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On Generators of Free Color Lie Superalgebras of Rank Two

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Abstract. Let L be a free color Lie superalgebra on two generators x, y. A criterion for two elements of L to be a generating set is given.

1. Introduction

P. M. Cohn proved in [3] that the t-automorphisms generate the group of all automorphisms of a free Lie algebra of finite rank. In [6], [7] Mikhalev obtained the following analogue of Cohn's theorem: The elementary automorphisms and linear changes generate a group of automorphisms of a free color Lie superalgebra of finite rank. The freedom of the subalgebras of free color Lie algebras [6], [7] gives rise to the following analogue of Nielsen's theorem: If n G-homogeneous elements generate a free color Lie superalgebra of rank n, then these elements are free generators of it. Now let $X = \{x, y\}$ and $L(X)$ be a free color Lie superalgebra freely generated by the set X. If two G-homogeneous elements h_1, h_2 generate $L(X)$, then they freely generate $L(X)$. So $[h_1, h_2]$ is a linear combination of the elements $[x, x], [y, y], [x, y]$. The main assertion of this note is the theorem that the subalgebra generated by h_1, h_2 is equal to the free color Lie superalgebra $L(X)$ if and only if $[h_1, h_2] = \alpha[x, x] + \beta[x, y] + \gamma[y, y]$, where α, β, $\gamma \in K^*$. In [4] Dicks obtained a similar criterion for free associative algebras of rank two.

2. Preliminaries

Let K and G be a field and abelian group respectively. Assume that $R = \bigoplus_{g\in G} R_g$ is a G-graded K-algebra. The homogeneous elements are those from some R_g. For each homogeneous $a \in R_g$ we shall write $d(a) = g$. The G -valued function d is called the degree map on R. Let K^* be the multiplicative group of the field K, $\varepsilon : G \times G \to K^*$ a skew-symmetric bilinear form, and $G_- = \{g \in G \mid \varepsilon(g, g) = -1\}$.

Definition 2.1. We say that a G -graded algebra R is a color Lie superalgebra if $[x, y] = -\varepsilon(d(x), d(y))[y, x]$, $[x, [y, z]] = [[x, y], z] + \varepsilon(d(x), d(y))[y, [x, z]]$,

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$[v,[v,v]]=0$, with $d(v)\in G_-$, for homogeneous $x,y,z,v\in R$.

Definition 2.2. Let $X=\bigcup_{g\in G} X_g$ be a G-graded set, again $d(x)=g$ for $x\in X_g$, $d(u)=\sum_{i=1}^{n} d(x_i)\in G$ for $u=x_1...x_n\in\langle X\rangle$, $x_i\in X$, $d(z)=d(\gamma(z))$ for $z\in V[X]$, $(\langle X\rangle)_g=\{u\in\langle X\rangle \mid d(u)=g\}$, $(V[X])_g=\{z\in V[X]\mid d(z)=g\}$, $(F[X])_g$ is the K-linear hull of the subsets $(V[X])g\in F[X]$, $F[X]=\bigoplus_{g\in G}(F[X])_g$, I the G -graded ideal in $F[X]$ generated by homogeneous elements of the form $[a,b]+\varepsilon(d(a),d(b))[b,a]$ and $[[a,b],c]-[a,[b,c]]+\varepsilon(d(a),d(b))[b,[a,c]]$, where $a,b,c\in V[X]$ then, $L[X]=F[X]/I$ is a free color Lie superalgebra (i.e., each G-map of degree zero from X to any color Lie superalgebra R uniquely extends to a G -homomorphism of degree zero of color Lie superalgebras. $L[X]=F[X]/I\to R$, for $z\in F[X]$).

If M is a G-graded associative algebra over K, then M with the operation $[a,b]=ab-\varepsilon(d(b),d(a))ba$ for homogeneous elements $a,b\in M$ is a color Lie superalgebra denoted by $[M]$.

Let $X=\{x_1,...,x_n\}$ be a G-graded set, and $A(X)$ be a free G-graded associative algebra with 1 over K. Then the subalgebra $L(X)$ in $[A(X)]$ generated by X is a free color Lie superalgebra with set X of free generators. The algebra $A(X)$ is the enveloping algebra of $L(X)$.

In order to proceed with the proof of our main result we have to introduce some more notation. By $U(L)$ we denote the universal enveloping algebra of L.

There is the augmentation homomorphism $\varepsilon' : U(L)\to K$ defined by $\varepsilon'(x_i)=0$, $i=1,2,...,n$. There are mappings $\frac{\partial}{\partial x_i}: U(L)\to U(L)$, $i=1,2,...,n$, satisfying the following conditions whenever $a,b\in K$ and $u,v\in U(L)$:

1. $\frac{\partial(x_j)}{\partial x_i}=\delta_{ij}$,
2. $\frac{\partial}{\partial x_i}(au+bv)=a\frac{\partial u}{\partial x_i}+b\frac{\partial v}{\partial x_i}$,
3. $\frac{\partial}{\partial x_i}(uv)=\varepsilon'(u)\frac{\partial v}{\partial x_i}+\frac{\partial u}{\partial x_i}v$.

