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Irreducible Linear Group-Subgroup Pairs with the Same Invariants

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Abstract. We consider the problem of finding all linear algebraic group-subgroup pairs such that the rational invariants of the group and of the subgroup coincide. In this paper the solution will be given for the case where both the group and the subgroup are connected complex irreducible linear groups.

1. Introduction

Let G be an algebraic group acting on an algebraic variety X. By *polynomial invariant* of G we mean a polynomial function on X that is constant on G-orbits. Similarly, a *rational invariant* of G is a rational function on X that is constant on those G-orbits where it is defined.

Suppose H is an algebraic subgroup of G. We refer to a group-subgroup pair (H, G) as simply *a pair*. Any invariant of G is an invariant of H, where the converse normally is not true if H is a proper subgroup.

Definition 1.1. We call a pair (H, G) acting on X exceptional, if the fields of rational invariants of H and G coincide: $k(X)^H = k(X)^G$. If $H = G$ we say that the pair (H, G) is trivial.

A general problem is to classify exceptional pairs. We consider the following specific situation.

Let $X = V$ be a finite dimensional vector space over $\mathbb{C}$, and suppose $G \subset GL(V)$ is a linear algebraic group. In this paper, we classify exceptional pairs of connected irreducible linear groups.

From now on, we talk only about linear groups (unless specified otherwise), so the letters G, $H, \ldots,$ now stand for linear groups, and the description "acting on V" is often omitted. For a detailed explanation of the notation, see section 2.

One can compare a classification of exceptional pairs to the main theorem of the Galois theory establishing a bijection between subgroups of the Galois group

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of a field extension L/K and subfields of L containing K. In other words, a finite group is uniquely determined by its invariants (even in the most general situation of any action on any algebraic variety). By classifying exceptional pairs we establish that a group is almost always uniquely determined by its invariants for other class of groups.

To the best of our knowledge, despite the naturality and simplicity of this problem, it has not been considered before. However, there are two classical results that play a major role in the classification of exceptional pairs.

In his diploma thesis in 1959, E.B.Vinberg classified simple irreducible complex linear groups $G \subset GL(V)$ acting with an orbit open in V, so-called *locally transitive* groups (alternatively, V is called a *prehomogeneous vector space*). A locally transitive group cannot have non-trivial polynomial or rational invariants. Thus, any locally transitive group with a locally transitive subgroup make an exceptional pair.

Almost 20 years later, Sato and Kimura [13], and, in a simpler way, Shpiz [14], completed the classification of irreducible locally transitive groups.

D.Montgomery and H.Samelson [7] in 1943, and A.Borel [1] in 1950, described real groups that act transitively on spheres. They also provided the list of inclusions between these groups. Any pair of a group and a subgroup transitive on a sphere is exceptional.

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2. Notation

All groups in this paper are connected complex reductive linear algebraic groups, unless mentioned otherwise.

For brevity, we say that a pair (H, G) is connected, irreducible, etc., if both H and G are connected, irreducible (as linear groups), etc.

$\mathbb{C}(V)^G$ and $\mathbb{C}[V]^G$ denote the field of *rational invariants*, and the algebra of *polynomial invariants* of G on V, respectively.

A *pair* (H, G) is a pair of a group G and a subgroup H.

$O(G)$ is an *orbit in general position* of G.

G_* is a *stationary subgroup in general position* of G, see section 3.

$Z(G)$ is the *center*, and G' is the *commutator subgroup* of G.

$V/G :=\operatorname{Spec}\mathbb{C}[V]^G$ denotes *the categorical quotient*.

$\tilde{G} \bowtie G$ means $\tilde{G}$ is a *castling transform* of G; $\breve{G}$ stands for an *immediate castling transform* of G; $(\tilde{H}, \tilde{G}) \bowtie (H, G)$ means $(\tilde{H}, \tilde{G})$ is a simultaneous castling transform of (H, G), see Definition 4.1.

$\nu(G) :=$ tr.deg.$\mathbb{C}(V)^G$.

For an exceptional pair (H, G), $\nu(H, G) :=$ tr.deg.$\mathbb{C}(V)^H =$ tr.deg.$\mathbb{C}(V)^G$.

Suppose $G_i \subset GL(V_i)$, $i = 1, 2$. Then $G_1 \otimes G_2$ denotes the representation of the group $G_1 \times G_2$ in the vector space $V_1 \otimes V_2$.

SL_n (SO_n, Sp_{2n}, resp.) as a linear group denotes the natural representation of SL_n (SO_n, Sp_{2n}, resp.) as an algebraic group.

For $G \subset GL(V)$, $\Lambda^k G$ ($S^k G$) denotes the representation of G in $\Lambda^k V$ ($S^k V$).

$Spin_k$ denotes the spin (or half-spin) representation of SO_k.

For A a graded algebra without zero divisors, we denote by PQA the subfield in QA spanned by the elements $\frac{f}{g}$, where f and g are homogeneous of the same degree, see section 3.2.

$I(G)$ denotes *the index set* of a semisimple group G, see section 6.

$\mathfrak{g}$, $\mathfrak{h}, \ldots$ denote the respective tangent Lie algebras of the groups G, $H, \ldots$.

A semisimple group G (or the corresponding algebra $\mathfrak{g}$) is *reduced* if for any $\tilde{G} \bowtie G$ we have $I(G) \leq I(\tilde{G})$; similarly, a pair (H, G) is *reduced* if for any $(\tilde{H}, \tilde{G}) \bowtie (H, G)$ we have $I(G) \leq I(\tilde{G})$, see Definition 6.1. Here, for two (unordered) number sets, $A = \{a_0, \ldots, a_s\}$, $B = \{b_0, \ldots, b_t\}$, we define $A < B$, if $s \leq t$, $a_i \leq b_i$ for all i (upon a proper renumeration) and either $s < t$, or $a_i < b_i$ for some i.

A triple of groups (G, H, S), where $H, S \subset G$, is called a *factorization* if $G = HS$, see section 7.

For a reductive algebra $\mathfrak{a}$, denote by $\mathfrak{a}'$ the semisimple part of $\mathfrak{a}$.

A semisimple algebra $\mathfrak{a}$ is called *strongly semisimple* if it contains no ideals of type A_1. For a reductive algebra $\mathfrak{a}$, we denote by $\mathfrak{a}^s$ the maximal strongly semisimple subalgebra of $\mathfrak{a}$. We say $\mathfrak{a}^s$ is *the strongly semisimple part* of $\mathfrak{a}$.

3. Outlines

1. Suppose (H, G) is a semisimple pair.

Geometrically, a semisimple pair (H, G) is exceptional if and only if the closures of orbits in general positions for H and G coincide. Indeed, Rosenlicht's theorem [12] states that rational invariants separate (both G- and H-) orbits in general position. It follows that an H-orbit in general position is open in a G-orbit.

Furthermore, for semisimple linear groups, the field of rational invariants is the quotient field of the algebra of polynomial invariants [15]. Therefore, (H, G) is exceptional if and only if the algebras of polynomial invariants of H and G coincide: $\mathbb{C}[V]^H = \mathbb{C}[V]^G$.

Let $G_x \subset G$ denote the stationary subgroup of a point $x \in V$.

Definition 3.1. Suppose there is a subgroup $G_* \subset G$ such that G_x is a conjugate of G_* for x in a Zariski open subset in V. Then we call G_* a stationary subgroup in general position for the action of G on V.

For a reductive (in particular, semisimple) linear group, a stationary subgroup in general position always exists [11], [5].

A large portion of our classification relies on the fact that for any nontrivial exceptional pair (H, G), $\dim G_* > 0$ (Lemma 5.6(b)). This is a strong restriction on the group G. In [3],[4], Elashvili classified irreducible linear groups G with $\dim G_* > 0$.

We consider separately the cases G_* is reductive (section 8.) or nonreductive (section 9.). We will see that if G_* is reductive, and the pair (H, G) is exceptional, then H acts transitively on the the homogeneous space G/G_*, i.e., $G = HG_*$. This leads to a further reduction. Namely, Onishchik [8],[9] classified triples of reductive groups (G, H, S), where H, S are subgroups of G such that $G = HS$. Combining [3],[4], [8] and [9], we prove that exceptional pairs with reductive G_* are only those listed in Table A, and their castling transforms (see Definition 4.1).

Suppose G_* is nonreductive. This case includes all irreducible linear groups with trivial algebra of invariants, i.e., locally transitive groups [3],[4].

Based on the classification of locally transitive groups ([13], [14]), we describe locally transitive pairs (H, G) (Table L.) After that, we only need to look for exceptional pairs with at least one nontrivial invariant. In reality, when G_* is nonreductive, there is only one type of G that has nontrivial algebra of invariants, and it requires special attention.

