x) I$$

for some measurable $\omega : X \to \mathbb{T}$. We are using $\circ$ to denote composition of operators when there is some risk of viewing $r(x)$ itself as an ordinary $\mathbb{C}$-valued function.

Because $R$ is just plain conjugation and $r(x)$ is antilinear, we also have

$$r(x) \circ R = \overline{\omega(x)} I.$$

Because $r(x) r(\check{x}) = 1$, one has

$$R \circ r(\check{x}) = R \circ r(x)^{-1} = R \circ (r(x) \circ R \circ R)^{-1} = R \circ (\overline{\omega(x)} I \circ R)^{-1} = \omega(x) I$$

so that

$$\omega(\check{x}) = \omega(x)$$

for almost all $x$. For $-\pi < \theta \leq \pi$ set $\sqrt{e^{i\theta}} = e^{i\frac{\theta}{2}}$. Then

$$u(x) = \sqrt{\omega(x)}$$

is a measurable $\mathbb{T}$-valued function satisfying

$$u(x)^2 = \omega(x), \qquad u(\check{x}) = u(x).$$

The operator

$$Uf(x) := u(x) f(x)$$

is unitary from $V$ to $V$ and

$$
\begin{aligned}
USU^{-1}f(x) &= u(x)r(x)TU^{-1}f(x) = u(x)r(x)\sqrt{\frac{d\mu(\check{x})}{d\mu(\check{x})}}u(\check{x})^{-1}f(\check{x}) \\
&= \sqrt{\frac{d\mu(\check{x})}{d\mu(\check{x})}}u(x)r(x)Ru(x)Rf(\check{x}) = \sqrt{\frac{d\mu(\check{x})}{d\mu(\check{x})}}Ru(x)\overline{\omega(x)}u(x)f(\check{x}) \\
&= \sqrt{\frac{d\mu(\check{x})}{d\mu(\check{x})}}u(x)\overline{u(x)}\;\overline{u(x)}u(x)Rf(\check{x}) = \sqrt{\frac{d\mu(\check{x})}{d\mu(\check{x})}}Rf(\check{x}) = RTf(x) \\
&= S_o f(x),
\end{aligned}
$$

so that $S = U^{-1}S_oU$. ∎

**Corollary 3.16.** *If a real form of $L^2(X,\mu)$ is invariant under some spinor structure, then it is of the form $UL^2(X,\mu)^{\mathbb{R}}U^{-1}$ for some unitary $U$.*

*Remark.* If $\pi$ is irreducible then $S$ is unique modulo sign. This follows from Shur's Lemma applied to the intertwining operator $S_1S_2$, which is $\mathbb{C}$-linear.

The "simplest" infinite-dimensional Majorana spinors are those in $V(\mu_X, 1, \{\rho_k\})$ with $\mu_X$ being the Haar measure of $X$ and the $\rho_k$ given by the dyadic Rademacher functions

$$\rho_{2\ell}(x) = 1, \quad \rho_{4\ell+1}(x) = (-1)^{x_{4\ell+3}}, \quad \rho_{4\ell+3}(x) = (-1)^{x_{4\ell+1}}.$$

**Theorem 3.17.** *$\pi_{(\mu_X,1,\{\rho_k\})}$ is irreducible over $\mathbb{C}$, but*

$$L^2(X)^{\mathbb{R}} = \{f \in L^2(X):\ f(\check{x}) = \overline{f(x)}\}$$

*is an invariant real form. The real representation obtained by restriction to $L^2(X)^{\mathbb{R}}$ is irreducible and does not arise from any representation of $\mathbb{C}\otimes\mathfrak{C}$ by restriction of the scalars.*

*Proof.* The irreducibility over $\mathbb{C}$ follows from the ergodicity of the Haar measure, exactly as in the proof of Corollary (3.10).

It is straightforward to check that the functions $c_k$ satisfy (2.2) and the conditions of Theorem (3.14), so the corresponding $J_k, J_k'$, must leave the real form $V^{\mathbb{R}}$ invariant. Of course, this can be deduced by direct calculation as well. $V^{\mathbb{R}}$ must be irreducible under $\mathfrak{C}$, since any closed invariant subspace generates a closed $\mathbb{C}\otimes\mathfrak{C}$-invariant subspace in $V$.

Finally, suppose that the representation of $\mathfrak{C}$ in $V^{\mathbb{R}}$ could be extended to one of $\mathbb{C}\otimes\mathfrak{C}$ in $V^{\mathbb{R}}$ itself. Denote by $\mathbb{J}$ the operation representing multiplication by $\sqrt{-1}$: $\mathbb{J}$ is an orthogonal complex structure in $V^{\mathbb{R}}$ commuting with $\mathfrak{C}$. Its unique $\mathbb{C}$-linear extension to all of $V = V^{\mathbb{R}} \oplus iV^{\mathbb{R}}$ is unitary and commutes with all the $J_k, J_k'$. As we have already mentioned, this implies that $\mathbb{J}$ is given pointwise, by an operator-valued measurable function: $(\mathbb{J}f)(x) = j(x)f(x)$. In the present case, $j(x)$ is complex valued. Since $j(x)^2 = -1$, we can write it as $j(x) = \epsilon(x)i$ for some measurable $\epsilon : X \to \{\pm 1\}$. The condition for $\mathbb{J}$ to leave invariant the real form $V^{\mathbb{R}}$ and to commute with the Clifford action amount to, respectively,

$$\epsilon(\check{x}) = -\epsilon(x), \qquad \epsilon(x+\delta_k) = \epsilon(x)$$

for almost all $x$ and all $k$. The second equation implies that $\epsilon$ is actually constant on each $\Delta$-equivalence class. By ergodicity of $\mu_X$, $\epsilon$ must then be constant almost everywhere, contradicting the first equation. ∎

**Corollary 3.18.** *Assume $\mu \cong \check{\mu}$. Then*

*(a) $L^2(X,\mu)^{\mathbb{R}}$ is a real form of $L^2(X,\mu)$ which is not unitarily conjugate to $L^2(X,\mu)_{\mathbb{R}}$*

*(b) If $\pi$ is a spin representation on $L^2(X,\mu)$, then $R\pi R$ is another, which is not unitarily equivalent to $\pi$.*

*Proof.* Let $\{J_k, J_k'\}$ represent a spinor structure on $L^2(X,\mu)$, which we can take in its GW form (2.3) with parameters $\mathcal{C}$. Let ${}^R c_k(x)$ denote the multipliers for the representation $RJ_kR$. By inspection, $RJ_kR = J_k$ implies ${}^R c_k(x) = -\overline{c_k(x)}$, while $RJ_k'R = J_k'$ implies ${}^R c_k(x) = \overline{c_k(x)}$, which is impossible since $|c_k(x)| = 1$. ∎

For more on the nature of the spinors that split over $\mathbb{R}$, see §5.

Now we will analyze the quaternionic structures on spinors. Recall that a quaternionic structure in a $\mathfrak{C}$-module $V$ is a $\mathbb{C}$-antilinear operator

$$Q : V \to V$$

that preserves norm, commutes with the action of $\mathfrak{C}$ and satisfies

$$Q^2 = -I.$$

**Theorem 3.19.** *$\pi_{(\mu,\nu,\mathcal{C})}$ admits an invariant quaternionic structure if and only if $\mu$ and $\check{\mu}$ are equivalent, $\check{\nu} = \nu$ almost everywhere, and there exist a measurable family of operators*

$$q(x) : \; V_x \to V_{\check{x}} \cong V_x$$

*which are $\mathbb{C}$-antilinear, preserve the norm and satisfy*

$$\begin{aligned} q(x)q(\check{x}) &= -I, \\ q(x)c_k(\check{x}) &= (-1)^k c_k(x)q(x+\delta_k) \end{aligned} \tag{3.20}$$

*for all $k \in \mathbb{N}$ and almost all $x \in X$.*

*Proof.* The argument exactly parallels that of Theorem 3.2, with the equation $r(x)r(\check{x}) = I$ replaced for $q(x)q(\check{x}) = -I$, as it fits the condition $Q^2 = -I$. We will not repeat it here, but will highlight the pointwise formula obtained for the quaternionic structure, for later reference:

$$Qf(x) = q(x)Tf(x) \tag{3.21}$$

where $T$ is as in (3.4). ∎

**Corollary 3.22.** *If $\mu$ is discrete and $\pi_{(\mu,\nu,\mathcal{C})}$ is irreducible over $\mathbb{C}$, then it admits no invariant quaternionic structure. In particular, the Fermi-Fock representations are of complex type.*

*Proof.* As we mentioned in (3.9), discreteness of $\mu$ and irreducibility of $\pi_{(\mu,\nu,\mathcal{C})}$ implies that $\mu$ is supported in some translate $x_o+\Delta$. Since $(x_o+\Delta)\check{}\cap(x_o+\Delta) = \emptyset$, $\mu$ cannot be equivalent to $\check{\mu}$. $\pi_{(\mu,\nu,\mathcal{C})}$ cannot admit then any real or quaternionic structures and, therefore, is of complex type. ■

There are families of representations $\pi_{(\mu,\nu,\mathcal{C})}$ whose $\mu$ and $\nu$ are consistent with checking, so that the operator $T$ is a well defined unitary involution, but whose $c_k(x)$ do not transform properly. Indeed, this is the case for $\pi_{(\mu_X,1,\{1\})}$ and, more generally,

**Corollary 3.23.** *The tensor product representations $\pi_{(\mu_X,1,\mathcal{C}^{\otimes})}$ are all of complex type.*

*Proof.* The Haar measure is ergodic and satisfies the condition of (3.10). Hence the same argument as in the proof of that Corollary shows that there are no measurable solutions $q(x)$ to the equations (3.20). ■

Most interesting are the quaternionic structures invariant under a spinor structure with $\nu = 1$, i.e., when the fibers $V_x$ have real dimension two and, therefore, do not admit any quaternionic structures themselves. To describe them, recall that in this case $V = L^2(X,\mu)$, which has the space of real-valued functions as a (non-invariant) real form; let, as in §2, denote the conjugation with respect to it by $v \mapsto \bar{v}$.