For any $a\in K$, $\frac{\partial a}{\partial x_i}=0$. We will call these mappings Fox derivatives [5]. We need some lemmas to be used throughout 3.

Lemma 2.3. *Let $L(X)$ be a color Lie superalgebra and $v_1,...,v_m,u$ be some elements of $L(X)$. Suppose u belongs to the left ideal of $U(L)$ generated by $v_1,...,v_m$. Then u belongs to the subalgebra of $L(X)$ generated by $v_1,...,v_m$.*

The assertion of the lemma is similar to the corresponding assertion for Lie algebras [9]. The proof follows by using analogue of the Poincare'-Birkhoff -Witt theorem for free color Lie superalgebras.

Definition 2.4. Suppose $S=\{s_\alpha\mid\alpha\in I\}$ is a G -homogeneous subset of the free color Lie superalgebra $L(X)$. By an elementary transformation of the set S we mean a mapping $\omega: S\to L(X)$ such that $\omega(s_\alpha)=s_\alpha$ for all $\alpha\in I\setminus\beta$,

$\omega(s_\beta) = \lambda s_\beta + \omega(s_{\alpha_1}...s_{\alpha_t})$, where $\lambda \in K$, $\lambda \neq 0$, $\alpha_1,...,\alpha_t \neq \beta$, $d(\omega(s_{\alpha_1}...s_{\alpha_t})) = d(s_\beta)$.

Lemma 2.5. *The elementary transformations and linear changes of a set of free generators induce automorphisms of a free color Lie superalgebra.*

Proof. The proof is analogous to that of Lemma 2.7.2 of [1]. ∎

In [7] Mikhalev obtained the following analogue of Cohn's theorem:

Theorem 2.6. *The elementary automorphisms and linear changes generate a group of automorphisms of a free color Lie superalgebra of finite rank.*

The following theorem is an analog of the Birman's result [2] for groups.

Theorem 2.7. *Let $X = \{x_1, ..., x_n\}$ and let $f_1, ..., f_n$ be G -homogeneous elements in $L(X)$, $d(x_i) = d(f_i)$. Then the endomorphism $\varphi : L(X) \to L(X)$, where $\varphi(x_i) = f_i$, $1 \leq i \leq n$ is an automorphism if and only if the matrix $\left(\frac{\partial f_i}{\partial x_j}\right)$, $1 \leq i, j \leq n$ is invertible over $A(X)$.*

Proof. The proof is along the lines of the proof of the theorem in [9]. ∎

Now let K be a field and L be the free color Lie superalgebra on two generators x, y. Clearly the group of automorphisms of L generated by the automorphisms of the form

$$\varphi : \begin{array}{ll} x \to & y \\ y \to & x \end{array}, \qquad \psi : \begin{array}{ll} x \to & x \\ y \to & \alpha y + x \end{array}$$

$\alpha \in K$, $\alpha \neq 0$, $d(x) = d(y)$.

Lemma 2.8. *Let h_1, $h_2 \in L$, $h_1 = \alpha x + \beta y$, $h_2 = \gamma x + \delta y$, where α, β, γ, $\delta \in K$. Then h_1 and h_2 freely generate L if and only if $\alpha\delta - \beta\gamma \neq 0$.*

The lemma follows immediately from the Theorem 2.7. We now come to our main result.

3. Main Theorem

Suppose K is a field of characteristic zero. Let $X = \{x, y\}$ and L be a free color Lie superalgebra on X.

Theorem 3.1. *Let h_1, h_2 be G -homogeneous elements of L and H the subalgebra they generate. Then $H = L$ if and only if $[h_1, h_2] = \alpha[x, y] + \beta[x, x] + \gamma[y, y]$, where α, β, $\gamma \in K^*$.*

Proof. If $H = L$ then the set $\{h_1, h_2\}$ can be transform into the set $\{x, y\}$ by the automorphisms of L. Then we can write h_1, h_2 as $h_1 = ax + by$ and $h_2 = cx + dy$, where $a, b, c, d \in K$, $d(x) = d(y)$, $ad - bc \neq 0$.

If $d(x)$, $d(y) \notin G_-$, then $[h_1, h_2] = \alpha[x, y]$, where $\alpha = ad - bc\varepsilon(d(y), d(x))$.

If $d(x)$, $d(y) \in G_-$, then $[h_1, h_2] = \alpha[x, y] + \beta[x, x] + \gamma[y, y]$, where $\alpha = ad + bc$, $\gamma = bd$, $\beta = ac$.

Now we prove "if" part. We will consider four cases:

Case I. Let $d(x)$, $d(y) \notin G_-$. In this case $[h_1, h_2]$ is a nonzero scalar multiple of $[x, y]$.

