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On the Structure of Graded Transitive Lie Algebras

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Abstract. We study finite-dimensional Lie algebras $\mathfrak{L}$ of polynomial vector fields in n variables that contain the vector fields $\frac{\partial}{\partial x_i}$ $(i = 1, \ldots, n)$ and $x_1 \frac{\partial}{\partial x_1} + \cdots + x_n \frac{\partial}{\partial x_n}$. We show that the maximal ones always contain a semi-simple subalgebra $\bar{\mathfrak{g}}$, such that $\frac{\partial}{\partial x_i} \in \bar{\mathfrak{g}}$ $(i = 1, \ldots, m)$ for an m with $1 \leq m \leq n$. Moreover a maximal algebra has no trivial $\bar{\mathfrak{g}}$-modules in the space spanned by $\frac{\partial}{\partial x_i}(i = m+1, \ldots, n)$. The possible algebras $\bar{\mathfrak{g}}$ are described in detail, as well as all $\bar{\mathfrak{g}}$-modules that constitute such maximal $\mathfrak{L}$. The maximal algebras are described explicitly for $n \leq 3$.

Keywords: Lie algebras, vector fields, graded Lie algebras.
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1. Introduction

The classification of Lie algebras of vector fields has a history of more than 100 years. Lie [7] was particularly interested in it, since he tried to create catalogues of differential equations with a fixed symmetry group. Apart from this, there are several motivations for considering this problem [1]. The problem in its full generality is (still) too difficult to attack. Our concern will be transitive algebras, that are also graded. Remember that $\mathcal{W}_n$ carries a natural filtering, determined by the lowest order terms that appear in the Taylor expansion of the coefficients of a vector field. If one considers with this filtering, the associated graded Lie algebra of a transitive Lie algebra, one obtains the graded transitive Lie algebra that we study here.

This paper is a continuation of [10]. There is studied the relation of the graded transitive Lie algebras $\mathfrak{L}$ with its structure as an abstract Lie algebra. Results are derived about the forms of the Levi subalgebras, the radical and the nilradical of $\mathfrak{L}$. Of particular interest is the product $\bar{\mathfrak{g}}$ of those Levi subalgebras that have elements of degree -1. It follows that $\bar{\mathfrak{g}} = \bar{\mathfrak{g}}_{-1} \oplus \bar{\mathfrak{g}}_0 \oplus \bar{\mathfrak{g}}_1$, where the subscript denotes the degree.

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There were two main conclusions in [10]. The first one was that when studying maximal algebras $\mathfrak{L}$, it is unnatural to take maximality in the class of Lie algebras of fixed maximal degree, cf. [4, 5, 6]. The second conclusion was that to make progression, one will have to study how $\mathfrak{L}$ is a representation over $\bar{\mathfrak{g}}$. Exactly these two points we take into account, and end up with a rather detailed description for the possible forms of $\mathfrak{L}$. It seems difficult to proceed any further in the general case. This is due to the possible presence of certain quadratic terms Q_α in $\bar{\mathfrak{g}}_1$. These terms can exist only if only $S^2(V^*) \otimes V$ contains a $\bar{\mathfrak{g}}_0$-module isomorphic to the $\bar{\mathfrak{g}}_0$-module $\bar{\mathfrak{g}}_1$. Here V is an arbitrary $\bar{\mathfrak{g}}_0$-representation (and $Q_\alpha \in S^2(V^*) \otimes V$). The lowest number of variables where this happens is $n = 3$. Another complication occurs when the space spanned by ∂_{x_i} $(i = 1, \ldots, n)$ contains trivial $\bar{\mathfrak{g}}$-modules. Such algebra $\mathfrak{L}$ can not be maximal, but we cannot prove that $\mathfrak{L}$ is contained in a maximal one in case $Q_\alpha \neq 0$ and $n \geq 5$. We give a more detailed overview of the results in section 2., after introducing the notions that play an important role.