2. Suppose G has a nontrivial center $Z(G)$.

Let A be a graded algebra without zero divisors. Denote by PQA the subfield in QA spanned by the elements $\frac{f}{g}$, where f and g are homogeneous of the same degree.

Geometrically, for a finitely generated algebra A, PQA is the field of functions on the projectivization of $SpecA$. In particular, tr.deg.$PQA =$tr.deg.$QA - 1$ if tr.deg.$QA \geq 1$, and tr.deg.$PQA = 0$ if tr.deg.$QA = 0$. Therefore, given two algebras A, B, such that tr.deg.$QA >$tr.deg.QB, and tr.deg.$PQA =$tr.deg.PQB, we conclude tr.deg.$QA = 1$, and tr.deg.$QB = 0$.

Denote the commutator subgroup of G by G'. Since G is irreducible, $Z(G) \cong \mathbb{C}^*$. Hence, by Lemmas 5.3 and 5.11, $\mathbb{C}(V)^G = PQ\mathbb{C}[V]^{G'}$.

Suppose H is semisimple. Then $H \subset G'$, and $\mathbb{C}(V)^H = Q\mathbb{C}[V]^H$. If tr.deg.$\mathbb{C}[V]^{G'} \geq 1$, then (H, G) is exceptional follows tr.deg.$\mathbb{C}[V]^{G'} - 1 =$ tr.deg.$\mathbb{C}[V]^H$, which is impossible. If tr.deg.$\mathbb{C}[V]^{G'} = 0$, then we obtain tr.deg.$\mathbb{C}[V]^H = 0$, i.e., (H, G') is a locally transitive semisimple pair.

Now suppose H is not semisimple, $Z(H) = Z(G) = \mathbb{C}^*$. If (H', G') is exceptional, then (H, G) is also exceptional. Suppose (H', G') is not exceptional, i.e., tr.deg.$\mathbb{C}[V]^{H'} >$tr.deg.$\mathbb{C}[V]^{G'}$. Then, as we saw above, (H, G) is exceptional if and only if tr.deg.$\mathbb{C}[V]^{H'} = 1$, and tr.deg.$\mathbb{C}[V]^{G'} = 0$. Using [13], [14], we describe such pairs (H', G') (Table R.)

4. The main result

In order to formulate the result, we first need to introduce an equivalence relation on the set of irreducible semisimple groups, as well as pairs, called *castling transform.*

Suppose $G_0 \subseteq SL(V)$. Consider the group $G = G_0 \otimes SL(W)$, $\dim W \leq \dim V$, and the group $\check{G} = G_0 \otimes SL(\check{W})$ that acts on $V^* \otimes \check{W}$, where $\dim \check{W} = \dim V - \dim W$. Then $\check{G}_* \cong G_*$ [4].

In particular, if $\dim V = \dim W + 1$, then $(G_0)_* \cong G_*$.

Definition 4.1. We say that the group $\check{G}$ is an immediate castling transform of the group G, and vice versa. We say that a group $\tilde{G}$ is a castling transform of G, and write $\tilde{G} \bowtie G$, if $\tilde{G}$ is a result of a sequence of immediate castling transforms of G. We write $(\tilde{H}, \tilde{G}) \bowtie (H, G)$ if the pair $(\tilde{H}, \tilde{G})$ is a simultaneous castling transform of the pair (H, G).

When talking about pairs, we sometimes omit the word "simultaneous".

For an exceptional (H, G), denote $\nu(H, G) = $tr.deg.$\mathbb{C}(V)^H = $tr.deg.$\mathbb{C}(V)^G$.

We will see in Lemma 5.7, that if (H, G) is an exceptional irreducible pair, and $(\tilde{H}, \tilde{G}) \bowtie (H, G)$, then the pair $(\tilde{H}, \tilde{G})$ is also exceptional, and $\nu(\tilde{H}, \tilde{G}) = \nu(H, G)$.

Note that if (H, G), and (G, K) are exceptional pairs, then (H, K) is exceptional. Hence, we will assume that H is a maximal subgroup in G, unless mentioned otherwise.

The following theorem is the main result of this paper. The notation is explained in detail in section 2.

Theorem 4.2. *Let (H, G) be a connected irreducible pair, where H is maximal in G. Denote $H_0 = SL_s \otimes SL_t \otimes X_k$, $G_0 = SL_{st} \otimes X_k$, where $X_k \subseteq SL_k$ is irreducible, and $st > k$.*

(i) *Suppose (H, G) is exceptional semisimple. If $\nu(H, G) = 0$ then (H, G) is isomorphic, up to castling transform, to one of the pairs from Table L, or to (H_0, G_0). If $\nu(H, G) > 0$ then (H, G) is isomorphic, up to castling transform, to one of the pairs from Table A.*

(ii) *Conversely, all pairs from Tables L and A, as well as their castling transforms, are exceptional.*

(iii) *Suppose (H, G) is exceptional, H', G' are the commutator subgroups of H, G, and $G \neq G'$. If $H = H'$ then (H, G') is isomorphic, up to castling transform, to one of the pairs from Table L, or to (H_0, G_0). If $H \neq H'$ then either (H', G') is an exceptional pair, or (H', G') is isomorphic, up to castling transform, to one of the pairs from Table R, or to (H_0, G_0).*

(iv) *Conversely, suppose $G \neq G'$. If $H = H'$ and (H, G') is, up to castling transform, from Table L, then (H, G) is exceptional. If $H \neq H'$ and (H', G') is, up to castling transform, from Table R, then (H, G) is exceptional.*

H is maximal in G in all pairs in all tables.

Remark 4.3. 1. The group H_0 may have zero, one, or more invariants, depending on s,t, and k values, and also on X_k.

2. Note that all exceptional pairs except for one have one or less invariant.

3. In Table A, the third column shows $\nu(H, G)$, and the last column shows the generators degrees for the algebra of invariants (which is always polynomial).

Table L.

	H	G
1	Sp_{2n}	SL_{2n}
2	$\Lambda^2 SL_{2n+1}$	$SL_{n(2n+1)}$
3	$Spin_{10}$	SL_{16}
4	$SL_s \otimes SL_t,\ s > t$	SL_{st}
5	$SL_n \otimes Y_k,\ k < n$	$SL_n \otimes X_k$, Y_k maximal in $X_k \subseteq SL_k$
6	$Sp_{2n} \otimes SL_{2k+1},\ 2k < n$	$SL_{2n} \otimes SL_{2k+1}$
7	$\Lambda^2 SL_{2n+1} \otimes SL_2$	$SL_{n(2n+1)} \otimes SL_2$

Table A.

	H	G	ν	deg
1	G_2	SO_7	1	2
2	$Spin_7$	SO_8	1	2
3	$Spin_9$	SO_{16}	1	2
4	$Spin_{11}$	$Spin_{12}$	1	4
5	$SL_n \otimes Y_n$	$SL_n \otimes X_n$, Y_n maximal in $X_n \subseteq SL_n$	1	n
6	$Sp_{2n} \otimes SL_2$	SO_{4n}	1	2
7	$G_2 \otimes SL_2$	$SO_7 \otimes SL_2$	1	4
8	$Spin_7 \otimes SL_2$	$SO_8 \otimes SL_2$	1	4
9	$Spin_7 \otimes SO_3$	$SO_8 \otimes SO_3$	3	2,4,6
10	$Spin_7 \otimes SL_3$	$SO_8 \otimes SL_3$	1	6

Table R.