**Proposition 3.24.** *If $\check{\mu} \cong \mu$, $\nu = 1$ and for a.a. $x$*

$$\overline{c_1(x)} = -c_1(\check{x}), \qquad \overline{c_k(x)} = (-1)^{k+1}c_k(\check{x}) \quad \forall k \geq 2,$$

*then*

$$Q_1 f(x) = (-1)^{x_1}\sqrt{\frac{d\mu(\check{x})}{d\mu(x)}}\,\overline{f(\check{x})}$$

*is a quaternionic structure in $L^2(X,\mu)$ invariant by $\pi_{(\mu,1,\mathcal{C})}$. In that case,*

$$q(x) = (-1)^{x_1}R.$$

*Proof.* Both ${}^{T}J_k = TJ_kT$ and ${}^{R}J_k = RJ_kR$ are GW representations whose multiplier operators are, respectively,

$${}^{T}c_k(x) = (-1)^{k+1}c_k(\check{x}), \qquad {}^{R}c_k(x) = -\overline{c_k(x)}.$$

Therefore,

$$\tilde{J}_k := TRJ_kRT$$

has

$$\tilde{c}_k(x) = {}^{T}({}^{R}c_k)(x) = (-1)^{k+1}\,{}^{R}c_k(\check{x}) = (-1)^{k+1}(-\overline{c_k(\check{x})}) = (-1)^k\overline{c_k(\check{x})}$$

as the parameter $\mathcal{C}$. The operator $\Phi(x) = (-1)^{x_1} I$ anticommutes with $J_1$ and $J_1'$ and commutes with $J_k$ and $J_k'$ for all $k > 1$. Since $Q = \Phi R T$ and, clearly, $\Phi R T = R \Phi T = -R T \Phi$,

$$QJ_kQ = \Phi RTJ_k\Phi TR = -\Phi RTJ_kTR\Phi = -\Phi \tilde{J}_k \Phi = \begin{cases} \tilde{J}_1 & k = 1 \\ -\tilde{J}_k & k > 1 \end{cases}$$

and similarly for the $J_k'$. It follows that the $c_k$'s for $QJ_kQ$ are

$${}^{Q}c_1(x) = \tilde{c}_1(x) = -\overline{c_1(\check{x})}, \qquad {}^{Q}c_k(x) = -\tilde{c}_k(x) = -(-1)^k\overline{c_k(\check{x})} \qquad (k > 1).$$

Hence, the representation commutes with $Q$ iff $c_1(x) = -\overline{c_1(\check{x})}$ and $c_k(x) = (-1)^{k+1}\overline{c_k(\check{x})}$ for $k > 1$. ∎

*Remark.* Once again, (3.24) holds for arbitrary $\nu$, provided we measurably fix a real structure $\sigma(x)$ on each $V_x$, invariant under translations by $\Delta$ and checking, and replace $R$ and the bars for $\sigma(x)$ throughout.

**Theorem 3.25.** *Every pair $(\pi, Q)$ consisting of a unitary representation of $\mathfrak{C}$ with $\nu = 1$, together with an invariant quaternionic structure, is unitarily equivalent to a GW representation on $L^2(X,\mu)$ having $Q_1$ as invariant quaternionic structure.*

*Proof.* Realize $\pi$ as a GW representation $\pi_{(\mu,1,\mathcal{C})}$. By Theorem (3.19), $\mu$ is equivalent to $\check{\mu}$, the derivative $d\mu(\check{x})/d\mu(\check{x})$ exists a.e. and $T$ is a well defined unitary operator on $V$. Let $q(x)$ be the operator-valued function associated to $(\pi, Q)$,

$$q(x)(f(x)) = (QTf)(x).$$

Since each $q(x)$ is antilinear and norm preserving, $R \circ q(x)$ is a linear, unitary operator on $V_x = \mathbb{C}$ and therefore has the form

$$R \circ q(x) = \alpha(x) I$$

for some measurable $\alpha : X \to \mathbb{T}$. Because $R$ is just plain conjugation and $q(x)$ is antilinear, we also have $q(x) \circ R = \overline{\alpha(x)} I$. Because $q(x)q(\check{x}) = -1$,

$$R \circ q(\check{x}) = -R \circ q(x)^{-1} = -R \circ (q(x) \circ R \circ R)^{-1} = -R \circ (\overline{\alpha(x)} I \circ R)^{-1} = -\alpha(x) I,$$

so that

$$\alpha(\check{x}) = -\alpha(x)$$

for almost all $x$. If we set $\beta(x) = (-1)^{x_1}\alpha(x)$, then

$$R \circ q(x) = (-1)^{x_1}\beta(x) I, \qquad q(x) \circ R = (-1)^{x_1}\overline{\beta(x)} I, \qquad \beta(\check{x}) = \beta(x).$$

Define

$$u(x) := \sqrt{\beta(x)},$$

where for any $-\pi < \theta \le \pi$, $\sqrt{e^{i\theta}} := e^{i\frac{\theta}{2}}$. Then $u(x)$ is a measurable $\mathbb{T}$-valued function satisfying

$$u(x)^2 = \beta(x), \qquad u(\check{x}) = u(x).$$

The operator

$$Uf(x) := u(x)f(x)$$

is unitary from $V$ to $V$. One has

$$\begin{aligned}
UQU^{-1}f(x) = u(x)q(x)TUf(x) &= u(x)q(x)\sqrt{\frac{d\mu(\check{x})}{d\mu(\check{x})}}u(\check{x})^{-1}f(\check{x}) \\
&= \sqrt{\frac{d\mu(\check{x})}{d\mu(\check{x})}}u(x)q(x)Ru(x)Rf(\check{x}) \\
&= \sqrt{\frac{d\mu(\check{x})}{d\mu(\check{x})}}u(x)(-1)^{x_1}\overline{\beta(x)}u(x)Rf(\check{x}) \\
&= (-1)^{x_1}\sqrt{\frac{d\mu(\check{x})}{d\mu(\check{x})}}u(x)\overline{\beta(x)}u(x)Rf(\check{x}) \\
&= (-1)^{x_1}\sqrt{\frac{d\mu(\check{x})}{d\mu(\check{x})}}u(x)\overline{u(x)}\ \overline{u(x)}u(x)Rf(\check{x}) \\
&= (-1)^{x_1}\sqrt{\frac{d\mu(\check{x})}{d\mu(\check{x})}}Rf(\check{x}) \\
&= (-1)^{x_1}RTf(x) = Qf(x)
\end{aligned}$$

■

**Corollary 3.26.** *If a quaternionic structure on $L^2(X,\mu)$ is invariant under some spinor structure, then it is unitarily equivalent to $Q_1$.*

*Remark.* If $\pi$ is irreducible then there is a most one invariant $Q$ up to sign. This follows from Schur's Lemma applied to the operator $Q_1Q_2$, which is $\mathbb{C}$-linear and commutes with $\pi$. We will ignore the sign ambiguity and talk in that case about the unique quaternionic (or real) structure. real or quaternionic structures, see §6.

## 4. Examples in $L^2(\mathbb{T})$

The representations

$$\pi_{\mathcal{C}} := \pi_{(\mu_X,1,\mathcal{C})}$$

where $\mu_X$ is the Haar measure, are realized on $L^2$ of the circle $\mathbb{T}$, as follows.

$$x \mapsto \sum_{k=1}^{\infty} \frac{x_k}{2^k}$$

from $X$ to the unit interval $[0,1)$ is a bijection off a countable set. Under it, the Haar measure $\mu_X$ corresponds to the Lebesgue measure on $[0,1)$. Hence, as a measure space, $(X,\mu_X)$ is a union of $(0,1)$ with a set of measure zero, or *Lebesgue space.* The same is true for the circle $\mathbb{T}$; in this case, the maps

$$\theta_k : \mathbb{T} \to \mathbb{Z}_2$$

such that

$$t = e^{2\pi i\theta} \;\leftrightarrow\; \theta = \sum_{k=1}^{\infty} \frac{\theta_k(t)}{2^k}.$$

induce an identification

$$L^2(\mathbb{T}) = L^2(X,\mu_X). \tag{4.1}$$

As a topological space, however, $X$ is homeomorphic to the Cantor set, via

$$x \mapsto \sum_{k=1}^{\infty} \frac{x_k}{3^k};$$

$X$ is sometimes called the Cantor group [12]. The two topologies are related by Cantor's function. We will often switch between $\mathbb{T}$ and $X$, but must keep in mind that translations in $X$ do *not* correspond to rigid rotations in $\mathbb{T}$ -they preserve the measure but not the metric. In the switching, Cantor's function will not be used explicitly, thanks to the fact that at the $L^2$-level, it is like switching between Fourier's and Walsh' basis.