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2. Definitions and description of results

Consider the Lie algebra $\mathcal{W}_n$ of all polynomial vector fields on $\mathbb{C}^n$. An element $X \in \mathcal{W}_n$ has the form

$$X = \sum_{i=1}^{n} P_i(x)\partial_{x_i}$$

where the P_i are polynomials in n variables $x = (x_1, x_2, \ldots, x_n)$, and $\partial_{x_i} = \frac{\partial}{\partial x_i}$. We say that X has *degree* $k-1$ if the *polynomials* P_i are all homogeneous of degree k. We write $X \in U_{k-1}$. If we define the Euler vector field $E = \sum_{i=1}^{n} x_i \partial_{x_i}$, then we have exactly that

$$U_k = \{X \in \mathcal{W}_n | [E, X] = kX\}. \tag{1}$$

Thus $\mathcal{W}_n$ becomes a $\mathbb{Z}$-graded Lie algebra,

$$\mathcal{W}_n = \bigoplus_{k=0}^{\infty} U_{k-1}; \qquad [U_i, U_j] \subset U_{i+j}$$

Our object is to study subalgebras $\mathfrak{L}$ of $\mathcal{W}_n$ that satisfy the following conditions:

(0) $\mathfrak{L}$ is finite-dimensional.

(1) $\mathfrak{L}$ is graded: if $\mathfrak{L}_k = \mathfrak{L} \cap U_k$ then $\mathfrak{L} = \oplus L_k$.

(2) $\mathfrak{L}$ is transitive: $\mathfrak{L}_{-1} = U_{-1}$.

We will pay special attention to the case

(3) $\mathfrak{L}$ is maximal: there exists no finite-dimensional Lie algebra $\mathfrak{L}' \subset \mathcal{W}_n$ containing $\mathfrak{L}$ properly.

We call $\mathfrak{L}$ satisfying (0)-(2) a *graded transitive Lie algebra*, and those satisfying (0)-(3) a *maximal graded transitive Lie algebra.* Note that linear changes of coordinates map $\mathfrak{L}$ to a Lie algebra that we call equivalent; more precisely the aim is to study all equivalence classes of maximal graded transitive Lie algebras of vector fields. If $\mathfrak{L}$ is maximal then $\mathfrak{L}$ contains E: if $E \notin \mathfrak{L}$ then $\mathfrak{L}' = \mathfrak{L} \rtimes \mathbb{C}E$ is a one-dimensional extension of $\mathfrak{L}$. Once a finite-dimensional Lie algebra of vector fields contains E it is automatically $\mathbb{Z}$-graded, thanks to equation (1). Throughout we will assume that $E \in \mathfrak{L}$.

Our method is based on results from [10]. These results concern the structure of algebras $\mathfrak{L}$, not necessarily maximal, as an abstract Lie algebra. Usually $\mathfrak{L}$ is not semi-simple; $\mathfrak{L}$ can only be semi-simple if it is contained in $U_{-1} \oplus U_0 \oplus U_1$. Let $\mathfrak{R}$ denote the radical of $\mathfrak{L}$. If $\mathfrak{R} \neq \{0\}$ then also $V = \mathfrak{R} \cap U_{-1} \neq \{0\}$. We can perform a linear change of coordinates such that

$$V = \langle \partial_{x_{m+1}}, \partial_{x_{m+2}}, \ldots, \partial_{x_n} \rangle$$

in the new coordinates (which are again denoted by $x_1, x_2, \ldots, x_n$). Let I_V denote the (graded) ideal[1] in $\mathfrak{L}$ given by

$$I_V = \{X \in \mathfrak{L} | X = \sum_{i=m+1}^{n} P_i(x) \partial_{x_i}\}$$

The quotient $\mathfrak{g} = \mathfrak{L}/I_V$ is semi-simple, since $\mathfrak{R}$ is contained in I_V, and also graded. Moreover the elements of $\mathfrak{g}$ are of the form (omitting $+I_V$)

$$\bar{X} = \sum_{i=1}^{m} P_i(x_1, x_2, \ldots, x_m) \partial_{x_i}.$$

Note that P_i is independent of x_j for $j > m$, since otherwise $[\partial_{x_j}, X] \notin I_V$, which contradicts the fact that $\partial_{x_j} \in \mathfrak{R}$.