	H'	G'
1	$S^2 SL_n$	$SL_{n(n+1)/2}$
2	$\Lambda^2 SL_{2n}$	$SL_{n(2n-1)}$
3	$\Lambda^3 SL_n,\ n = 7, 8$	$SL_{n(n-1)(n-2)/6}$
4	$Spin_{14}$	SL_{64}
5	E_6	SL_{27}
6	$SL_n \otimes SL_n$	SL_{n^2}
7	$\Lambda^3 SL_6$	Sp_{20}
8	$S^3 SL_2$	Sp_4
9	$Spin_{12}$	Sp_{32}
10	E_7	Sp_{56}
11	$SO_n \otimes SL_2$	Sp_{2n}
12	$Sp_{2n} \otimes SO_3$	Sp_{6n}
13	$Spin_9$	$Spin_{10}$
14	$Spin_7 \otimes SL_2$	$Spin_{10}$
15	$\Lambda^2 SL_6 \otimes SL_2$	$SL_{15} \otimes SL_2$
16	$\Lambda^2 SL_5 \otimes SL_n,\ n = 3, 4$	$SL_{10} \otimes SL_n$
17	$S^2 SL_3 \otimes SL_2$	$SL_6 \otimes SL_2$
18	$Spin_{10} \otimes SL_n,\ n = 2, 3$	$SL_{16} \otimes SL_n$
19	$E_6 \otimes SL_2$	$SL_{27} \otimes SL_2$
20	$SO_n \otimes SL_k,\ n > k > 1$	$SL_n \otimes SL_k$
21	$Sp_{2n} \otimes SL_{2k},\ n > k > 1$	$SL_{2n} \otimes SL_{2k}$
22	$Sp_{2n} \otimes SO_3,\ n > 1$	$SL_{2n} \otimes SO_3$
23	$Sp_{2n} \otimes SO_3$	$Sp_{2n} \otimes SL_3$

5. Preliminaries

Lemma 5.1. *Suppose (H, G) is an exceptional pair acting on an algebraic variety X. Suppose G acts on another variety Y, and $\pi : X \longrightarrow Y$ is a surjective G-equivariant homomorphism. Then (H, G) acting on Y is also an exceptional pair.*

Proof. Consider $\pi^* : \mathbb{C}(Y) \longrightarrow \mathbb{C}(X)$, where $(\pi^* f)(v) = f(\pi(v))$. Then π^* is G-equivariant and injective. Hence, $\mathbb{C}(Y)^H \neq \mathbb{C}(Y)^G$ would imply $\mathbb{C}(X)^H \neq \mathbb{C}(X)^G$. ■

Lemma 5.2. *Suppose (H, G) is an exceptional reductive linear pair acting on a vector space V, $U \subset V$ is a G-invariant subspace. Then (H, G) acting on U is also an exceptional pair.*

Proof. Since G is reductive, there exists a G-invariant subspace U' such that $V = U \oplus U'$. Denote by $\pi : V \longrightarrow U$ the projection on U parallel to U'. Then π is G-equivariant, and, by Lemma 5.1, (H, G) acting on U is exceptional. ■

Till the end of this section, assume H and G are semisimple linear groups acting on a vector space V, unless mentioned otherwise.

Denote $V/G :=\mathrm{Spec}\,\mathbb{C}[V]^G$ (*the categorical quotient*).

Lemma 5.3. *Let $N \subset G$ be a normal subgroup. Then $\mathbb{C}[V]^G \cong \mathbb{C}[V/N]^{G/N}$. In particular, if $N \subset H$, and (H, G) is exceptional, then the pair $(H/N, G/N)$ acting on V/N is exceptional.*

Corollary 5.4. *Suppose (H, G) is an exceptional pair, $G = G_1 \times G_2$, $H = H_1 \times G_2$. Take $G' = G_1 \times G_2'$, $H' = H_1 \times G_2'$, so that $G \subset G' \subset GL(V)$. Then (H', G') is also exceptional.*

Proof. The homomorphism $\pi : V/G_2 \longrightarrow V/G_2'$ is surjective and G_1-equivariant. By Lemma 5.3 , (H_1, G_1) acting on V/G_2 is exceptional. Hence, by Lemma 5.1, (H_1, G_1) acting on V/G_2' is also exceptional, and, therefore, (H', G') is exceptional. ■

For a linear group G, denote by $O(G) \subset V$ a G-orbit in general position. Also, denote $\nu(G) = \mathrm{tr.deg.}\,\mathbb{C}(V)^G$. For a semisimple G, $\nu(G) = \mathrm{tr.deg.}\,\mathbb{C}[V]^G$.

Lemma 5.5. *(a)* $\nu(G) = \mathrm{codim}_V O(G)$.

(b) (H, G) is an exceptional pair if and only if $\dim O(H) = \dim O(G)$, *or, equivalently,* $\nu(G) = \nu(H)$.

Proof. (a) By Rosenlicht's theorem [12], there exists a finite set of rational invariants that separates (both G- and H-) orbits in general position. By Lemma 2.1 [15], this finite set generates the field of rational invariants. Since G is semisimple, the field of fractions of the algebra of polynomial $G-$invariants coincides with the field of rational $G-$invariants: $Q\mathbb{C}[V]^G = \mathbb{C}(V)^G$ (Th. 3.3 [15]). Thus (Corollary 2.3, [15]), $\nu(G) = \mathrm{tr.deg.}\,\mathbb{C}[V]^G = \mathrm{tr.deg.}\,\mathbb{C}(V)^G = \mathrm{codim}_V O(G)$.

(b) $\mathbb{C}[V]^H = \mathbb{C}[V]^G$ implies $\overline{O(H)} = \overline{O(G)}$, and, therefore, by (a) $\nu(H) = \mathrm{codim}_V O(H) = \mathrm{codim}_V O(G) = \nu(G)$. ■

Lemma 5.6. *(a)* $\nu(G) = \dim V - \dim G + \dim G_*$;

(b) if (H, G) *is a nontrivial exceptional pair, then* $\dim G_* > 0$.

Proof. For $x \in O(G)$, G_x is a conjugate of G_*. Hence, $\dim G = \dim O(G) + \dim G_*$. Combined with Lemma 5.5, this implies (a). Also, by Lemma 5.5, (b), $\dim O(H) = \dim O(G)$, hence $\dim G_* = \dim G - \dim O(G) > \dim H - \dim O(H) = \dim H_* \geq 0$. ■

Lemma 5.7. *If* (H, G) *is an exceptional irreducible pair, and* $(\tilde{H}, \tilde{G}) \bowtie (H, G)$, *then the pair* $(\tilde{H}, \tilde{G})$ *is also exceptional, and* $\nu(\tilde{H}, \tilde{G}) = \nu(H, G)$.

Proof. We use the notation introduced in Definition 4.1. We may assume $(\tilde{H}, \tilde{G}) = (\check{H}, \check{G})$ is an immediate castling transform of (H, G), i.e., $\tilde{H} = \check{H} = H_0 \otimes SL(\check{W}) \subset \tilde{G} = \check{G} = G_0 \otimes SL(\check{W})$, where $H_0 \subset G_0$. Denote $\dim V = n$, $\dim W = k$, then $\dim \check{W} = n - k$.

It's enough to prove that $\nu(\check{G}) = \nu(G)$. Indeed, since $\nu(H) = \nu(G)$, this would imply $\nu(\check{H}) = \nu(H) = \nu(G) = \nu(\check{G})$, and, therefore, $(\check{H}, \check{G})$ is exceptional by Lemma 5.5.

We have $\check{G}_* \cong G_*$. By Lemma 5.6, $\nu(\check{G}) = \dim(V^* \otimes \check{W}) - \dim \check{G} + \dim \check{G}_* = n(n-k) - \dim SL_{n-k} - \dim G_0 + \dim \check{G}_* = nk - (k^2 - 1) - \dim G_0 + \dim \check{G}_* = \dim(V \otimes W) - \dim G + \dim G_* = \nu(G)$. ■

Lemma 5.8. *([2]). Suppose* $G = G_1 \times \ldots \times G_k$, *where each* G_i *is a simple normal subgroup. If* G *acts irreducibly on* V, *then there exist vector spaces* $V_1, \ldots, V_k$ *such that* G_i *acts irreducibly on* V_i, *and* $V \cong V_1 \otimes \ldots \otimes V_k$ *(as* G*-representations).*

Lemma 5.9. *Suppose* $G = G_1 \times \ldots \times G_n$ *is a semisimple group, and* $H \subset G$ *is a maximal semisimple subgroup. Then, under a proper renumeration, either (A)* $H = H_1 \times G_2 \times \ldots \times G_n$, *where* H_1 *is a maximal subgroup in* G_1; *or (B)* $G_1 \cong G_2$, *and* $H = H_1 \times G_3 \times \ldots \times G_n$, *where* $H_1 \cong G_1$ *is embedded in* $G_1 \times G_2$ *diagonally. In particular, if* G *is an irreducible linear group, and* H *is an irreducible subgroup, then (A) holds.*

Lemma 5.10. *([2]). (1) Any maximal irreducible linear subgroup of* SL_n *is either simple, or a conjugate of* $SL_s \otimes SL_t$, $st = n$;

(2) Any maximal irreducible linear subgroup of Sp_{2n} *is either simple, or a conjugate of* $SO_s \otimes Sp_{2t}$, $st = n$;

(3) Any maximal irreducible linear subgroup of SO_n *is either simple, or a conjugate of* $SO_s \otimes SO_t$, *or* $Sp_s \otimes Sp_t$, $st = n$.