The group of unitary characters of $X$ -the *continuous* homomorphisms $X \to \mathbb{T}$, can be identified with $\Delta$, the subgroup of $X$ of elements with finite support. The character corresponding to $\alpha \in \Delta$ is

$$\phi_\alpha(x) = (-1)^{\sum \alpha_k x_k}.$$

In particular,

$$\hat{X} = \{\phi_\alpha\}_{\alpha\in\Delta}$$

is an orthonormal basis of $L^2(X,\mu_X)$. Via the identification (4.1) the $\phi_\alpha$ become the classical periodic Walsh functions $w_0, w_1, \dots$, defined by

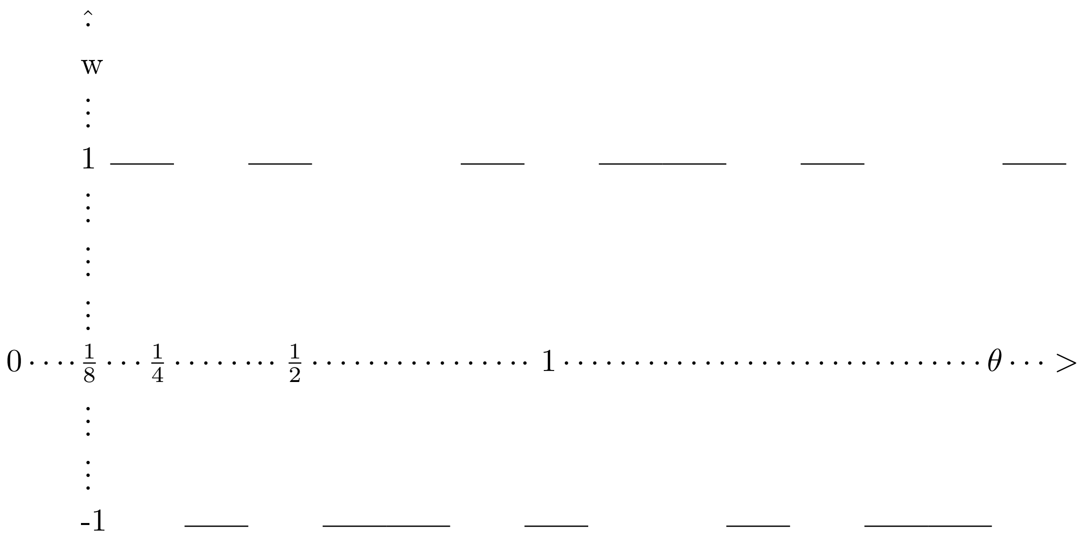
$$w_n(t) = (-1)^{\sum_{k=1}^{\infty} n_{k-1}\theta_k(t)} \tag{4.2}$$

for $t \in \mathbb{T}$ and $n = \sum_{k=0}^{\infty} n_k 2^k$ is the dyadic expansion of the integer $n$. The correspondence is

$$w_n \leftrightarrow \phi_\alpha \qquad \text{iff} \qquad n = \sum_{k=0}^{\infty} \alpha_{k+1} 2^k.$$

We will refer to both the $w_n$ and the $\phi_\alpha$ as Walsh functions.

Of course, $\hat{X} \neq \hat{\mathbb{T}}$, since on $\mathbb{T}$ the $\phi_\alpha$ are not even continuous. Periodic Walsh functions jump between 1 and $-1$, with the jumps occurring at the points of the form $j2^k$ with $j, k \in \mathbb{Z}$. As an illustration, here is $w_5(e^{2\pi i\theta}) = (-1)^{\theta_1+\theta_3}$ ($\leftrightarrow \phi_{\delta^1+\delta^3}$) for $\theta > 0$:




Define

$$\sigma^k = \delta^1 + \cdots + \delta^k$$

and $\tau_k f(x) = f(x+\delta^k)$ for $x \in X$. Then $\pi_{\mathcal{C}}$ is defined by

$$J_k = -i\phi_{\sigma^{k-1}} c_k \tau_k, \qquad J'_k = \phi_{\sigma^k} c_k \tau_k.$$

For simplicity, we shall refer to these representations as *spinor structures, on* $L^2(\mathbb{T})$. Since the Haar measure on $\mathbb{T}$ is ergodic and $\nu = 1$,

**Proposition 4.3.** *The representations* $(\pi_{\mathcal{C}}, L^2(\mathbb{T}))$ *are irreducible.*

*Remarks.* (a) The operation $x \mapsto \check{x}$ in $X$ corresponds to the symmetry in $(0,1)$ with respect to the midpoint which, on $\mathbb{T} \subset \mathbb{C}$, becomes ordinary complex conjugation. The real form $V^{\mathbb{R}}$ is

$$L^2(\mathbb{T})^{\mathbb{R}} = \{f \in L^2(\mathbb{T}) : \overline{f(t)} = f(\bar{t})\}$$

and $\pi_{\mathcal{C}}$ leaves it invariant if and only if the $c_k$'s, which are now functions from $\mathbb{T}$ to itself, satisfy

$$\overline{c_k(t)} = (-1)^k c_k(\bar{t}).$$

An analogous statement can be made for the invariant quaternionic structure defined by

$$Qf(t) = (-1)^{\theta_1(t)}\overline{f(\bar{t})}$$

(b) The function $(-1)^{\theta_1(t)}$ is the periodic Haar's mother wavelet.

(c) $L^2(\mathbb{T})^{\mathbb{R}}$, the typical spin-invariant real form, is the real span of the Fourier basis $\{e^{2\pi ik\theta}\}$. The ordinary real form $L^2(\mathbb{T})_{\mathbb{R}}$ is the real span of the Walsh basis $\{w_n\}$.

Infinite matrices of 0's and 1's are a source of an interesting family of examples.

**Definition.** *The representation* $(\pi_{\mathcal{C}}, L^2(\mathbb{T}))$ *is a character representation if* $\mathcal{C} \subset \hat{X}$.

Explicitly, the assumption is that

$$c_k(x) = \phi_{\gamma^k}(x) = (-1)^{\sum_{j\geq 1}\gamma_j^k x_j}$$

for appropriate $\gamma^k \in \Delta$. These can be can be regarded as the rows of an infinite matrix

$$\gamma = \begin{bmatrix} \gamma_1^1 & \gamma_2^1 & \cdots \\ \gamma_1^2 & \gamma_2^2 & \cdots \\ \vdots & \vdots & \end{bmatrix} \tag{4.4}$$

of 0's and 1's with finitely many 1's in each row. Regarding $X$ as a $\mathbb{Z}_2$-vector space, $\Delta$ is a subspace and the set of such $\gamma$'s can be identified with $\mathrm{End}_{\mathbb{Z}_2}(\Delta)^*$.

Given such $\gamma$, define unitary operators on $L_2(X,\mu_X)$ by

$$J_k^\gamma f(x) = -i\phi_{\sigma^{k-1}+\gamma^k}(x)f(x+\delta^k)$$

$$J_k^{\gamma\prime} f(x) = \phi_{\sigma^{k}+\gamma^k}(x)f(x+\delta^k),$$

$k = 1, 2, \ldots$.

**Proposition 4.5.** $J_k^\gamma, J_k^{\gamma\prime}$, *define a spinor representation if and only if*

$$\gamma_\ell^k = \gamma_k^\ell, \qquad \gamma_k^k = 0 \tag{4.6}$$

*for all* $k, \ell$. *In that case, they act on the Walsh basis by:*

$$J_k^\gamma \phi_\alpha = -i(-1)^{\alpha_k}\phi_{\alpha+\gamma^k+\sigma^{k-1}}, \qquad J_k^{\gamma\prime}\phi_\alpha = (-1)^{\alpha_k}\phi_{\alpha+\gamma^k+\sigma^k}.$$

*The corresponding spinor representation* $\pi^\gamma$ *is irreducible.*

*If*

$$\sum_j \gamma_j^k \equiv k \mod(2) \ \ \forall k \tag{4.7}$$

*then* $\pi^\gamma$ *is of real type and has*

$$L^2(\mathbb{T})^{\mathbb{R}} = \{f \in L^2(\mathbb{T}) : \ \overline{f(t)} = f(\bar{t})\}$$

*as the unique invariant real form.*

*If, instead,*

$$\sum_j \gamma_j^1 \equiv 0, \qquad \sum_j \gamma_j^k \equiv k \mod(2) \ \ \forall k \geq 2 \tag{4.8}$$

$\pi^\gamma$ *is of quaternionic type and*

$$Qf(t) = (-1)^{\theta_1(t)}\ \overline{f(\bar{t})},$$

*is the unique invariant quaternionic structure.*

*Proof.* The operators $J_k$, $J'_k$ are of the form (2.3), with the characters $c_k(x) = \phi_{\gamma^k}(x)I$ as multipliers. We verify equations (2.2):

$$\begin{aligned} c_k(x+\delta^k) &= \phi_{\gamma^k}(x+\delta^k) = \phi_{\gamma^k}(\delta^k)\phi_{\gamma^k}(x) = (-1)^{\sum_j \gamma_j^k \delta_j^k}\phi_{\gamma^k}(x) \\ &= (-1)^{\gamma_k^k}\phi_{\gamma^k}(x) = \phi_{\gamma^k}(x) = c_k(x) \\ &= c_k(x)^* \end{aligned}$$

since the $c_k$ are real. Also,

$$\begin{aligned} c_k(x)c_l(x+\delta^k) &= \phi_{\gamma^k}(x)\phi_{\gamma^l}(x+\delta^k) = \phi_{\gamma^k}(x)\phi_{\gamma^l}(x)\phi_{\gamma^l}(\delta^k) \\ &= \phi_{\gamma^k}(x)\phi_{\gamma^l}(x)(-1)^{\gamma_k^l} = (-1)^{\gamma_l^k}\phi_{\gamma^k}(x)\phi_{\gamma^l}(x) \\ &= \phi_{\gamma^k}(x+\delta^l)\phi_{\gamma^l}(x) = c_l(x)c_k(x+\delta^l) \end{aligned}$$

It is clear that the converse also holds. The calculation of the action on Walsh functions is straightforward and irreducibility follows from (4.2).