Let $[h_1, h_2] = \alpha[x, y]$, $\alpha \in K^*$. Take Fox derivative of both sides:

$$\frac{\partial[h_1, h_2]}{\partial x} = \alpha\frac{\partial[x, y]}{\partial x} \text{ and } \frac{\partial[h_1, h_2]}{\partial y} = \alpha\frac{\partial[x, y]}{\partial y}.$$

By the chain rule for Fox derivativations [5],

$$\frac{\partial[h_1, h_2]}{\partial x} = \frac{\partial h_1}{\partial x}\frac{\partial[h_1, h_2]}{\partial h_1} + \frac{\partial h_2}{\partial x}\frac{\partial[h_1, h_2]}{\partial h_2}$$

and

$$\frac{\partial[h_1, h_2]}{\partial y} = \frac{\partial h_1}{\partial y}\frac{\partial[h_1, h_2]}{\partial h_1} + \frac{\partial h_2}{\partial y}\frac{\partial[h_1, h_2]}{\partial h_2}.$$

Therefore,

$$\frac{\partial[h_1, h_2]}{\partial h_1} = \frac{\partial(h_1h_2 - \varepsilon(d(h_1), d(h_2))h_2h_1)}{\partial h_1} = h_2, \quad \frac{\partial[h_1, h_2]}{\partial h_2} = -\varepsilon(d(h_1), d(h_2))h_1$$

and

$$\begin{aligned} \frac{\partial h_1}{\partial x}h_2 - \varepsilon(d(h_1), d(h_2))\frac{\partial h_2}{\partial x}h_1 &= \alpha y, \\ \frac{\partial h_1}{\partial y}h_2 - \varepsilon(d(h_1), d(h_2)\frac{\partial h_2}{\partial y}h_1 &= -\varepsilon(d(x), d(y))\alpha x. \end{aligned}$$

Set $k = -\varepsilon(d(x), d(y))$. Then we have

$$\begin{aligned} \alpha^{-1}\frac{\partial h_1}{\partial x}h_2 - \alpha^{-1}\varepsilon(d(h_1), d(h_2))\frac{\partial h_2}{\partial x}h_1 &= y, \\ k^{-1}\alpha^{-1}\frac{\partial h_1}{\partial y}h_2 - k^{-1}\alpha^{-1}\varepsilon(d(h_1), d(h_2))\frac{\partial h_2}{\partial y}h_1 &= x. \end{aligned}$$

We see that x and y are belong to the left ideal of A generated by h_1 and h_2. From the Lemma 2.3. we conclude that x and y are belong to the subalgebra H generated by h_1 and h_2. Hence $H = L$.

Case II. Let $d(x) \in G_-$ and $d(y) \notin G_-$. In this case

$$[h_1, h_2] = \alpha[x, y] + \beta[x, x],$$

where $\alpha, \beta \in K^*$. It follows that

$$\frac{\partial[h_1, h_2]}{\partial x} = \alpha\frac{\partial[x, y]}{\partial x} + \beta\frac{\partial[x, x]}{\partial x}$$

and

$$\frac{\partial[h_1,h_2]}{\partial y} = \alpha\frac{\partial[x,y]}{\partial y} + \beta\frac{\partial[x,x]}{\partial y}.$$

Then

$$\begin{aligned}\frac{\partial h_1}{\partial x}h_2 - \varepsilon(d(h_1),d(h_2))\frac{\partial h_2}{\partial x}h_1 &= \alpha y + 2\beta x,\\ \frac{\partial h_1}{\partial y}h_2 - \varepsilon(d(h_1),d(h_2))\frac{\partial h_2}{\partial y}h_1 &= -\varepsilon(d(x),d(y))\alpha x.\end{aligned}$$

By the Lemma 2.8. the elements $\alpha y + 2\beta x$ and $-\varepsilon(d(x),d(y))\alpha x$ freely generate L. These elements belong to the left ideal of A generated by h_1 and h_2. So by the Lemma 2.3. they belong to H.

Case III. Let $d(x) \notin G_-$ and $d(y) \in G_-$. Since $d(x) \notin G_-$, $[x,x]=0$ and $[h_1,h_2] = \alpha[x,y]+\gamma[y,y]$, where $\alpha,\gamma \in K-\{0\}$. If we take Fox derivative of both sides and we replace the roles of x and y we obtain the result as in Case II.

Case IV. Let $d(x)$ and $d(y) \in G_-$. In this case

$$[h_1,h_2] = \alpha[x,y] + \beta[x,x] + \gamma[y,y].$$

If we take Fox derivative of both sides we get

$$\begin{aligned}\frac{\partial h_1}{\partial x}h_2 - \varepsilon(d(h_1),d(h_2))\frac{\partial h_2}{\partial x}h_1 &= \alpha y + 2\beta x,\\ \frac{\partial h_1}{\partial y}h_2 - \varepsilon(d(h_1),d(h_2))\frac{\partial h_2}{\partial y}h_1 &= -\varepsilon(d(x),d(y))\alpha x + 2\gamma y.\end{aligned}$$

If $\alpha^2\varepsilon(d(x),d(y)) + 4\beta\gamma \neq 0$, then by the Lemma 2.8. the elements $\alpha y + 2\beta x$ and $-\varepsilon(d(x),d(y))\alpha x + 2\gamma y$ freely generate L and they belong to the free color Lie superalgebra H generated by h_1 and h_2. So $H = L$ as claimed. ■

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