Let A be a graded algebra without zero divisors. As in section 3.2, we denote by PQA the subfield in QA spanned by the elements $\frac{f}{g}$, where f and g are homogeneous of the same degree.

Lemma 5.11. *Suppose* $\mathbb{C}^*$ *acts on* A *so that* $\lambda(f) = \lambda^{deg(f)} f$. *Then* $(QA)^{\mathbb{C}^*} = PQA$.

6. Castling transform growth

Here we are going to prove a number of technical facts regarding the dimension growth for the representation space of a semisimple linear group under castling transform, to be used mostly in section 9.

Suppose $G = G_{k_0} \otimes SL_{k_1} \otimes \ldots \otimes SL_{k_s},\ G_{k_0} \subset SL_{k_0},\ s \geq 0,\ k_i \geq 2$, where G_{k_0} does not contain a normal subgroup isomorphic (as a linear group) to SL_k for any k. If $G_{k_0} = \{id\}$, let $k_0 = 1$. Then we say that the set of numbers $I(G) = \{k_0, k_1, \ldots, k_s\}$ is *the index set of* G.

Denote by $G_0, \ldots, G_s$ the immediate castling transforms of G. Namely, $I(G_0) = \{k_0, \overline{k}_0, \ldots, k_s\},\ \ I(G_i) = \{k_0, \ldots, k_{i-1}, \overline{k}_i, k_{i+1}, \ldots, k_s\}$, where $\overline{k}_0 = (\prod_{0 \leq j \leq s} k_j) - 1,\ \overline{k}_i = (\prod_{j \neq i} k_j) - k_i,\ i = 1 \ldots s$. If $\overline{k}_i$ is negative, we imply that the castling transform G_i is not defined. Note that $I(G_i) \setminus I(G) = \{\overline{k}_i\}$. We say that $\overline{k}_i$ is *the new element* of $I(G_i),\ i = 0 \ldots s$.

Suppose we have two (unordered) number sets, $A = \{a_0, \ldots, a_s\},\ B = \{b_0, \ldots, b_t\}$. Define $A < B$, if $s \leq t,\ a_i \leq b_i$ for all i (upon a proper renumeration) and either $s < t$, or $a_i < b_i$ for some i.

Definition 6.1. We say that a group G (or the corresponding algebra $\mathfrak{g}$) is reduced if for any $\tilde{G}$ such that $\tilde{G} \bowtie G$ we have $I(G) \leq I(\tilde{G})$. Similarly, we say that a pair (H, G) is reduced if for any $(\tilde{H}, \tilde{G})$ such that $(\tilde{H}, \tilde{G}) \bowtie (H, G)$ we have $I(G) \leq I(\tilde{G})$.

Clearly, for any G there is a sequence of immediate castling transforms $G \bowtie \ldots$, such that at each step the index set decreases. Hence, there exists a reduced $\tilde{G}$ such that $\tilde{G} \bowtie G$.

Denote $m(G) = \max_{k \in I(G)} k,\ i(G) = \{i \in I(G) |\quad k_i = m(G)\}$. Denote $A = \{k_0, k_1\},\ A_1 = \{k_0, k_0 - k_1\},\ B = \{k_0\},\ B_0 = \{k_0, k_0 - 1\}$.

Lemma 6.2. *(a) For $i \notin i(G)$, the new element $\overline{k}_i$ is the maximal one in $I(G_i) : \overline{k}_i = m(G_i)$, and $I(G) < I(G_i)$, except for the case $I(G) = A,\ I(G_1) = A_1$, $k_0 \leq 2k_1$.*

(b) If $i(G) \ni 0$, then (a) also holds for $i = 0$, except for the case $I(G) = B$, $I(G_0) = B_0$.

(c) If $I(G) < I(G_i)$, then $\overline{k}_i = m(G_i)$ with the same two exceptions.

Proof. Suppose $s \geq 2$. Take any $i_0 \in i(G)$. For any $i \notin i(G),\ i \neq 0$, we have $|I(G)| = |I(G_i)|$, and $\prod_{j \neq i} k_j \geq 2k_{i_0} \geq k_i + k_{i_0}$. Hence, $\overline{k}_i = (\prod_{j \neq i} k_j) - k_i \geq k_{i_0} \geq k_j$ for all $j \leq s$, and, therefore, $m(G_i) = \overline{k}_i \geq m(G)$.

We have $|I(G_0)| = |I(G)| + 1$ for all s. If $s \neq 0$, then $\overline{k}_0 = (\prod_{0 \leq j \leq s} k_j) - 1 > k_{i_0}$ implying $m(G_0) = \overline{k}_0$.

For $i \notin i(G)$, or $i = 0$, (c) is stated in (a) and (b). Consider $i \in i(G) \neq 0$. Then $|I(G_i)| = |I(G)|$, and $\{\overline{k}_i\} = I(G_i) \setminus I(G)$. We have $\overline{k}_i > k_i \geq k_j$ for all $j \in I(G)$. Hence, $\overline{k}_i = m(G_i)$ ■

Suppose $G^0 \bowtie \ldots \bowtie G^l$ is a sequence of immediate castling transforms, where G^0 is reduced. Assume that the sequence doesn't contain a "loop" $\ldots \bowtie S \otimes SL(U) \bowtie S \otimes SL(U') \bowtie S \otimes SL(U) \bowtie \ldots$. In terms of index sets, this means that at any step of the sequence, the new element of the index set stays unchanged under the next immediate castling transform.

Corollary 6.3. $I(G^0) < \ldots < I(G^l)$.

Proof. Induction on l. Since G^0 is reduced, $I(G^0) < I(G^1)$. Assume $l \geq 2$, and $\{I(G^{l-2}), I(G^{l-1})\} \neq \{A, A_1\}$, or $\{B, B_0\}$ (exceptional cases from the Lemma). Suppose, $G^{l-1} = G_i^{l-2}$. By assumption, $I(G^{l-2}) < I(G^{l-1})$, and, therefore, by Lemma (c), the maximal element of G^{l-1} is the new one: $i \in i(G^{l-1})$. Since there are no loops, $G^l = G_j^{l-1}$ for some $j \neq i$. Hence, by Lemma (a), $I(G^l) > I(G^{l-1})$. If $\{I(G^{l-2}), I(G^{l-1})\} = \{A, A_1\}$, or $\{B, B_0\}$, then $G^l = G_0^{l-1}$, and $I(G^l) > I(G^{l-1})$. ∎

Corollary 6.4. *Suppose* $0 \in i(G)$, $s \neq 1$, *and* $\tilde{G} \bowtie G$, $\tilde{G} \neq G$. *Then* $\min(I(\tilde{G}) \setminus I(G)) \geq \min_{i \leq s} m(G_i) = \overline{k}_{j_0}$, *where* $k_{j_0} = \max_{1 \leq j \leq s} k_j$ *if* $s \geq 2$, *or* $j_0 = 0$ *if* $s = 0$. *If* $s = 1$, *and* G *is reduced, then* $\min(I(\tilde{G}) \setminus I(G)) \geq \overline{k}_1 = k_0 - k_1$.

Corollary 6.5. *Let* (H, G) *be a reduced pair. Assume* G *is not reduced, and* H *is maximal in* G. *Then (i)* $(H, G) \cong (G_n \otimes H_m, G_n \otimes SL_m)$, *where* $H_m \subset SL_m$ *is a maximal subgroup,* $G_n \subset SL_n$, $m < n$. *(ii) In particular,* H *is reduced.*

Proof. (i) follows from Lemma 5.9, except for the fact $m < n$. Take $\tilde{G} \bowtie G$, $\tilde{G}$ reduced. Consider the shortest sequence of immediate castling transforms $\tilde{G} \bowtie \ldots \bowtie G$. Then Corollary 6.3 implies that m is the new element in $I(G)$, and, therefore, $m \leq n - 1 < n$.

(ii) Take any $\tilde{H} \bowtie H$. Suppose there exists $\tilde{G} \bowtie G$ such that $(\tilde{H}, \tilde{G}) \bowtie (H, G)$. Then we have $I(G) \leq I(\tilde{G})$, implying $I(H) \leq I(\tilde{H})$. If there exists no $\tilde{G} \bowtie G$ such that $(\tilde{H}, \tilde{G}) \bowtie (H, G)$, then $H_m = SL_s \otimes SL_t$ (see Lemma 5.10) and $\tilde{H} = G_n \otimes SL_{nt-s} \otimes SL_t$. Since $n > m$, we have $s < m < n \Longrightarrow nt - s > nt - n > n > s \Longrightarrow I(\tilde{H}) > I(H)$. ∎

7. Factorizations

The following facts are implications of [8],[9] that will be used in section 8.