Now we are left with $\mathfrak{g}$, which is semi-simple. We can therefore decompose $\mathfrak{g}$ into, say r, simple ideals:

$$\mathfrak{g} = \mathfrak{g}^{(1)} \times \mathfrak{g}^{(2)} \times \cdots \times \mathfrak{g}^{(r)}$$

Each summand $\mathfrak{g}^{(i)}$ is graded, and we have

$$\mathfrak{g}^{(i)} = \mathfrak{g}_{-1}^{(i)} \oplus \mathfrak{g}_0^{(i)} \oplus \mathfrak{g}_1^{(i)},$$

with $\dim(\mathfrak{g}_{-1}^{(i)}) = \dim(\mathfrak{g}_1^{(i)}) = m_i \neq 0$ and $\sum m_i = m$. Since the summands $\mathfrak{g}^{(i)}$ mutually commute, each $\mathfrak{g}^{(i)}$ itself can be considered to be a graded Lie algebra of vector fields on $\mathbb{C}^{m_i}$. The Euler operator on $\mathbb{C}^{m_i}$ is a derivation of $\mathfrak{g}^{(i)}$: therefore it is contained in $\mathfrak{g}^{(i)}$.

Now we can describe the contents of this paper. First (section 3.) we describe all possible $\mathfrak{g}^{(i)}$. This description can already be found in [8], and is summarized in table 1. Therefore the Lie algebra $\mathfrak{g}$ can be described explicitly.

[1] I_V is the maximal ideal J such that $J \cap U_{-1} = V$; this statement, noted by Jan Draisma, simplifies some proofs in [10].

Of great technical importance is lemma 3.6, which describes the elements of $\mathfrak{g}_1$ in terms of those of $\mathfrak{g}_0$.

The Levi-Malcev theorem states that there exists a Lie subalgebra $\bar{\mathfrak{g}}$ in $\mathfrak{L}$ such that $\mathfrak{g} = \mathfrak{L}/I_V$. We call $\bar{\mathfrak{g}}$ an extension of $\mathfrak{g}$; we can assume that $\bar{\mathfrak{g}}$ is graded, see section 4.. Let us denote the remaining variables $(x_{m+1}, \ldots, x_n)$ by $(y_1, \ldots, y_{n-m})$, so that $V = \langle \partial_{y_1}, \ldots, \partial_{y_{n-m}} \rangle$. The new terms that appear in $\bar{\mathfrak{g}}$ are completely determined by the action on V. After some calculations one ends up with the form given in proposition 4.1. In particular $\bar{\mathfrak{g}}_{-1} = \langle \partial_{x_1}, \ldots, \partial_{x_m} \rangle$ is unchanged (after maybe a linear change of coordinates), and $X \in \bar{\mathfrak{g}}_1$ takes the form

$$X = \sum_{i=1}^{m} P_i(x)\partial_{x_i} + \sum_{i=1}^{n-m} L_i(x,y)\partial_{y_i} + \sum_{i=1}^{n-m} Q_i(y)\partial_{y_i},$$

where the terms of P_i, L_i and Q_i are of the forms $cx_a x_b, cx_a y_b$ and $cy_a y_b$, respectively.

Then we come to the study of the module: I_V is a module over $\mathfrak{g}$. This we study in section 5., for the case that V contains no trivial $\bar{\mathfrak{g}}_0$-modules. In this case we prove that there exists a unique maximal algebra (assuming $\bar{\mathfrak{g}} \neq \{0\}$ is fixed).

If V contains a trivial $\bar{\mathfrak{g}}_0$-module, the situation is more complicated, see section 6.. Though $\mathfrak{L}$ is never maximal, we only can prove that it is contained in a maximal algebra, in case that $Q_i = 0$ for all $i = 1, \ldots, m$. If some $Q_i \neq 0$, it is possible that $\mathfrak{L}$ is contained in a maximal algebra by suitably extending $\bar{\mathfrak{g}}$. Essentially this depends on the existence of a suitable $\mathbb{Z}$-grading, called the zdegree, in (an extension of) $\mathfrak{L}$.

Finally section 7. discusses the cases $n \leq 3$, while section 8. contains the conclusion.

3. Lie algebras of depth 1

We discuss a class of $\mathbb{Z}$-graded Lie algebras, which are closely related to Lie algebras of vector fields. For convenience we use

Definition 3.1. Suppose $\mathfrak{g} = \oplus \mathfrak{g}_k$ is a $\mathbb{Z}$-graded Lie algebra. We say that $\mathfrak{g}$ is of depth 1 if

- $\mathfrak{g}_k = \{0\}$ for all $k < -1$ and $\mathfrak{g}_{-1} \neq \{0\}$.
- $\mathfrak{g}_+ = \oplus_{k \geq 0} \mathfrak{g}_k$ contains no non-zero ideal of $\mathfrak{g}$.