According to Theorem (3.14), $\pi^\gamma$ will leave $L^2(\mathbb{T})^{\mathbb{R}}$ invariant if and only if $c_k(\check{x}) = (-1)^k\overline{c_k(x)}$, which translates into the equation

$$\phi_{\gamma^k}(\check{x}) = (-1)^k\overline{\phi_{\gamma^k}(x)}.$$

Since $\phi_{\gamma^k}(\check{x}) = \phi_{\gamma^k}(1+x) = \phi_{\gamma^k}(1)\phi_{\gamma^k}(x) = (-1)^{\sum_j \gamma_j^k}\phi_{\gamma^k}(x)$ and $\phi$ is real, the equation is satisfied exactly when $\sum_j \gamma_j^k \equiv k \mod(2)$, i.e., when $\gamma \in \Gamma_{\mathbb{R}}$.

A similar computation shows that $\pi^\gamma$ leaves $Q$ invariant exactly when $\gamma \in \Gamma_{\mathbb{H}}$. The uniqueness follows from the irreducibility of $\pi^\gamma$. ■

Because of (4.6), the columns of $\gamma$ also involve finitely many ones, so $\gamma \in \mathrm{End}_{\mathbb{Z}_2}(\Delta)$. Define

$$\Gamma = \{g \in \mathrm{End}_{\mathbb{Z}_2}(\Delta) : \quad \gamma_\ell^k = \gamma_k^\ell, \quad \gamma_k^k = 0\}$$

$$\Gamma_{\mathbb{R}} = \{g \in \Gamma : \text{ satisfying (4.7)}\}$$

$$\Gamma_{\mathbb{H}} = \{g \in \Gamma : \text{ satisfying (4.8)}\}$$

In other words, $\gamma \in \Gamma$ belongs to $\Gamma_{\mathbb{R}}$ if and only if the parity of the number of 1's in the $k$th. row (or column) equals the parity of $k$, while $\gamma \in \Gamma_{\mathbb{H}}$ if and only if the same condition holds except for the first row, which must have an even number of 1's.

For $\gamma = 0$, $\pi^\gamma$ is of complex type, by (4.5). Set, instead,

$$\mathbf{B} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \beta = \begin{bmatrix} \mathbf{B} & \mathbf{0} & \mathbf{0} & \dots \\ \mathbf{0} & \mathbf{B} & \mathbf{0} & \dots \\ \mathbf{0} & \mathbf{0} & \mathbf{B} & \dots \\ \vdots & \vdots & \vdots & \end{bmatrix}, \quad \gamma = \begin{bmatrix} 0 & 0 & \mathbf{0} & \mathbf{0} & \dots \\ 0 & 0 & \mathbf{0} & \mathbf{0} & \dots \\ \mathbf{0} & \mathbf{0} & \mathbf{B} & \mathbf{0} & \dots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{B} & \dots \\ \vdots & \vdots & \vdots & \vdots & \end{bmatrix}$$

where bold letters denote matrices and non-bold scalars. Then $\beta \in \Gamma_{\mathbb{R}}$ and $\gamma \in \Gamma_{\mathbb{H}}$, so that $\pi^{\beta}$ is of real type while $\pi^{\gamma}$ is of quaternionic type.

For any (vector-valued) function $f$ on $X$ define

$$\partial_k f(x) := \phi_{\delta^k}(x)(f(x+\delta^k) - f(x))$$

where, as usual, addition in $X$ is modulo 2. These difference operators are natural in two ways: they are the partial derivatives in $X = \mathbb{Z}_2^{\infty}$ once we fix the motion from 0 to 1 as positive and, via $X \approx \mathbb{T}$, the ordinary derivative on $\mathbb{T}$ with respect to the angular parameter is $f'(\theta) = \lim_{k\to\infty} 2^k \partial_k f(x(\theta))$ or, equivalently,

$$\frac{d}{d\theta} = \sum_{k=0}^{\infty} 2^k(2\partial_{k+1} - \partial_k).$$

This follows by taking incremental quotients of the form $\Delta\theta = (-1)^{\theta_k}2^{-k}$ and noting that the translation $x \mapsto x+\delta^k$ in $X$, corresponds to the operation $\theta \mapsto \theta+(-1)^{\theta_k}2^{-k}$ in $\mathbb{T}$.

This suggests some deformations of the derivative operator that, aside from the obvious one $\sum_{k=0}^{\infty} z^k(2\partial_{k+1} - \partial_k)$, are directly related to spinors. For example, the operators $\partial_k$ can be expressed in terms of the $J_k^{\kappa}, J_k^{\kappa\prime}$ of the special character representation $\kappa = \pi_{(\mu_X,1,\{1\})}$; a small calculation shows that $\partial_k = i(\phi_{\sigma^{k-1}}J_k^{\kappa} - J_k^{\kappa}J_k^{\kappa\prime})$. Replacing now $\kappa$ by any $\pi = \pi_{\mathcal{C}}$ -indeed, by any $\pi$ whatsoever, one obtains corresponding "twisted derivatives"

$$\frac{d}{d_{\pi}\theta} = i\lim_{k\to\infty} 2^k(\phi_{\sigma^{k-1}}J_k^{\pi} - J_k^{\pi}J_k^{\pi\prime}).$$

We will not discuss these here but will concentrate instead in the following first-order differential-like operators which relate directly to the main subject of this paper.

Given a spin structure $\pi^{\mathcal{C}}$ on $L^2(\mathbb{T})$, consider the associated operators

$$\mathcal{D} = \sum_k J_k\partial_k, \qquad \sum_k \mathcal{D}' = J_k'\partial_k$$

or, better yet, their linear combinations $D = (-\mathcal{D}'+i\mathcal{D})/2$, $D' = (\mathcal{D}'+i\mathcal{D})/2$. Evidently,

$$D = \sum_{k=0}^{\infty} a_k\partial_k, \qquad D' = \sum_{k=0}^{\infty} a_k^*\partial_k.$$

We will not attempt to motivate them *a priori*. They are, of course, linear wherever defined and annihilate constants, but their resemblance to Dirac operators does not go very far because the $\partial_k$ do not commute with the spinor representation. Moreover, they -even their domain and spectra- depend on the specific representation, not just on its equivalence class.

**Proposition 4.9.** *For the standard Fermi-Fock representation the domain of $D'$ consists of $\{0\}$ alone. For the character representations, the domains of both $D$ and $D'$ are dense in $L^2(\mathbb{T})$.*

**Proof.** The characteristic functions of points in $\Delta$, $\chi_\alpha$, are an orthonormal basis of $V_{(\mu_\Delta,1,\{1\})}$, where the Fermi-Fock representation acts. One has

$$\partial_k\chi_\alpha = (-1)^{1+\alpha_k}(\chi_\alpha+\chi_{\alpha+\delta_k})$$

so

$$D'\chi_\alpha = \sum_k(-1)^{1+\alpha_k}\phi_{\sigma^k}\chi'_k(\chi_\alpha+\chi_{\alpha+\delta_k}).$$

This is nonzero only at the points of the form $x=\alpha$, $x=\alpha+\delta_l$, so

$$D'\chi_\alpha = C^0_\alpha\ \chi_\alpha+\sum_{k\geq 1}C^k_\alpha\ \chi_{\alpha+\delta^k}$$

with $C^k_\alpha = D'\chi_\alpha(\alpha+\delta^k)\in\mathbb{Z}$. Therefore

$$C^0_\alpha = D'\chi_\alpha(\alpha) = \sum_k(-1)^{\alpha_k}\phi_{\sigma^k}(\alpha)\chi'_k(\alpha) = -\sum_{k\notin\mathrm{supp}(\alpha)}\phi_{\sigma^k}(\alpha)$$

Since $\alpha$ has finite support, $\phi_{\sigma^k}(\alpha)$ is a constant, $=1$ or $-1$, for all $k>>0$, so the series diverges.