Throughout this section, let G be a connected reductive algebraic group, $H, S \subset G$ be algebraic subgroups, and $\mathfrak{g}, \mathfrak{h}, \mathfrak{s}$ be the respective tangent Lie algebras. A triple (G, H, S) is called a *factorization* if $G = HS$, i.e., $\forall g \in G$ there exist $h \in H$ and $s \in S$ such that $g = hs$. We say that a factorization (G, H, S) is trivial if $G = H$ or $G = S$.

Correspondingly, a triple of Lie algebras $(\mathfrak{g}, \mathfrak{h}, \mathfrak{s})$ is called a factorization if $\mathfrak{g} = \mathfrak{h} + \mathfrak{s}$.

Lemma 7.1. *If* (G, H, S) *is a factorization, then* $(\mathfrak{g}, \mathfrak{h}, \mathfrak{s})$ *is also a factorization. If* H *and* S *are reductive, then the converse is also true.*

Proof. Follows from Lemma 1.3 [8], Th.3.1 [9]. ∎

Let $\mathfrak{g}$ and $\mathfrak{h} \subset \mathfrak{g}$ be semisimple Lie algebras, $\mathfrak{s} \subset \mathfrak{g}$ be a reductive Lie algebra.

For a reductive algebra $\mathfrak{a}$, denote by $\mathfrak{a}'$ the semisimple part of $\mathfrak{a}$.

Table O.

$\mathfrak{g}$	$\mathfrak{h}$	$\mathfrak{s}$
$\mathfrak{sl}_{2n},\ n>1$	$\mathfrak{sp}_{2n}$	$\mathfrak{sl}_{2n-1}$
$\mathfrak{sl}_{2n},\ n>1$	$\mathfrak{sl}_{2n-1}$	$\mathfrak{sp}_{2n}$
$\mathfrak{so}_{2n},\ n>1$	$\mathfrak{sl}_n \oplus \mathfrak{sl}_n^*$	$\mathfrak{so}_{2n-1}$
$\mathfrak{so}_{2n},\ n>1$	$\mathfrak{so}_{2n-1}$	$\mathfrak{sl}_n \oplus \mathfrak{sl}_n^*$
$\mathfrak{so}_{4n},\ n>1$	$\mathfrak{sp}_{2n} \oplus \mathfrak{sl}_2$	$\mathfrak{so}_{4n-1}$
$\mathfrak{so}_{4n},\ n>1$	$\mathfrak{so}_{4n-1}$	$\mathfrak{sp}_{2n} \oplus \mathfrak{sl}_2$
$\mathfrak{so}_7$	$\mathfrak{g}_2$	$\mathfrak{so}_6$
$\mathfrak{so}_7$	$\mathfrak{so}_6$	$\mathfrak{g}_2$
$\mathfrak{so}_7$	$\mathfrak{g}_2$	$\mathfrak{so}_5$
$\mathfrak{so}_8$	$\mathfrak{spin}_7$	$\mathfrak{so}_5$
$\mathfrak{so}_8$	$\mathfrak{spin}_7$	$\mathfrak{so}_6$
$\mathfrak{so}_8$	$\mathfrak{spin}_7$	$\mathfrak{so}_7$
$\mathfrak{so}_8$	$\mathfrak{so}_7$	$\mathfrak{spin}_7$
$\mathfrak{so}_8$	$\mathfrak{so}_5 \oplus \mathfrak{so}_3$	$\mathfrak{spin}_7$
$\mathfrak{so}_{16}$	$\mathfrak{spin}_9$	$\mathfrak{so}_{15}$
$\mathfrak{so}_{16}$	$\mathfrak{so}_{15}$	$\mathfrak{spin}_9$

Lemma 7.2. *The triple $(\mathfrak{g}, \mathfrak{h}, \mathfrak{s})$ is a factorization $\Longleftrightarrow$ $(\mathfrak{g}, \mathfrak{h}, \mathfrak{s}')$ is a factorization. In particular, if $\mathfrak{s}$ is commutative, then the factorization $(\mathfrak{g}, \mathfrak{h}, \mathfrak{s})$ is trivial.*

Proof. Follows from Th.1.1[8], Th.3.2 [9]. ■

Definition 7.3. ([8], [9]). A semisimple algebra $\mathfrak{a}$ is called strongly semisimple if it contains no ideals of type A_1. For a reductive algebra $\mathfrak{a}$, we write $\mathfrak{a} = \mathfrak{a}^s \oplus \mathfrak{a}^r$, where $\mathfrak{a}^s$ is a sum of all simple ideals not of type A_1, and $\mathfrak{a}^r$ is a sum of the center, and of all simple ideals of type A_1. We call $\mathfrak{a}^s$ the strongly semisimple part of $\mathfrak{a}$.

Suppose $\mathfrak{s}$ is semisimple.

Lemma 7.4. *Let $(\mathfrak{h}^r)_{\mathfrak{g}^r}, (\mathfrak{s}^r)_{\mathfrak{g}^r}$ be the projections of $\mathfrak{h}^r, \mathfrak{s}^r$ on $\mathfrak{g}^r$. Then $(\mathfrak{g}, \mathfrak{h}, \mathfrak{s})$ is a factorization if and only if both $(\mathfrak{g}^s, \mathfrak{h}^s, \mathfrak{s}^s)$ and $(\mathfrak{g}^r, (\mathfrak{h}^r)_{\mathfrak{g}^r}, (\mathfrak{s}^r)_{\mathfrak{g}^r})$ are factorizations. In particular, if $\mathfrak{s}^s = 0$ and either (a) $\mathfrak{g}^r = 0$, or (b) $\mathfrak{s}^r_{\mathfrak{g}^r} = 0$, then the factorization $(\mathfrak{g}, \mathfrak{h}, \mathfrak{s})$ is trivial.*

Proof. Follows from Th.5.1 [8], Th.3.3 [9]. ■

Suppose $\mathfrak{s} \subset \mathfrak{g}$ are strongly semisimple algebras, and let $\mathfrak{g} = \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_k$, $\mathfrak{s} = \mathfrak{s}_1 \oplus \ldots \oplus \mathfrak{s}_l$, where $\mathfrak{g}_1 \ldots \mathfrak{g}_k, \mathfrak{s}_1 \ldots \mathfrak{s}_l$ are simple ideals. Assume $\mathfrak{h} \subset \mathfrak{g}$ is a maximal semisimple subalgebra of the form $\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{g}_2 \oplus \ldots \oplus \mathfrak{g}_k$. Denote by $\mathfrak{s}_{ij}$ the projection of $\mathfrak{s}_i$ on $\mathfrak{g}_j$.

Lemma 7.5. *Suppose $(\mathfrak{g}, \mathfrak{h}, \mathfrak{s})$ is a nontrivial factorization. Then, for some i, either $\mathfrak{s}_{i1} = \mathfrak{g}_1$, or $(\mathfrak{g}_1, \mathfrak{h}_1, \mathfrak{s}_{i1})$ is a factorization from Table O.*

Proof. Follows from Th.4.3 [9]. ■

8. G semisimple, G_* reductive

In this section we assume that (H, G) is a nontrivial exceptional connected irreducible semisimple pair such that G_* is reductive. The result of this section is a list of such pairs (H, G) up to castling transform (Table A).

If G_* is reductive, then the action of H on $O(G) \cong G/G_*$ is stable [6]. On the other hand, this action is locally transitive, since, by Lemma 5.5, $\dim O(H) = \dim O(G)$. Hence, H acts transitively on G/G_*.

It's easy to see (Prop.5.1 [8]) that H acts transitively on a G-homogeneous space G/G_* if and only if the triple (G, H, G_*) is a factorization if and only if (see Lemma 7.1) $(\mathfrak{g}, \mathfrak{h}, \mathfrak{g}_*)$ is a factorization, where $\mathfrak{g}_*$ is the tangent algebra of G_*.

By Lemma 5.6(b), $\dim \mathfrak{g}_* > 0$. The works of Elashvili [3], [4] provide a list of irreducible semisimple algebras $\mathfrak{g}$ with $\dim \mathfrak{g}_* > 0$ (we refer to it as Elashvili list).

In the following, we use the notation of sections 6., 7. See also section 2.

Assume $(\mathfrak{g}, \mathfrak{h}, \mathfrak{g}_*)$ is a nontrivial factorization. We may assume H is maximal in G.