It follows that for $X, Y \in \mathfrak{g}_k$: $[\partial, X] = [\partial, Y]$ for all $\partial \in \mathfrak{g}_{-1} \Rightarrow X = Y$ $(k \geq 0)$. Inductively, we obtain that any $X \in \mathfrak{g}_k$ is uniquely determined by the products

$$[\partial_1, [\partial_2, \ldots, [\partial_{k+1}, X] \ldots]],$$

where $\partial_1, \partial_2, \ldots, \partial_{k+1} \in \mathfrak{g}_{-1}$ and $k \geq 0$. This way $X \in \mathfrak{g}_k$ is given by the linear map $A = \varphi_k(X)$:

$$A: \underbrace{\mathfrak{g}_{-1} \times \cdots \times \mathfrak{g}_{-1}}_{k+1} \to \mathfrak{g}_{-1}$$

given by

$$A(\partial_1, \dots, \partial_{k+1}) = \frac{1}{k!}[\partial_1, [\partial_2, \dots, [\partial_{k+1}, X] \dots]].$$

Due to Jacobi's identity and $[\partial_i, \partial_j] = 0$, A is symmetric, hence $A \in S^{k+1}(\mathfrak{g}_{-1}^*) \otimes \mathfrak{g}_{-1}$. Choosing a basis in $\mathfrak{g}_{-1}$, we obtain a linear map $\varphi_k : \mathfrak{g}_k \to U_k$, where U_k denotes the space of all vector fields on $\mathbb{C}^m$, $m = \dim(\mathfrak{g}_{-1})$, of degree k.

Composing all φ_k, we obtain a linear map $\varphi : \mathfrak{g} \to \mathcal{W}_m$, with $\varphi|_{\mathfrak{g}_k} = \varphi_k$. Clearly φ is injective. One can prove (see [5]) that φ is a Lie algebra morphism, of degree 0. Hence we find that $\mathcal{W}_m$ contains all Lie algebras of depth 1 with $\dim(\mathfrak{g}_{-1}) = m$ as subalgebras.

This point of view gives us the opportunity to *compare* two Lie algebras $\mathfrak{g}$ of depth 1 and with $\dim(\mathfrak{g}_{-1}) = m$. Since all are embedded in $\mathcal{W}_m$ it can happen that one is contained in another; those finite-dimensional ones that are not contained in any other of finite dimension, we call maximal.

Next we turn to the case that $\mathfrak{g}$ is simple. Since the Killing form is a non-degenerate pairing between $\mathfrak{g}_k$ and $\mathfrak{g}_{-k}$, we see that necessarily

$$\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \text{ with } \dim(\mathfrak{g}_{-1}) = \dim(\mathfrak{g}_1).$$

Conversely, suppose $\mathfrak{g}$ is a simple Lie algebra with a $\mathbb{Z}$-grading such that $\mathfrak{g}_k = 0$ if and only if $|k| > 1$. Since $\mathfrak{g}$ contains no non-trivial ideals at all, $\mathfrak{g}$ is of depth 1. Therefore we will investigate now which simple (abstract) Lie algebras $\mathfrak{g}$ can be given a $\mathbb{Z}$-grading such that $\mathfrak{g}_k = 0$ if and only if $|k| > 1$. Let us fix a Cartan subalgebra $\mathfrak{h}$ in $\mathfrak{g}$, of dimension ℓ. Now $\mathfrak{g}$ is $\mathbb{Z}^\ell$-graded by the (coefficients of) the roots. Suppose $\theta = n_1\alpha_1 + \cdots + n_\ell\alpha_\ell$ is the highest root. If $n_s = 1$ for a certain s, we can put a $\mathbb{Z}$-grading on $\mathfrak{g}$ by $\deg(e_\alpha) = m_s$ if $\alpha = \sum_{i=1}^{\ell} m_i\alpha_i$. Since $|m_s| \le n_s$, we have indeed that $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$.

The simple Lie algebras of depth 1, obtained in this way, are given in table 1; we depict the Dynkin diagram of $\mathfrak{g} = X_\ell$ (where $X = A, B, C, D$ or E) with the point s colored black and write $X_{\ell;s}$ for this graded Lie algebra.