For the character representations, the Hilbert space is $V_{(\mu_X,1,\mathcal{C})}$, where the $\phi_\alpha$ form an orthonormal basis -or, equivalently, $L^2(\mathbb{T})$, where the $w_n$ form an orthonormal basis. With $c_k=\phi_{\gamma^k}I$,

$$J_kf(x) = -i\phi_{\sigma^{k-1}+\gamma^k}(x)f(x+\delta^k), \qquad J'_kf(x) = \phi_{\sigma^k+\gamma^k}(x)f(x+\delta^k)$$

wherefrom

$$J_k\phi_\alpha = -i(-1)^{\alpha_k}\phi_{\alpha+\gamma^k+\sigma^{k-1}}, \qquad J'_k\phi_\alpha = (-1)^{\alpha_k}\phi_{\alpha+\gamma^k+\sigma^k}. \tag{4.10}$$

On the other hand,

$$\partial_k\phi_\alpha = -2\chi_k(\alpha)\phi_{\alpha+\delta^k}$$

where, as before, $\chi_k$ is the characteristic function of the set $X_k$. Therefore

$$\begin{aligned} D\phi_\alpha &= 2\sum_{k\in\ \mathrm{supp}(\alpha)}(-1)^{\alpha_k}\chi_k\phi_{\alpha+\gamma^k+\sigma^k} \\ &= \sum_{k\in\ \mathrm{supp}(\alpha)}(-1)^{\alpha_k}(\phi_{\alpha+\gamma^k+\sigma^k}-\phi_{\alpha+\gamma^k+\sigma^{k-1}}) \end{aligned}$$

(4.11)

$$\begin{aligned} D'\phi_\alpha &= 2\sum_{k\in\ \mathrm{supp}(\alpha)}(-1)^{\alpha_k}\chi'_k\phi_{\alpha+\gamma^k+\sigma^k} \\ &= \sum_{k\in\ \mathrm{supp}(\alpha)}(-1)^{a_k}(\phi_{\alpha+\gamma^k+\sigma^k}+\phi_{\alpha+\gamma^k+\sigma^{k-1}}). \end{aligned}$$

$D$ and $D'$ are therefore well defined in the linear span of the Walsh functions, which is dense in $L^2$. ∎

**Proposition 4.12.** *Let $\pi = \pi_{(\mu,\nu,\mathcal{C})}$ be a GW representation with $\mu$ equivalent to $\check{\mu}$ and let $D_\pi, D'_\pi$, be the associated operators. Then*

$$TD_\pi T = -D'_{\tilde{\pi}}, \qquad TD'_\pi T = -D_{\tilde{\pi}}$$

*where $T$ is the operator of (3.14) and $\tilde{\pi} = \pi_{(\mu,\nu,\tilde{\mathcal{C}})}$ with*

$$\tilde{c}_k(x) := (-1)^{k+1} c_k(\check{x}).$$

*In particular, a spinor structure $\pi$ on $L^2(\mathbb{T})$ leaves invariant the real form $L^2(\mathbb{T})^{\mathbb{R}}$ if and only if*

$$TD_\pi T = D'_\pi.$$

*Proof.* As we saw in the proof of (3.14),

$$TJ_kTf(x) = \tilde{J}_k f(x)$$

with

$$\tilde{c}_k(x) = (-1)^{k+1} c_k(\check{x}).$$

Since $J'_k f(x) = i(-1)^{x_k} J_k f(x)$ and $T$ anticommutes with multiplication by $(-1)^{x_k}$, one has

$$TJ_kTf(x) = -\tilde{J}'_k f(x)$$

and, therefore,

$$Ta_kT = \tilde{a}^*_k, \qquad Ta^*_kT = \tilde{a}_k.$$

These identities have a meaning and are valid for any GW triple, as long as $T$ is invertible. Under the same assumption and for identical reasons,

$$T\partial_k = -\partial_k T.$$

Therefore

$$TDT = \sum_k Ta_kTT\partial_kT = -\sum_k \tilde{a}^*\partial_k = -\tilde{D}'$$

and, similarly, $TD'T = -\tilde{D}$.

The last assertion follows by comparing the formula for $\tilde{c}_k$ with (3.14).■

**Corollary 4.13.** *If a spinor structure $\pi$ on $L^2(\mathbb{T})$ leaves invariant some real structure, then*

$$\mathrm{Spec}(D'_\pi) = -\mathrm{Spec}(\mathrm{D}_\pi).$$

Whenever defined, either operator determines the representation. For example, if $\{a_k\}$ are the creation operators corresponding to a character representation, then

$$2a_kf = D_\pi - \phi_{\delta^k} D_\pi(\phi_{\delta^k} f)$$
$$2a^*_kf = D'_\pi f - \phi_{\delta^k} D'_\pi(\phi_{\delta^k} f).$$

*Remark.* For character representations, the matrices of $D$ and $D'$ in the Walsh basis involve only $0$ and $\pm 1$, as is evident from (4.11). An intriguing aspect of these matrices is that, although very non-symmetric, they appear to be always diagonalizable. More remarkably, *the diagonalization can be done over $\mathbb{Z}$* in the sense that all eigenvalues are integers and all eigenspaces can be spanned by integral linear combinations of the Walsh functions. The diagonalizability condition is equivalent to a combinatorial property of the matrices $\gamma$ which we have been able to verify in some, but not all, cases.

## 5. Kaplansky's infinite-dimensional numbers

The real finite-dimensional division algebras, -associative or not, with or without a $1$, occur only in dimensions 1,2,4 and 8. If we require a multiplicative identity and that $||ab|| = ||a||\ ||b||$ for some norm (be *normed*), one obtains the usual algebras of real, complex, quaternionic and octonionic numbers.

In [13], Kaplansky proved that

there are no infinite dimensional normed division algebras,

no "infinityonic numbers". Of course, in infinite dimensions there are many division algebras, even associative and commutative ones, like $\mathbb{R}[t]$, as well as many normed algebras, because $U \otimes U \cong U$ for any linear space. But none will satisfy both conditions simultaneously.

A normed algebra has no zero-divisors, a condition that is often used as the definition of division algebra on the basis of the equivalence that exists in finite dimensions. So, let us make our terminology more precise.

For the rest of the section, an *algebra* is any vector space $U$ endowed with a bilinear operation $\star$. It is *normed* if $U$ is a real Hilbert space and $||v \star w|| = ||v||\ ||w||$. It is *left-division* if $\forall v \neq 0,\ \exists v_L^{-1}$ such that

$$v_L^{-1} \star (v \star w) = w \qquad \forall w.$$

It is *right-division* if $\forall v \neq 0,\ \exists v_R^{-1}$ such that

$$(w \star v) \star v_R^{-1} = w \qquad \forall w;$$

it is simply *division* if it is both left- and right-division. An *equivalence*, is a change $\star \mapsto \tilde{\star}$ of the form

$$v \tilde{\star} w = A(B(v) \star C(w))$$

with $A, B, C \in O(U)$. Then, up to equivalence, one may assume that in a division algebra there is a two-sided unit and that left and right inverses agree.

Kaplansky then shows that weakening "division" to, say, "left-division" does nothing in finite-dimensions, i.e., that

a finite-dimensional real left-division normed algebra, is a division algebra

and speculates about the situation in infinite dimensions. A counterexample could claim the role of infinite-dimensional relatives of the quaternions and octonions. The first counterexamples were found 30 years later by Cuenca and Rodriguez-Palacios [7],[17].

Now we can describe all such structures, i.e., all the left-division normed algebras on an infinite-dimensional separable real Hilbert space -or *ILNA*'s, as we will be calling them for short. It turns out that there are mazes of inequivalent ones, as implied by

**Theorem 5.1.** *The ILNA's are naturally parametrized up to equivalence by the triples* $(\mu, \nu, \mathcal{C})$ *of 3.15 and 3.15. In fact, if* $U$ *is a separable real Hilbert space, then there is a one-to-one correspondence between real representations of* $\mathfrak{C}$ *on* $U$ *and structures of ILNA's on* $U$ *having a left-identity.*

*Proof.* We only need to show that such algebras are in correspondence with the real orthogonal representations of $\mathfrak{C}$; the examples of [7][17] were also implicitly or explicitly built from the CAR's, the new ingredient here being the role of the GW parametrization.

Let $(U, \star)$ be given, where $U$ is a separable Hilbert space. Pick a unit vector $v_o \in U$ and define $v \hat{\star} w = v_o^{-1} \star (v \star w)$, where we drop the subscript $L$ in $v_L^{-1}$. Then $v_o \hat{\star} w = v_o^{-1} \star (v_o \star w) = w$. Therefore, up to equivalence, we may assume that $\star$ has a left-identity element $\epsilon$, i.e.,

$$\epsilon \star u = u$$

for all $u$. Let $H$ denote the orthogonal complement of $\epsilon$ in $U$. Then

$$\pi(h)v = h \star v, \qquad (h \in H)$$

defines a unitary representation of $\mathfrak{C}$ on $U$. Indeed, if $h, h'$, are orthogonal to $1$, then polarizing the norm condition yields

$$h \star (h' \star v) + h' \star (h \star v) = -2 < h, h' > v$$

for all $v \in U$.

Conversely, let $\pi$ be an real orthogonal representation of $\mathfrak{C}$ on $U$ and choose an isomorphism of real Hilbert spaces

$$F : U \tilde{\to} H \oplus \mathbb{R} \ \subset \mathfrak{C}.$$

It is precisely when

$$\dim_{\mathbb{R}} H = 1, 3, 7, \infty,$$

that such isomorphism exists, i.e., that $\mathfrak{C}$ has a non-trivial module of dimension $\dim H + 1$.

Define $\star = \star_\pi$ by

$$u \star v = \pi(F(u))v. \tag{5.2}$$

By the orthogonality of $\pi$, the resulting algebra is normed. Finally, if $u \in U$, $u \neq 0$, write $F(u) = h + \lambda$ with $h \in H$ and $\lambda \in \mathbb{R}$. Then

$$\begin{aligned} F^{-1}(h - \lambda) \star (u \star v) &= \pi(FF^{-1}(h - \lambda))(u \star v) = \pi(h - \lambda)(\pi(F(u)v) \\ &= \pi(h - \lambda)(\pi(h + \lambda)v) = (\pi(h - \lambda)\pi(h + \lambda))v \\ &= \pi((h - \lambda)(h + \lambda))v = \pi(h^2 - \lambda^2)v = (-||h||_H^2 - \lambda^2)v \\ &= -(||h||_H^2 + \lambda^2)v = -||h + \lambda||_{\mathfrak{C}}^2 v = -||F^{-1}(h + \lambda)||_{\mathfrak{C}}^2 v \\ &= -||u||_U^2 v \end{aligned}$$

In terms of the involution in $H \oplus \mathbb{R}$ ("conjugation") $K(h + \lambda) = -h + \lambda$,

$$u_L^{-1} := ||u||^{-2} F^{-1} K F(u)$$

is a left-inverse of $u$. ∎

We next look a bit more closely at the structure of $\star$ and show how its groups of symmetries reflects various classes spinors that exist in infinite dimensions. The exceptional properties of the symmetries of the Octonions [2] leads to some conjectures. Here we discuss only aspects directly related to real and quaternionic structures and related matters.