By Lemma 7.2, $\mathfrak{g}_*$ cannot be commutative. Lemma 7.4 in turn eliminates algebras $\mathfrak{g}$ where the strongly semisimple part $(\mathfrak{g}_*)^s$ of $\mathfrak{g}_*$ is trivial and either (a) $\mathfrak{g}$ is strongly semisimple, or (b) the strongly semisimple part $\mathfrak{g}^s$ of $\mathfrak{g}$ contains all type A_1 ideals of $\mathfrak{g}_*$.

Assume $\mathfrak{g}_*$ has a semisimple ideal not of type A_1.

Lemma 8.1. *Suppose (H, G) is a reduced pair. Then G is reduced.*

Proof. Assume G is not reduced. By Corollary 6.5, $(H, G) \cong (G_n \otimes H_m, G_n \otimes SL_m)$, where $H_m \subset SL_m$ is a maximal subgroup, $G_n \subset SL_n$, $m < n$. Note that since G is not reduced, and G_* is reductive, we have $m > 2$.

Take a reduced $\tilde{G}$, $\tilde{G} \bowtie G$. Denote $Lie(\tilde{G}_*) = \tilde{\mathfrak{g}}_*$, $Lie(H_m) = \mathfrak{h}_m$, $\mathfrak{s} = (\mathfrak{g}_*)^s$. We have $\tilde{\mathfrak{g}}_* \cong \mathfrak{g}_*$. Let $\mathfrak{s}_m$ denote the projection of $\mathfrak{s}$ into $\mathfrak{sl}_m$. Since $(\mathfrak{g}, \mathfrak{h}, \mathfrak{s})$ is a factorization (Lemma 7.4), the triple $(\mathfrak{sl}_m, \mathfrak{h}_m, \mathfrak{s}_m)$ is also a factorization. Lemma 7.5 implies then that either $\mathfrak{s}_m = \mathfrak{sl}_m$, or $(\mathfrak{sl}_m, \mathfrak{h}_m, \mathfrak{s}_m) \cong (\mathfrak{sl}_{2k}, \mathfrak{sp}_{2k}, \mathfrak{sl}_{2k-1})$ for some k (note that $\mathfrak{h} = \mathfrak{sl}_{2k-1}$ cannot hold due to $(\mathbb{C}^m)^H = 0$). This is only possible if $\mathfrak{s}$ has an ideal isomorphic to $\mathfrak{sl}_t$ for some $t > 2$. In Table 1 we list all reduced algebras $\tilde{\mathfrak{g}}$ such that $\mathfrak{s} = (\tilde{\mathfrak{g}}_*)^s$ satisfies this condition [3],[4].

We see that $t = 3$ in all cases in Table 1 except for case 3, where $t = 5$, and case 4, where $t = 6$. However, Corollary 6.4 implies that $m > 6$ except for cases 7 and 8, where $m \geq 6$. This contradiction finishes the proof of Lemma 8.1. ∎

Furthermore, suppose $\mathfrak{g}$ has a simple ideal $\bar{\mathfrak{g}}$ that contains $(\mathfrak{g}_*)^s$. Then, unless $\bar{\mathfrak{g}}$ appears in Table **O** (as $\mathfrak{g}$) together with $(\mathfrak{g}_*)^s$ (as $\mathfrak{s}$), a triple $(\mathfrak{g}, \mathfrak{h}, \mathfrak{g}_*)$ is not a factorization by Lemma 7.5 for any $\mathfrak{h} \subset \mathfrak{g}$. Hence, we can disregard such algebras.

Table 2 presents the remaining part of Elashvili list.

Table 1.

	$\tilde{\mathfrak{g}}$	$(\tilde{\mathfrak{g}}_*)^s$
1	$\Lambda^3\mathfrak{sl}_6$	$\mathfrak{sl}_3 \oplus \mathfrak{sl}_3$
2	$\Lambda^3\mathfrak{sl}_8$	$\mathfrak{sl}_3$
3	$\mathfrak{spin}_{11}$	$\mathfrak{sl}_5$
4	$\mathfrak{spin}_{12}$	$\mathfrak{sl}_6 \oplus \mathfrak{sl}_6$
5	$\mathfrak{spin}_{13}$	$\mathfrak{sl}_3 \oplus \mathfrak{sl}_3$
6	$\Lambda^3_0\mathfrak{sp}_6$	$\mathfrak{sl}_3$
7	$\mathfrak{g}_2$	$\mathfrak{sl}_3$
8	$\mathfrak{spin}_7 \oplus \mathfrak{sl}_2$	$\mathfrak{sl}_3$
9	$\mathfrak{spin}_9 \oplus \mathfrak{sl}_2$	$\mathfrak{sl}_3$
10	$\mathfrak{f}_4 \oplus \mathfrak{sl}_2$	$\mathfrak{sl}_3$
11	$\mathfrak{e}_6 \oplus \mathfrak{sl}_3$	$\mathfrak{sl}_3$
12	$\mathfrak{e}_6 \oplus \mathfrak{sl}_2$	$\mathfrak{sl}_3$

Table 2.

$\mathfrak{g}$	$(\mathfrak{g}_*)^s$
$\Lambda^2\mathfrak{sl}_{2n}$	$\Lambda^2\mathfrak{sp}_{2n}$
$\mathfrak{so}_7$	$\mathfrak{so}_6$
$\mathfrak{so}_7$	$\mathfrak{g}_2$
$\mathfrak{so}_{2n}$	$\mathfrak{so}_{2n-1}$
$\mathfrak{spin}_{12}$	$\mathfrak{sl}_6 \oplus \mathfrak{sl}_6^*$
$\mathfrak{sl}_n \oplus \mathfrak{x}_n$	$\mathfrak{x}_n,\ \mathfrak{x}_n \subseteq \mathfrak{sl}_n$
$\mathfrak{so}_7 \oplus \mathfrak{sl}_2$	$\mathfrak{so}_5$
$\mathfrak{so}_8 \oplus \mathfrak{sl}_2$	$\mathfrak{so}_6$
$\mathfrak{so}_8 \oplus \mathfrak{so}_3$	$\mathfrak{so}_5$
$\mathfrak{so}_8 \oplus \mathfrak{sl}_3$	$\mathfrak{so}_5$

Applying Lemma 7.5 to the algebras from Table 2, we obtain the exceptional pairs listed in Table A.

9. G semisimple, G_* nonreductive

In this section we classify semisimple locally transitive irreducible pairs, and then show that any nontrivial exceptional semisimple irreducible pair (H, G) with nonreductive G_* is a locally transitive pair.

Suppose (H, G) is a semisimple irreducible locally transitive pair.

Lemma 9.1. *Up to simultaneous castling transform, (H, G) is isomorphic to one of the pairs from Table L, or to (H_0, G_0) as defined in Theorem 4.2.*

Proof. We may assume that (H, G) is reduced (see Definition 6.1). First, suppose G is reduced.

Lemma 9.2. *$G \cong SL_n$, or $G \cong SL_n \otimes X_m$, where $X_m \subseteq SL_m$, $m < n$.*

Proof. According to [14], the following is the complete list of reduced semisimple irreducible locally transitive linear groups:

(1)SL_n, (2)Sp_{2n}, (3)$\Lambda^2 SL_{2k+1}$, (4)$Spin_{10}$, (5)$SL_n \otimes X_m$, where $X_m \subseteq SL_m$, $m < n$, (6)$Sp_{2n} \otimes SL_{2k+1}$, where $2k < n$, (7)$\Lambda^2 SL_{2k+1} \otimes SL_2$.

In Th.2.3 [2], Dynkin listed all inclusions $A \subset B$ between irreducible linear groups such that B is not isomorphic to SL_n, Sp_{2n}, or SO_n (as a linear group). This theorem, combined with Lemmas 5.9 and 5.10, implies that if G is isomorphic to (2), (3), (4), (6) or (7), then it does not have a proper locally transitive subgroup. Hence, G has to be isomorphic to (1) or (5). ∎

Suppose $G \cong SL_n$. By Lemma 5.10, H is isomorphic to one of Sp_{2n}, $\Lambda^2 SL_{2k+1}$, $Spin_{10}$, $SL_s \otimes SL_t$, $s > t$ (entries $1-4$ of Table L). [1]

Now suppose $G \cong SL_n \otimes X_m$. By Lemma 5.9, either $H \cong SL_n \otimes Y_m$, where $Y_m \subset X_m$ is maximal, or $H \cong X_n \otimes X_m$, where $X_n \subset SL_n$ is maximal.

The group $SL_n \otimes Y_m$ is locally transitive (entry 5 of Table L). Consider $H = X_n \otimes X_m$. By Lemma 5.10, X_n is either simple, or $X_n = SL_s \otimes SL_t$, $st = n$.