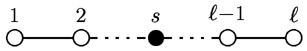
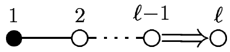
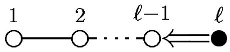
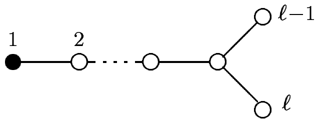
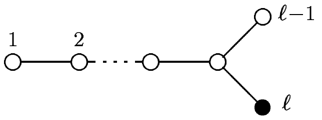
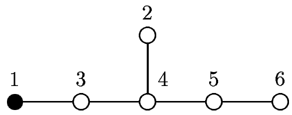
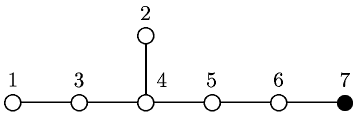
$\mathfrak{g}$	Diagram	$\mathfrak{g}_0$	$\dim(\mathfrak{g}_{-1})$
$A_{\ell;s}$		$A_{s-1} \times \mathbb{C} \times A_{\ell-s}$	$s(\ell+1-s)$
B_ℓ		$B_{\ell-1} \times \mathbb{C}$	$2\ell-1$
C_ℓ		$A_{\ell-1} \times \mathbb{C}$	$\binom{\ell+1}{2}$
$D_{\ell;1}$		$D_{\ell-1} \times \mathbb{C}$	$2\ell-2$
$D_{\ell;\ell}$		$A_{\ell-1} \times \mathbb{C}$	$\binom{\ell}{2}$
E_6		$D_5 \times \mathbb{C}$	16
E_7		$E_6 \times \mathbb{C}$	27

Table 1. Simple Lie algebras of depth 1

In cases where s is unique, we omitted it in the notation; moreover we have put $A_0 = \{0\}$. Obviously $A_{\ell;s}$ and $A_{\ell;\ell-s+1}$ are isomorphic, as well as $D_{\ell;\ell}$ and $D_{\ell;\ell-1}$.

The structure of $\mathfrak{g}_0$ is described by the Dynkin diagram of $\mathfrak{g}$, with s omitted; the summand $\mathbb{C}$ corresponds to the space spanned by the Euler vector field; it is central in $\mathfrak{g}_0$. Finally $\dim(\mathfrak{g}_{-1})$ can be calculated by

$$\dim(\mathfrak{g}_{-1}) = \frac{1}{2}(\dim \mathfrak{g} - \dim \mathfrak{g}_0).$$

The following theorem tells that the simple Lie algebras of depth one are all (gradedly) isomorphic to one from the table above.

Theorem 3.2. *(Morozov, [8],§6)*
The simple Lie algebras of depth 1 are those given in table 1.

Example 3.3. Let us construct $A_{\ell;s}$ in terms of vector fields. Generators for A_ℓ are $\{E_{ij}\}$ with $i,j = 1,\ldots,\ell+1$ and there is one linear relation among them: $E_{11} + E_{22} + \cdots + E_{\ell+1,\ell+1} = 0$. The commutator is defined by

$$[E_{ij}, E_{ab}] = \delta_{ja} E_{ib} - \delta_{ib} E_{aj}.$$

Moreover

$$\mathfrak{g}_{-1} = \langle E_{ij} | i \geq s+1 \text{ and } j \leq s \rangle.$$

Let us denote

$$\partial_{ij} = \partial_{x_{ij}} = E_{s+i,j} \qquad (i \leq l-s+1, j \leq s)$$

and

$$X_{ij} = E_{j,s+i} \qquad (i \leq l - s + 1, j \leq s).$$

For E_{ab} $(a, b \leq s)$, we have

$$[\partial_{ij}, E_{ab}] = [E_{s+i,j}, E_{ab}] = \delta_{aj} E_{s+i,b} = \delta_{aj} \partial_{ib}$$

so that

$$E_{ab} = \sum_{i=1}^{\ell-s+1} x_{ia} \partial_{ib} \qquad (a, b \leq s).$$

Similarly, $[\partial_{ij}, E_{s+a,s+b}] = -\delta_{bi} E_{s+a,j} = -\delta_{bi} \partial_{aj}$, and we find

$$E_{s+a,s+b} = -\sum_{j=1}^{s} x_{bj} \partial_{aj} \qquad (a, b \leq \ell - s + 1).$$

Finally,

$$[\partial_{ij}, X_{ab}] = [E_{s+i,j}, E_{b,s+a}] = \delta_{jb} E_{s+i,s+a} - \delta_{ai} E_{bj}.$$

Hence by integrating we find

$$X_{ab} = -\sum_{i,j} x_{aj} x_{ib} \partial_{ij}.$$

Note that

$$E_{11} + E_{22} + \cdots + E_{\ell+1,\ell+1} = \sum_{k=1}^{s} \left(\sum_{i=1}^{\ell-s+1} x_{ik} \partial_{ik} \right) + \sum_{k=1}^{\ell-s+1} \left(-\sum_{j=1}^{s} x_{kj} \partial_{kj} \right) = 0.$$

The procedure above shows how inductively we can obtain the realization of $\mathfrak{g}$ in terms of polynomial vector fields. Once a (graded) basis of $\mathfrak{g}$ is known, as well as the corresponding structure constants, the realization is fixed.