To an ILNA $(U, \star)$ we associate its group of equivalences

$$Eq(\star) \subset O(U)^3$$

consisting of the triples $(g_0, g_1, g_2)$ of orthogonal transformations of $U$ satisfying

$$g_o(u \star v) = (g_1 u) \star (g_2 v) \tag{5.3}$$

$\forall u, v \in U$. Then

$$Aut(\star) = \{(g_0, g_1, g_2) \in Eq(\star) : g_0 = g_1 = g_2\}.$$

As we saw in the proof of (5.1), $(U, \star)$ can be assumed to have a left-identity $e$. Set

$$\Im(U) = e^{\perp} \qquad \text{so that} \qquad U = \mathbb{R}e \oplus \Im(U)$$

We digress briefly on the peculiarities of the infinite-dimensional case and on the choice of paired basis $\{h_k, h_k'\}$ which, so far, has remained as an implicit parameter.

This choice determines a subspace

$$H_r = \mathrm{span}_{\mathbb{R}}\{h_k\}, \tag{5.4}$$

and an orthogonal complex structure $j$ on $H$

$$j(h_k) = h_k', \qquad j(h_k') = -h_k,$$

so that

$$H = H_r \oplus jH_r.$$

This data is equivalent to an isomorphism of real Hilbert spaces

$$H \cong \mathbb{C} \otimes H_r.$$

The equivalence class of $\pi$ depends on $H_r$ and $j$ -this is the reason we use the plural when talking about Fock representations. In QFT, $H$ is taken complex from the start; the physical meaning of the complex structure (or lack thereof) is discussed in [3].

Let

$$\tau : H \to H, \qquad \tau(h_1 + jh_2) = h_1 - jh_2$$

$h_i \in H_r$, which is an orthogonal involution. Obviously, $H_r$ and $j$ determine $\tau$, and (5.4) is the decomposition of $H$ into $\pm 1$-eigenspaces of $\tau$

$$H = H_+ \oplus H_-. \tag{5.5}$$

Conversely, let $\tau$ be an orthogonal involution of $H$. It determines an orthogonal decomposition (5.5) but the summands be of different size. In finite dimensions one sometimes says that an involution is a polarization if $\dim H_+ = \dim H_-$. This condition is equivalent to the existence of an isomorphism of $H$ which anticommutes with $\tau$. For example, when $\tau$ comes from a pair $(j, H_r)$, then $j$ is such an isomorphism. In infinite dimensions we adopt this as a definition of polarization.

As we saw in the proof of (5.1), $(U, \star)$ can be assumed to have a left-identity $e$; then left multiplication by elements of

$$\Im(U) := e^{\perp}$$

defines the Clifford action of $C(\Im(U))$ on $U$: $\Im(U)$ is identified with $H$ and inherits the structures above:

$$\Im(U) \cong \mathbb{C} \otimes W, \tag{5.6}$$

for some subspace $W \subset U$ and, writing also $j$ for the induced complex structure in $\Im(U)$,

$$U = \mathbb{R}e \oplus W \oplus jW.$$

We will keep denoting by $\tau$ the corresponding complex-conjugation.

The existence of these conjugations characterizes infinite-dimensional left-division normed algebras, since in the finite-dimensional case $\dim_{\mathbb{R}} \Im(U)$ is odd and, therefore, cannot support a complex structure. It is a bit one gains in exchange for giving up two-sided inverses.

Another infinite-dimensional phenomenon is the fact that *right multiplication is never invertible.*

$$R_v : u \mapsto u \star v$$

is injective whenever $v \neq 0$, because there are no zero-divisors. But, as shown in [13], if $R_v$ is onto for some $v$ then $(U, \star)$ would be two-sided division, which is ruled out. Hence,

$$R_v(U) = U \star v$$

is always a proper subspace of $U$. In particular, $R_e$ fixes $e$ and leaves $e^{\perp} = \Im(U)$ invariant. From the above, $\Im(U) \star e \neq \Im(U)$ so that

$$\mathrm{coker} R_e \neq (0)$$

Next we exemplify how algebraic properties of $\Im(U) \star e$ and $\tau$ discriminate among ILNA's and relate to analytic properties of the associated spinors. We will say that a given $(U, \star)$ "comes from" a given spin structure $\pi$ if it is equivalent to the one constructed from $\pi$ by the procedure of (5.1).

**Proposition 5.7.** *$(U, \star)$ comes from a Fermi-Fock representation if and only it has no proper left-ideals, $U$ is a complex Hilbert space and*

*(a) $\star$ is $\mathbb{C}$-linear in the right-slot:*

$$u \star iv = i(u \star v)$$

*(b)* $\Im(U) \star e$ *is a complex subspace; equivalently,*

$$(jw) \star e = i(w \star e)$$

*for all* $w \in \Im(U)$*.*

*Proof.* The Fermi-Fock representations $\pi$ are characterized by the fact that $U$ is a complex Hilbert space, the operators $\pi(h)$ are $\mathbb{C}$-linear, $\pi$ is irreducible and there exists a non-zero $v_o \in U$ annihilated by all $a_k^*$'s. From §3 we know that such $\pi$ is irreducible over $\mathbb{R}$, which translates into $(U, \star)$ not having proper left-ideals. We can choose $v_o = e$, so the condition $a_k^* v_o = 0$ becomes $J_k' e = iJ_k e$, i.e.,

$$\pi(h_k')e = i\pi(h_k)e$$

$\forall k$. In terms of the complex structure $j$ this amounts to $\pi(jh_k)e = i\pi(h_k)e$, or

$$\pi(jh)e = i\pi(h)e$$

$\forall h \in H_r$. Via the identification $H \leftrightarrow \Im(U)$ this becomes

$$(jw) \star e = i(w \star e) \tag{5.8}$$

$\forall w \in \Im(U)$. It follows that $\Im(U) \star e$ is closed under multiplication by $i$.

Conversely, suppose that the latter is the case: $\forall w \in \Im(U)\ \exists! w' \in \Im(U) : i(w \star e) = w' \star e$; uniqueness is assured because $R_e$ is injective. Then $jw := w'$ defines a linear operator $j : \Im(U) \to \Im(U)$ with the property that for all $w \in \Im(U)$,

$$(jw) \star e = i(w \star e).$$

$j$ is norm-preserving, because $\star$ is normed, and $j^2 = -I$, because

$$(j^2 w) \star e = i(jw \star e) = -w \star e.$$

Under $j$, $\Im(U)$ becomes a complex Hilbert space, with hermitian inner product $h(u, v) =< u, v > +\sqrt{-1} < ju, v >$. Let $\{w_k\}$ be a complex unitary basis of it. Then $\{w_k, jw_k\}$ is a real orthonormal basis and $W := \mathrm{span}_{\mathbb{R}}\{w_k\}$ is a real form of $(\Im(U), j)$, totally isotropic for $< ju, v >$. Then

$$J_k u := w_k \star u, \qquad J_k' u := (jw_k) \star u$$

defines a $C(\Im(U))$-spin structure on $U$. The equation $(jw) \star e = i(w \star e)$ translates into

$$a_k^* e = 0$$

$\forall k$, so $e$ is a vacuum vector. ■

Because of the evident lack of symmetry between the two slots, worsened in infinite dimensions, true automorphisms of ILNA's do not come easily. We will see that how the operator

$$Tf(x) = \sqrt{\frac{d\check{\mu}(x)}{d\mu(x)}} f(\check{x}), \qquad f \in \int_X^{\oplus} V_x \, d\mu(x)$$

used to typify spinors, can be used to construct automorphisms of $\star$.

Recall that $T : V \to V$, $V = \int_X^{\oplus} V_x \, d\mu(x)$, is well defined and invertible if and only if the measure $\mu$ and the multiplicity function $\nu$ are quasi-invariant and invariant, respectively, under the operation $x \mapsto \check{x}$. For simplicity we will assume

$$\mu = \check{\mu}, \qquad \nu = \check{\nu} = 1, \qquad \mu(X) = 1.$$

Then

$$V = L^2(X, \mu), \qquad Tf(x) = f(\check{x})$$

and the constant function $\mathbf{1}$ lies in $V$, is a unit vector and is fixed by $T$.

In our algebra $(U, \star_\pi)$, $U$ always arises as a $\pi$-invariant real form of $V$. By (3.13), we may assume that

$$U = V^{\mathbb{R}} = \{f \in L^2(X, \mu) : f(\check{x}) = \overline{f(x)} \;\; \text{a.e.}\}.$$

Choose the left-identity $e$ to be the constant function $1$, which lies in $U$. Then

$$H = \Im(U) = \{f \in L^2(X, \mu) : \; f(\check{x}) = \overline{f(x)}, \quad \int_X f(x) d\mu(x) = 0\}.$$

It is clear that on $U$, $T$ coincides with the conjugation $(\tau f)(x) = \overline{f(x)}$ and that $H$ is invariant under this operation (but not under multiplication by $i$!). The eigenspaces of $T = \tau$ actually polarize $H$:

$$H = H_+ \oplus H_- \tag{5.9}$$

where the 1-eigenspace $H_+$ consists of the real, even (relative to the checking symmetry) functions and the $-1$-eigenspace $H_-$ consists of the purely imaginary, odd functions. For emphasis, we can rewrite (5.9) as

$$H = H_{\text{real,even}} \oplus iH_{\text{real,odd}}$$

Under $X \approx (-\frac{1}{2}, \frac{1}{2})$, checking coincides with the symmetry with respect to $0$, so "even" and "odd" acquire their ordinary meaning.