Suppose X_n is simple. Since $SL_n \otimes X_m$ is reduced, H is locally transitive if and only if $X_n \cong \Lambda^2 SL_{2k+1}$, $X_m \cong SL_2$, or $X_n \cong Sp_{2l}$, $X_m \cong SL_{2k+1}$ (entries $6-7$ of Table L).

Now suppose $X_n \cong SL_s \otimes SL_t$. Then H may be locally transitive, and may be not, depending on s, t, m, and, for certain s, t, m values , on X_m (see Remark to Theorem 4.2).

The following Lemma 9.3 finishes the proof of Lemma 9.1. ∎

Lemma 9.3. *Suppose G is not reduced. Then $(H, G) \bowtie (SL_n \otimes X_m, SL_n \otimes SL_m)$, $2m > n > m$ (entry 5 of Table L).*

Proof. By Corollary 6.5, $(H, G) \cong (G_n \otimes H_m, G_n \otimes SL_m)$, where H_m is maximal in SL_m, $G_n \subseteq SL_n$, $n > m$, and, in particular, H is reduced.

Since H is reduced, the only option is $H = SL_r \otimes X_k$, $k < r$. Suppose $I(X_k) = \{k_0, \ldots, k_t\}$. Then either (a) $m = k_0$, $n = k_1 \ldots k_t r$, or (b) $m = k_1 k_2$, $n = k_0 k_3 \ldots k_t r$. Since G is not reduced, we have $m > n - m$, i.e., $2m > n$. This implies in case (a) $2k_0 > k_1 \ldots k_t r \implies k_1 = \ldots = k_t = 1$, $2m = 2k_0 > r$, and in case (b), analogously, $k_0 = k_3 = \ldots = k_t = 1$, $2m = 2k_1 k_2 > r$. Hence, $H \cong SL_n \otimes X_m \subset G \cong SL_n \otimes SL_m$, $2m > n$. ∎

Now we are going to look for exceptional pairs with nontrivial algebra of invariants, i.e., with $\nu(H, G) > 0$.

Any simple G with nonreductive G_* is locally transitive [3]. According to [4], there is only one type of semisimple (and not simple) reduced groups G with nonreductive G_* that are not locally transitive. Namely, this is $G^0 = X \otimes Sp(W)$, where $X \subsetneq SL(U)$, $\dim U < \dim W$, $\dim U$ is odd. Let $V^0 = U \otimes W$ denote the representation space.

Assume (H, G) is an exceptional pair such that $G \bowtie G^0$. Lemmas 9.4 and 9.6 show that (H, G) is trivial.

Lemma 9.4. *Suppose $G = G^0$. Then $H = G$.*

[1] Note that if $s = t$, then the group $SL_s \otimes SL_t$ has one invariant, i.e. not locally transitive.

Proof. By Lemma 5.8, we have $H = H(U) \otimes H(W)$, $H(U) \subseteq X$, $H(W) \subseteq Sp(W)$.

Lemma 9.5. $H(U) = X$.

Proof. We have $V^0 = U \otimes W \cong \mathrm{Hom}(U^*, W)$. Suppose $\psi \in \mathrm{Hom}(U^*, W)$. Let $b : W \times W \to \mathbb{C}$ denote the non-degenerate skew-symmetric bilinear form preserved by $Sp(W)$.

Define a mapping $\mu : \mathrm{Hom}(U^*, W) \to \Lambda^2 U^*$ by $\mu(\psi)(x, y) = b(\psi(x), \psi(y))$, $x, y \in U$. Since $\dim W > \dim U$, μ is surjective.

The action of the group G on $U \otimes W$ induces, by means of μ, the natural action of X on $\Lambda^2 U^*$. Since (H, G) is exceptional, the pair $(\Lambda^2 H(U), \Lambda^2 X)$ should be also exceptional. If $H(U) \neq X$ then $\dim \Lambda^2 X_* > 0$, and, by Th.7 [4], this implies that $X = SO(U^*)$. As follows from section 8., any exceptional pair of the form $(\Lambda^2 H(U), \Lambda^2 SO(U))$ is trivial. Hence, $H(U) = X$. ■

We have $(H, G) = (X \otimes H(W), X \otimes Sp(W))$. By Corollary 5.4, the pair $(H_1, G_1) = (SL(U) \otimes H(W), SL(U) \otimes Sp(W))$ is also exceptional. Since $\dim U$ is odd, G_1 is locally transitive. By Lemma 9.1, $H_1 = G_1$, $H(W) = Sp(W)$, and $H = G$. ■

The rest of the section is devoted to the proof of

Lemma 9.6. *Suppose $G \neq G^0$. Then $H = G$.*

Proof. We have $G_* \cong (G^0)_*$. We will show that if (H, G) is a nontrivial pair, then this equality cannot hold.

We may assume that (H, G) is a reduced pair, and, as always, that H is a maximal subgroup of G. Then, by Corollary 6.5, $(H, G) \cong (G_n \otimes H_m, G_n \otimes SL_m)$, where H_m is maximal in SL_m, $G_n \subseteq SL_n$, $n > m$, and, in particular, H is reduced.

Denote $\dim U = M < \dim W = 2N$.

Lemma 9.7. $m \geq \frac{1}{3}(16N + 1)$. *In particular,* $m \geq 11$.

Proof. We use the notation of Lemma 6.2. Since $X \neq SL(U)$, G^0 satisfies the conditions of Corollary 6.4 with $s \neq 1$. If $s = 0$, then Corollary 6.4 implies $m \geq 2NM - 1 \geq 3(2N) - 1 \geq \frac{1}{3}(16N + 1)$ for all $N > 1$. If $s \geq 2$, then, since $k_i \geq 3$ for all i, we have $k_{j_0} \leq \frac{1}{3}M$ and, therefore, $m \geq 2N \cdot 3 - \frac{1}{3}M \geq 2N \cdot 3 - \frac{1}{3}(2N - 1) = \frac{1}{3}(16N + 1)$. ■

Lemma 9.8. $\dim (G^0)_* \leq 2N^2 - 5N + 4$.

Proof. For a given N, $\dim G^0_*$ is maximal if $X = SO(U)$ ([4]). Hence, $\dim (G^0)_* \leq \frac{1}{2}(2N - M)^2 + \frac{1}{2}(2N - 1)$. Since $M \geq 3$, we obtain $\dim (G^0)_* \leq \frac{1}{2}(2N - 3)^2 + \frac{1}{2}(2N - 1) = 2N^2 - 5N + 4$. ■

Corollary 9.9. $\dim (G^0)_* < \frac{1}{2}m^2 - m - 1$.

Proof. The function $2N^2 - 5N + 4$ monotone increases for $N \geq 2$. Since $N \leq \frac{1}{16}(3m-1)$ by Lemma 9.7, we get $\dim (G^0)_* \leq 2(\frac{1}{16}(3m-1))^2 - 5(\frac{1}{16}(16N + 1)) + 4 = \frac{18}{16^2}m^2 - (\frac{12}{16^2} + \frac{15}{16})m + \frac{2}{16^2} + \frac{5}{16} + 4 < \frac{1}{14}m^2 - \frac{15}{16}m + 4$. Since $m \geq 11$, this implies $\dim (G^0)_* < \frac{1}{2}m^2 - m - 1$. ∎

Lemma 9.10. *For all* $m \geq 11$, *and for any proper irreducible subgroup* $H_m \subset SL_m$, $\dim H_m \leq \frac{1}{2}(m^2 + m)$.

Proof. It's enough to consider maximal subgroups $H_m \subset SL_m$, that is, H_m simple, or $H_m = SL_s \otimes SL_t$, $st = m$.

Lemma 3.2 [8] implies that if $H_m = SL_k$ as an algebraic group (in a representation different from the natural one), then $m > 2k + 2$, and, therefore, $k < \frac{1}{2}m - 1 < \frac{1}{2}m$ implying $\dim H_m = k^2 - 1 < (\frac{1}{2}m)^2 - 1 < \frac{1}{2}(m^2 + m)$.

If $H_m = Sp_{2k}$ (as an algebraic group), then $m \geq 2k$, and, therefore, $\dim H_m = 2k^2 + k \leq \frac{1}{2}(m^2 + m)$.

If $H_m = SO_k$ (as an algebraic group), then $m \geq k$, and, therefore, $\dim H_m = \frac{1}{2}(k^2 - k) < \frac{1}{2}(m^2 + m)$.

For $H_m = E_6, E_7, E_8, F_4$, and G_2 (again, as an algebraic group), $m \geq 27, 56, 248, 26$, and 7 respectively, and, therefore, $\dim H_m \leq \frac{1}{2}(m^2 + m)$ holds for all these groups.