We now describe this construction in terms of roots and root vectors.

Let $\{e_\alpha, h_i\}$ be a Chevalley basis of $\mathfrak{g}$, where $\alpha \in \mathcal{R}$ and $i = 1, 2, \ldots, \ell$. As usual, we denote $h_\alpha = \sum m_i h_i$ for $\alpha = \sum m_i \alpha_i$. Let, as above, s be such that $\deg(e_{\alpha_s}) = 1$. We define

$$\mathring{\mathcal{R}} = \{\alpha = \sum m_i \alpha_i \in \mathcal{R} | m_s = 0\} = \{\alpha \in \mathcal{R} | \deg(e_\alpha) = 0\}$$

and

$$\hat{\mathcal{R}} = \{\alpha = \sum_{i \neq s} m_i \alpha_i | m_i \geq 0 \text{ for all } i \text{ and } \alpha + \alpha_s \in \mathcal{R}\}.$$

Note that $0 \in \hat{\mathcal{R}}$. Clearly,

$$\mathcal{R} = (-\alpha_s - \hat{\mathcal{R}}) \cup \mathring{\mathcal{R}} \cup (\alpha_s + \hat{\mathcal{R}}) \qquad \text{(disjoint union)},$$

corresponding to the degrees -1,0 and 1, respectively. For $\alpha \in \hat{\mathcal{R}}$ we denote

$$\partial_\alpha = e_{-\alpha_s - \alpha} \qquad \text{and} \qquad X_\alpha = e_{\alpha_s + \alpha}.$$

The realization of $\mathfrak{g}$ is by vector fields in the variables x_α, $\alpha \in \hat{\mathcal{R}}$. Hence $|\hat{\mathcal{R}}| = m$ and

$$\mathfrak{g}_{-1} = \langle \partial_\alpha | \alpha \in \hat{\mathcal{R}} \rangle, \qquad \mathfrak{g}_1 = \langle X_\alpha | \alpha \in \hat{\mathcal{R}} \rangle,$$

while $\mathfrak{g}_0$ consists of vector fields of the form

$$\sum_{\alpha,\beta \in \hat{\mathcal{R}}} c_{\alpha\beta} x_\alpha \partial_\beta.$$

Hence $\mathfrak{g}_0$ acts naturally on the space $S^1 = \langle x_\alpha | \alpha \in \hat{\mathcal{R}} \rangle$.

Lemma 3.4. *The $\mathfrak{g}_0$-modules $\mathfrak{g}_1$ and S^1 are isomorphic.*

Proof. The space $\mathfrak{g}_1$ is the module dual to $\mathfrak{g}_{-1}$ by its non-degenerate pairing with respect to the Killing form $(\partial | X)$, with $\partial \in \mathfrak{g}_{-1}$ and $X \in \mathfrak{g}_1$; the $\mathfrak{g}_0$-action on $\mathfrak{g}_1$ is dual to the $\mathfrak{g}_0$-action on $\mathfrak{g}_{-1}$:

$$(\partial | [g, X]) = -([g, \partial] | X).$$

We show that S^1 satisfies the same relation, with respect to the pairing $(.,.)$ given by

$$(\partial_\alpha, x_\beta) = \delta_{\alpha\beta}.$$

Indeed, for $g = \sum c_{\gamma\epsilon} x_\gamma \partial_\epsilon$ on one hand we have

$$(\partial_\alpha, [g, x_\beta]) = (\partial_\alpha, \sum_\gamma c_{\gamma\beta} x_\gamma) = c_{\alpha\beta},$$

while

$$([g, \partial_\alpha], x_\beta) = -c_{\alpha\beta}.$$

Consequently, also S^1 is the module dual to $\mathfrak{g}_{-1}$. ∎

Remark 3.5. In the ADE cases, the Killing form can be normalized such that the isomorphism of the $\mathfrak{g}_0$-module S^1 and $\mathfrak{g}_1$ is given by $x_\alpha \mapsto X_\alpha$. In general an isomorphism is given by $x_\alpha \mapsto n_\alpha X_\alpha$, where $n_\alpha = 2/(\alpha|\alpha)$.