For the Walsh functions,

$$\phi_\alpha(\check{x}) = (-1)^{p(\alpha)} \phi_\alpha(x)$$

where $p(\alpha)$ is the parity of (the numbers of $1$'s in) $\alpha$. The linear combinations of Walsh functions are dense in $L^2(X, \mu)$, because this is true of the finite products of the $\chi_k$, $\chi'_k$, and $2\chi_k = 1 - \phi_{\delta_k}$, $2\chi'_k = 1 + \phi_{\delta_k}$. It follows that $H_+$ is spanned as a Hilbert space by the $\phi_\alpha$ with $\alpha$ even, while $H_-$ is so by the $i\phi_\beta$ with $\beta$ odd.

Pick an orthonormal basis $\{h_k\}$ of $H_+$ and $h'_k$ of $H_-$; only now can $\pi$ be specified, by

$$\pi(h_k) f = J_k f, \qquad \pi(h'_k) f = J'_k f$$

with $J_k, J'_k$, as in (2.3).

According to the proof of (3.14), whenever $T$ is well defined and invertible one has

$$TJ_kT = \tilde{J}_k, \qquad TJ'_kT = -\tilde{J}'_k \tag{5.10}$$

where, if $\pi = \pi_{(\mu,\nu,\{c_k\})}$, $\tilde{J}$ corresponds to the spin structure $\tilde{\pi} = \pi_{(\check{\mu},\nu,\{\tilde{c}_k\})}$, with

$$\tilde{c}_k(x) := (-1)^{k+1} c_k(\check{x}) \tag{5.11}$$

as multipliers. Let $\tilde{\star}$ be the associated product. Then (5.10), which is equivalent to $T(\pi(h)v) = \tilde{\pi}(\tau(h))Tv$, translates into $T(w \star v) = \tau(w) \mathbin{\tilde{\star}} Tv$ for $w \in \Im(U)$. Extending $\tau$ to all of $U$ so that $\tau(e) = e$, one obtains

$$T(u \star v) = \tau(u) \mathbin{\tilde{\star}} T(v).$$

But on $U$, $T = \tau$. Moreover, suppose that $\pi = \tilde{\pi}$, i.e., $\tilde{J}_k = J_k$. In that case,

$$T(u \star v) = T(u) \star T(v).$$

According to (3.14) the multipliers $c_k$ must satisfy $c_k = (-1)^k\ \overline{c_k}$, while according to (5.11) they must also satisfy $\check{c}_k = (-1)^{k+1} c_k$. In the terminology used above, this is equivalent to asking that $c_k$ be real and odd for $k$ even and imaginary and even for $k$ odd. We have proved

**Proposition 5.12.** . *Assume that $\check{\mu} = \mu$, $\mu(X) = 1$, $\nu = 1$ and that $c_k$ is real and odd for $k$ even, and imaginary and even for $k$ odd. Let $\pi_r$ be the real spin structure in $L^2(X,\mu)^{\mathbb{R}}$ associated to the GW parameters $\mu, \nu, \{c_k\}$, together with the conjugation $\tau = T$ and let $(L^2(X,\mu)^{\mathbb{R}}, \star)$ be the corresponding ILNA with the function $1$ as left identity. Then*

$$T(f \star g) = T(f) \star T(g)$$

*for all $f, g \in L^2(X,\mu)^{\mathbb{R}}$.*

**Example .** Let $\gamma$ an infinite matrix of $0$'s and $1$'s and set

$$c_{2n} = \phi_{\gamma^{2n}}, \qquad c_{2n+1} = i\phi_{\gamma^{2n+1}}$$

with $\gamma^{2n} \in \Delta$ odd and $\gamma^{2n+1} \in \Delta$ even. The condition $c_k(x)^* = c_k(x + \delta^k)$ translates into

$$\phi_{\gamma^{2n}}(\delta^{2n}) = 1, \qquad \phi_{\gamma^{2n+1}}(\delta^{2n+1}) = -1,$$

that is,

$$\gamma^{2n}_{2n} = 0, \qquad \gamma^{2n+1}_{2n+1} = 1.$$

The condition $c_k(x)c_l(x + \delta^k) = c_l(x)c_k(x + \delta^l)$ translates, exactly as in the case of the character representations, into $\gamma$ being symmetric.

Finally, the conditions of (5.12)

$$\check{c}_{2n} = -c_{2n}, \qquad \check{c}_{2n+1} = c_{2n+1}$$

translate into $\check{\phi}_{\gamma^{2n}} = -\phi_{\gamma^{2n}}$ and $i\phi_{\gamma^{2n+1}} = i\check{\phi}_{\gamma^{2n+1}}$ respectively. This is equivalent to $\phi_{\gamma^{2n}}(\mathbf{1}) = -1$, $\phi_{\gamma^{2n+1}}(\mathbf{1}) = 1$, or to

$$p(\gamma^{2n}) = 1, \qquad p(\gamma^{2n+1}) = 0.$$

We conclude: *$\gamma$ is to be symmetric, with diagonal* $(1,0,1,0,\dots)$ *and the parity of $\gamma^k$ opposite to that of $k$.*

$$\gamma = \begin{bmatrix} \mathbf{B} & \mathbf{0} & \mathbf{0} & \dots \\ \mathbf{0} & \mathbf{B} & \mathbf{0} & \dots \\ \mathbf{0} & \mathbf{0} & \mathbf{B} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \qquad \text{where} \qquad \mathbf{B} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}.$$

is the simplest example.

Next we give the elementary description announced earlier for the algebras $(L^2(\mathbb{T}), \star_\gamma)$ arising from Character Representations. According to (4.10), the algebraic span of the Walsh basis is preserved by the operators $J_k, J'_k$, thus $\star$ must be given by such expressions.

To make them explicit, we extend the sum modulo 2 in the set $\{0,1\} \subset \mathbb{N}$ to an abelian group structure in all of $\mathbb{N}$:

$$m \tilde{+} n := \sum_{j \geq 0} (m_j \tilde{+} n_j) 2^j$$

where

$$m = \sum_{j \geq 0} m_j 2^j, \qquad n = \sum_{j \geq 0} n_j 2^j, \qquad (m_j, n_j \in \{0,1\})$$

are the dyadic expansions of the positive integers $m, n$. The operation $\tilde{+}$ is just addition in $\Delta$, transported to $\mathbb{N}$ via $\mathfrak{n} : \Delta \to \mathbb{N}$,

$$\mathfrak{n}(\alpha) = \sum_{j \geq 0} \alpha_{j+1} 2^j.$$

Since the Walsh functions $w_n$ correspond to characters, one has

$$w_m(\theta) w_n(\theta) = w_{m \tilde{+} n}(\theta). \tag{5.13}$$

Note that $m \mapsto m \tilde{+} 2^j$ switches $m_{j-1}$ between 0 and 1 and

$$m^{(j)} := m \tilde{+} (2^j - 1) = m \tilde{+} (1 + 2 + 3^2 + \ldots + 2^{j-1})$$

is the integer obtained from $m$ by changing its first $j$ binary digits $m_0, \dots, m_{j-1}$.

Define a function $N_\gamma : \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ by

$$N_\gamma(k, m) = \mathfrak{n}(\gamma^k) \tilde{+} m^{(k-1)}.$$

Regarding $L^2(\mathbb{T})$ as a real space, a straightforward calculation shows that the product is given by the following table

$$\begin{aligned} w_0 \star_\gamma w_m &= w_m \\ (iw_0) \star_\gamma w_m &= w_{N_\gamma(1,m) \tilde{+} 1} \\ w_k \star_\gamma w_m &= (-1)^{1+m_{k-1}} i w_{N_\gamma(k,m)} \\ (iw_k) \star_\gamma w_m &= (-1)^{m_{k-1}} w_{N_\gamma(k+1,m) \tilde{+} 2^k} \end{aligned}$$

for all $k \geq 1$ and all $m \geq 0$, together with the requirement that $u \star iv = i(u \star v)$. For emphasis:

**Proposition 5.14.** *This table defines an* $\mathbb{R}$*-bilinear operator*

$$\star : L^2(\mathbb{T}) \times L^2(\mathbb{T}) \to L^2(\mathbb{T})$$

*satisfying*

$$||f \star g|| = ||f||\ ||g||$$

*and such that* $\forall f \neq 0\ \ \exists f_\ell^{-1}$ *:*

$$f_\ell^{-1} \star (f \star g) = g, \quad \forall g.$$

For example, take $\gamma = \mathbf{0}$. Then $N_{\mathbf{o}}(k,m) = m^{(k-1)}$ so

$$\begin{aligned} w_0 \star_o w_m &= w_m \\ (iw_0) \star_o w_m &= w_{m^{(0)} \tilde{+} 1} = w_{m \tilde{+} 1} \\ w_k \star_o w_m &= (-1)^{1+m_{k-1}} i w_{m^{(k-1)}} \\ (iw_k) \star_o w_m &= (-1)^{m_{k-1}} w_{m^{(k)} \tilde{+} 2^k} = (-1)^{m_{k-1}} w_{m^{(k+1)}} \end{aligned}$$

The representation $\pi_{(\mu_X,1,1)}$ being irreducible $/\mathbb{R}$, this algebra has no proper left ideals.