For $H_m = SL_s \otimes SL_t$, we have $\dim H_m = s^2 + t^2 - 2$, and $\max_{st=m}(t^2 + s^2 - 2) = 2^2 + \frac{m^2}{2^2} - 2 = \frac{m^2}{4} + 2$, that is, $\dim H_m \leq \frac{m^2}{4} + 2 < \frac{1}{2}(m^2 + m)$. ∎

Lemma 9.11. $\dim H_m > \frac{1}{2}m^2 + m$.

Proof. Since (H, G) is exceptional, we have $\dim H \geq \dim O(H) = \dim O(G) = \dim G - \dim G_*$, and, therefore, $\dim G - \dim H \leq \dim G_* = \dim G^0_* < \frac{1}{2}m^2 - m - 1$ by Corollary 9.9. On the other hand, $\dim G - \dim H = \dim SL_m - \dim H_m = m^2 - 1 - \dim H_m$. Combining, we get $\dim H_m > m^2 - 1 - (\frac{1}{2}m^2 - m - 1) = \frac{1}{2}m^2 + m$. ∎

However, since $\frac{1}{2}(m^2 + m) < \frac{1}{2}m^2 + m$, Lemma 9.11 and Lemma 9.10 contradict each other. ∎

10. G not semisimple

In this section we classify connected irreducible exceptional pairs (H, G), where G is not semisimple.

As we saw above (section 3.2), a classification of such pairs reduces to a classification of semisimple irreducible pairs (H', G'), where G' is locally transitive, and H' has exactly one invariant, i.e., $\nu(H') = 1$. The latter is similar to the classification of locally transitive pairs, section 9.

As always, we will assume H' is maximal in G'.

Lemma 10.1. *Up to simultaneous castling transform, (H', G') is isomorphic to one of the pairs from Table R, or to (H_0, G_0) as defined in Theorem 4.2.*

Proof. We may assume (H', G') is reduced. First, suppose G' is reduced.

In [14],[13], we find the list of reduced semisimple irreducible linear groups G with tr.deg$\mathbb{C}[V]^G \leq 1$. Similarly to the proof of Lemma 9.1, we use [2] to find all reduced pairs (H', G') with tr.deg$\mathbb{C}[V]^{H'} = 1$ and tr.deg$\mathbb{C}[V]^{G'} = 0$, see Table R and Remark to Theorem 4.2.

The following Lemma 10.2 (analogous to Lemma 9.3) finishes the proof of Lemma 10.1. ∎

Lemma 10.2. *Suppose G' is not reduced. Then $(H', G') \bowtie (Sp_4 \otimes SO_3, Sp_4 \otimes SL_3)$ (entry 23 of Table L).*

Proof. By Corollary 6.5, $(H', G') \cong (G_n \otimes H_m, G_n \otimes SL_m)$, where H_m is maximal in SL_m, $G_n \subseteq SL_n$, $n > m$, and H' is reduced.

We have (Lemma 5.10) either $H_m \cong SL_s \otimes SL_t$, $st = m$, or H_m is simple. Since H' is reduced and $n > m$, the only option, according to [14], is $H' = Sp_{2k} \otimes SO_3$, $k > 1$. Since G' is not reduced, we have $3 > 2k - 3$, i.e., $k = 2$. ∎

11. Attempt to generalize

The next step naturally would be to try to classify reducible semisimple exceptional pairs. However, we face the phenomenon of "blinking kernels", which makes such classification look unapproachable. To explain that, let us first define two "good" special cases of reducible pairs, namely, *direct sums* and *locally faithful* pairs.

Let $G_1, \dots, G_n$ be linear groups acting on $V_1, \dots, V_n$ respectively. Suppose $H_1 \subset G_1, \dots, H_n \subset G_n$ are subgroups. Denote $G = G_1 \times \dots \times G_n$, and $H = H_1 \times \dots \times H_n \subset G$.

Definition 11.1. The action of the group G on the sum $V = V_1 \oplus \dots \oplus V_n$ is called a direct sum of the actions of $G_1, \dots, G_n$. The pair (H, G) is called a direct sum of pairs (H_i, G_i).

It's easy to see that a direct sum of exceptional pairs is an exceptional pair, and vice versa, if a direct sum of pairs is exceptional then every summand is exceptional.

Definition 11.2. An action of G on V is called locally faithful if all invariant irreducible subspaces $U \subseteq V$ are faithful (as G-representations).

For a classification of locally faithful exceptional pairs, one can extend the ideas exploited for irreducible exceptional pairs.

Now suppose (H, G) is an arbitrary semisimple exceptional pair, $V = V_1 \oplus \dots \oplus V_n$, where V_i is irreducible G-invariant for all i. By Lemma 5.2, (H, G) acting on V_i is an exceptional pair. Denote by α_i the restriction map $\alpha_i : G \longrightarrow GL(V_i)$, and let $G \cong G_i \times \mathrm{Ker}(\alpha_i)$. There exist three options:

1) $G_i \cap G_j = \{id\}$ for all $i \neq j$. Then G is a direct sum of $G_1, \dots G_n$. Hence, in this case, we reduce the problem of classification for reducible exceptional pairs to the irreducible case.

2) $G_i = G_j$ for all i, j. Then G is locally faithful.

3) $\exists i \neq j$ such that $G_i \cap G_j \neq \{id\}$ and $G_i \neq G_j$. This case is what we call "blinking kernels". When it happens, the structure of the action of (H, G) on V in general becomes hardly observable.

References

[1] Borel, A., *Le plan projectif des octaves et les sphères comme espaces homogènes de groupes de Lie compacts*, C. R. Acad. Sci. Paris **230** (1950), 1378–1380.

[2] Dynkin, E. B., *Maximal subgroups of classical groups*, Tr. Mosk. Mat. O-va **1** (1952), 39–166 (in Russian). English translation: Amer. Math. Soc. Transl. (2)**6** (1957), 245–378.

[3] Elashvili, A. G., *A canonical form and stationary subalgebras of points in general position for simple linear Lie groups*, Funkts. Anal. Prilozh. **6(1)** (1972), 51–62 (in Russian). English translation: Funct. Anal. Appl. **6** (1972), 44–53.

[4] —, *Stationary subalgebras of points in general position for irreducible linear Lie groups*, Funkts. Anal. Prilozh. **6(2)** (1972), 65–78 (in Russian). English translation: Funct. Anal. Appl. **6** (1972), 139–148.

[5] Luna, D., *Slices étales*, Bull. Soc. Math. France, Suppl., Mem. **33** (1973), 81–105.

[6] —, *Sur les orbites fermées des groupes algébriques réductifs*, Invent. math. **16** (1972), 1–5.

[7] Montgomery, D., and H. Samelson, *Transformation groups on spheres*, Ann. Math. **44** (1943), 454–470.

[8] Onishchik, A. L., *Inclusion relations among transitive compact transformation groups*, Tr. Mosk. Mat. O-va **11** (1962), 199–242 (in Russian). English translation: Amer. Math. Soc. Transl. (2) **50** (1966), 5–58.

[9] —, *Decompositions of reductive Lie groups*, Mat. Sbornik Tom 80 **122** (1969) No.4 (in Russian). English translation: Math.USSR-Sbornik **9** (1966), No.4, 515–554.

[10] Popov, V. L., *A stability criterion for an action of a semisimple group on a factorial variety*, Izv. Akad. Nauk SSSR, Ser. Mat. **36(2)** (1970) 523–531 (in Russian). English translation: Math USSR Izv. **4** (1971), 527–535.

[11] Richardson, R. V., *Principal orbit types for algebraic transformation spaces in characteristic zero*, Invent. Math. **16** (1972), 6–14.

[12] Rosenlicht, M., *Some basic theorems on algebraic groups*, Amer. J. Math. **78** (1956), 401–443.

[13] Sato, M., and T. Kimura, *Classification of irreducible prehomogeneous vector spaces and their invariants*, Nagoya Math. J. **65** (1977), 1–155.

[14] Shpiz, G. B., *Classification of irreducible locally transitive linear Lie groups*, Geometric methods in Problems of Algebra and Analysis, Interuniv. Thematic Work Collect., Yaroslavl, 1978, 152–160 (in Russian).

[15] Vinberg, E. B., and V. L. Popov, "Invariant Theory," Algebraic Geometry **4.** Vsesoyuz. Inst. Nauchn. i Tekhn. Inform., Akad. Nauk SSSR (1989) 137–315 (in Russian). English translation: "Algebraic geometry IV: Linear algebraic groups, Invariant theory" Springer, 1994.

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