If, instead, $\gamma \in \Gamma_{\mathbb{R}}$ (see (4.5)), then $\star_\gamma$ has exactly two complementary left-ideals, $\mathcal{I}$, $i\mathcal{I}$, while if $\gamma \in \Gamma_{\mathbb{H}}$, there exist an antilinear, norm-preserving operator $Q$ such that

$$Q^2 = -I, \qquad f \star_\gamma Qg = Q(f \star_\gamma g).$$

We end this section relating these algebras and their equivalences to spin representations of orthogonal groups. The key identity is the following weak form of commutativity/associativity. Given $(U, \star_\pi)$ with left-identity $e$, we identify $H$ with $\Im(U)$, so the spin structure on $U$ is identified with left-multiplication by elements of $\Im(U)$. For any $0 \neq h \in \Im(U)$ let

$$r_h : U \to U$$

be the reflection through the 2-plane spanned by $h$ and $e$, i.e.,

$$r_h(e) = e, \qquad r_h(h) = h, \qquad r_h(v) = -v \quad \forall v \perp \{h, e\}.$$

So, $r_h|_{\Im(U)}$ is minus the reflection trough the hyperplane $h^\perp$. Extend $h \mapsto r_h$ to all of $U$ by setting

$$r_e := I.$$

Since for $h' \perp h$, it holds that

$$-\pi(h)\pi(h') = \pi(h')\pi(h)$$

it is easy to see that for any $u, h \in \Im(U)$

$$\pi(r_h u) = \pi(h)\pi(u)\pi(h)^{-1}.$$

Let

$$\mathcal{R}(H) \subset O(H)$$

be the (ordinary) group generated by the reflections $r_h$, i.e., the operators of the form

$$g = r_{h_1} \cdots r_{h_m} \tag{5.15}$$

with $h_i \in H,\ |h_i| = 1$. Since

$$\begin{aligned}\pi(r_h r_k u) &= \pi(h)\pi(r_k u)\pi(h)^{-1} = \pi(h)\pi(k)\pi(u)\pi(k)^{-1}\pi(h)^{-1}\\ &= \pi(hk)\pi(u)\pi(hk)^{-1},\end{aligned}$$

setting

$$M_\pi(r_{h_1} \cdots r_{h_m}) = \pi(h_1) \cdots \pi(h_m)$$

defines a projective representation of $\mathcal{R}(H)$ on $U$:

$$M_\pi : \mathcal{R}(H) \to O(U)$$

such that

$$\pi(g(h)) = M_\pi(g)\pi(h)M_\pi(g)^{-1}. \tag{5.16}$$

Because (5.15) is not unique, $M_\pi(g)$ is only projectively defined. Indeed, it is unique up to sign: $r_{-h} = r_h$ but $\pi(-h) = -\pi(h)$. According to common language, $M_\pi$ should be called the *spin representation of the group* $\mathcal{R}(H)$.

Although not a Lie group, $\mathcal{R}(H)$ is, in a sense, the largest subgroup of $O(H)$ for which a spin representation can be defined so that (5.16) holds without further restrictions when $\dim H = \infty$. $O(H)$ is generated by $\mathcal{R}(H)$ in the strong topology, but $M_\pi$ does *not* extend to a projective representation of it.

An interesting fact is that *some* $\pi$'s do induce spin representation of *some* Lie subgroups $K \subset O(H)$. For example, if $\pi = \pi_{(\mu_\Delta,1,1)}$ (Fermi-Fock) and $K$ consists of all $g \in O(H)$ such that $[g, i_H]$ is Hilbert-Schmidt, then $M_\pi$ is well defined on $K$ and is, in fact, the infinite-dimensional spin representation that appears in QFT [3][16]. The problem of describing all the possible spin (and metaplectic) pairs $(M_\pi, K)$ seems well fit to treatment by the GW parametrization.

## 6. Appendix

In this section we give the main lines of the proof of Theorem 2.4 following [8].

**Theorem 2.4.** *The operators $J_1, J_1', J_2, J_2', \ldots$ are mutually anticommuting orthogonal complex structures and, therefore, $\pi = \pi_{(\mu,\nu,\mathcal{C})}$ extends to a unitary representation of $\mathfrak{C}$ on $V$. Conversely, every spinor structure on a separable complex Hilbert space is unitarily equivalent to some $\pi_{(\mu,\nu,\mathcal{C})}$.*

*Proof.* The first implication is verified by a straightforward computation using the functional equations for the $c_k$.

Let now a countable collection of mutually anticommuting unitary complex structures on a separable complex Hilbert space $V$ be given. If infinite (or even) We can pair them arbitrarily so as to list them as $J_1, J_1', J_2, J_2', \dots$. The assumed properties

$$||J_k u|| = ||u|| = ||J_k' u||, \qquad J_k^2 = -I = {J_k'}^2,$$

$$J_k J_l + J_l J_k = J_k J_l' + J_l' J_k = J_k' J_l' + J_l' J_k' = 0,$$

when written in terms of the "creation" and "annihilation" operators

$$a_k = \frac{1}{2}\left(J_k' + iJ_k\right) \qquad\qquad a_k^* = \frac{1}{2}\left(-J_k' + iJ_k\right)$$

become

$$a_k a_l + a_l a_k = 0 = a_k^* a_l^* + a_l^* a_k^*, \qquad a_k a_l^* + a_l^* a_k = \delta_{kl}.$$

Therefore, the products

$$N_k = a_k^* a_k, \qquad N_k' = a_k a_k^*$$

are mutually commuting bounded self-adjoint operators, that are projectors:

$$N_k^2 = N_k, \qquad {N_k'}^2 = N_k.$$

According to the Spectral Theorem for self-adjoint operators, there exist a $\sigma$-algebra of sets $\mathcal{B}$ and a measure $\mu$ on $\mathcal{B}$ such that

$$V = \int_{\cup\mathcal{B}}^{\oplus} V_b \; d\mu(b), \tag{6.1}$$

where each operator in the set $\mathcal{N} = \{N_k, N_k'\}$, acts as multiplication by an essentially bounded function, $f \mapsto \phi f$. Indeed, any selfadjoint operator $P$ that commutes with all elements in $\mathcal{N}$ is of the same form.

If the operator $P$ is a projection, the corresponding function must satisfy $\phi^2 = \phi$ and, therefore, be the characteristic function of some set $Y_P \in \mathcal{B}$:

$$P(f) = \chi_{Y_P} f, \qquad \forall f \in \int_{\cup\mathcal{B}}^{\oplus} V_b \; d\mu(b).$$

In this way, each of the operators $N_k, N_k'$, corresponds to a set $X_k, X_{k'} \in \mathcal{B}$, so that, in the decomposition (6.1),

$$N_k f = \chi_{X_k} f, \qquad N_k' f = \chi_{X_k'} f.$$

But because our $\mathcal{N}$ comes from a Clifford representation, one has the identities

$$N_k + N_k^* = I, \qquad N_k N_k^* = 0,$$

So, $X_k^c = X_k'$.

The direct integral representation (6.1) is not unique. Moreover one can consider as the underlying space the set $X$ whose points are the subset in $\mathcal{B}$ of the form

$$x = \bigcap_{k=1}^{\infty} Z_k$$

where $Z_k = X_k$ or $X_k'$ and $\mathcal{B}$ is replaced by the Borel algebra generated by the sets $\{X_k, X_k'\}$. We assign the number 1 to each set $X_k$ and 0 to each $X_k'$. So that to each point $x \in X$ correspond an infinity binary sequence. In this way, we identify $X$ with $\mathbb{Z}_2^\infty$.

We may now write

$$V = \int_X^\oplus V_x \; d\mu(x)$$

with the operators $N$ acting by

$$N_k f(x) = \chi_k(x) f(x) = x_k f(x), \qquad N_k' f(x) = \chi_k'(x) f(x) = (1 - x_k) f(x)$$

if we view $x_k$ in $\mathbb{Z}$.

The fact that $x \mapsto \nu(x) = \dim V_x$ is measurable, is part of the spectral theorem. The quasi-invariance of $\mu$ follows from the identity

$$J_k L_\phi = -L_{{}^k\phi} J_k, \tag{6.2}$$

where $L_\phi$ is the operator of multiplication by the $\mathbb{C}$-valued, bounded measurable function $\phi$ and ${}^k\phi(x) = \phi(x+\delta^k)$. When $\phi$ is a characteristic function of a set $X_k$ or $X_k'$, (6.2) is a formal consequence of the relations between the $a_k$'s and the $J_k, J_k'$. Hence the formula holds for any measurable characteristic function. How to go from this to the quasi-invariance of $\mu$ and the $\Delta$-invariance of $\nu$ is explained in [8] and the main idea was used in the proof of the invariance statements of Theorem (3.2), so we will skip that here.

As to the operators $c_k(x) : V_x \to V_{x+\delta^k} = V_x$, they are defined explicitly by

$$c_k(x) f(x) = i(-1)^{x_1 + \ldots + x_{n-1}} J_k \tau_k f(x), \qquad \tau_k f(x) = \sqrt{\frac{d\mu(x+\delta^k)}{d\mu(x)}} \; f(x+\delta^k).$$

That they satisfy the invariance property is, again, a formal consequence of (6.2) and of the commutation relations satisfied by the $J_k, J_k'$. ∎

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