For $\alpha, \beta \in \mathcal{R}$, with $\alpha \neq -\beta$ we put

$$[e_\alpha, e_\beta] = c(\alpha, \beta) e_{\alpha+\beta}.$$

We have $c(\alpha, \beta) \neq 0$ if and only if $\alpha + \beta \in \mathcal{R}$. Moreover we put $c(\alpha, \beta) = 0$ if $\alpha + \beta \notin \mathcal{R}$. Now we can describe $\mathfrak{g}_0$ in terms of vector fields. For $e_\alpha \in \mathfrak{g}_0$ we have

$$[\partial_\beta, e_\alpha] = c(-\alpha_s - \beta, \alpha) e_{-\alpha_s - \beta + \alpha} = c(-\alpha_s - \beta, \alpha) \partial_{\beta - \alpha}.$$

Hence

$$e_\alpha = \sum_{\beta - \alpha \in \hat{\mathcal{R}}} c(-\alpha_s - \beta, \alpha) x_\beta \partial_{\beta - \alpha}.$$

Similarly for $h_i \in \mathfrak{h}$ we obtain

$$h_i = \sum_\beta \langle \beta, \alpha_i \rangle x_\beta \partial_\beta$$

where

$$\langle \beta, \alpha_i \rangle = \frac{2(\beta|\alpha_i)}{(\alpha_i|\alpha_i)}.$$

Next we consider $X_\alpha \in \mathfrak{g}_1$. The following expression is essential to us.

Lemma 3.6. *Suppose for $X_\alpha \in \mathfrak{g}_1$ and $\partial_\beta \in \mathfrak{g}_{-1}$ we have*

$$[\partial_\beta, X_\alpha] = L_\alpha^\beta \in \mathfrak{g}_0.$$

Then

$$X_\alpha = \frac{1}{2} \sum_{\beta \in \hat{\mathcal{R}}} x_\beta L_\alpha^\beta.$$

Proof. We have

$$X_\alpha = \sum_{\beta,\gamma,\epsilon} c_{\beta\gamma\epsilon}^\alpha x_\beta x_\gamma \partial_\epsilon;$$

we can assume that $c_{\beta\gamma\epsilon}^\alpha = c_{\gamma\beta\epsilon}^\alpha$. Now

$$[\partial_\beta, X_\alpha] = \sum_{\gamma,\epsilon} c_{\beta\gamma\epsilon}^\alpha x_\gamma \partial_\epsilon + \sum_{\gamma,\epsilon} c_{\gamma\beta\epsilon}^\alpha x_\gamma \partial_\epsilon = L_\alpha^\beta.$$

Hence

$$\frac{1}{2} \sum x_\beta L_\alpha^\beta = \frac{1}{2} \sum x_\beta \left(\sum_{\gamma,\epsilon} c_{\beta\gamma\epsilon}^\alpha x_\gamma \partial_\epsilon + \sum_{\gamma,\epsilon} c_{\gamma\beta\epsilon}^\alpha x_\gamma \partial_\epsilon \right) = X_\alpha \qquad \blacksquare$$

We can express L_α^β in the Chevalley basis by

$$L_\alpha^\beta = \begin{cases} c(-\alpha_s - \beta, \alpha_s + \alpha) e_{\alpha-\beta} & (\alpha \neq \beta) \\ -h_{\alpha_s+\alpha} & (\alpha = \beta). \end{cases}$$

Moreover in view of remark 3.5 we have for $e_\gamma \in \mathfrak{g}_0$:

$$\begin{aligned} [e_\gamma, X_\alpha] &= \frac{1}{2} \sum [e_\gamma, x_\beta] L_\alpha^\beta + \frac{1}{2} \sum x_\beta [e_\gamma, L_\alpha^\beta] \\ &= \frac{1}{2} \sum c(\gamma, \alpha_s + \beta) n_\beta n_{\beta+\gamma}^{-1} x_{\beta+\gamma} L_\alpha^\beta + \frac{1}{2} x_\beta [e_\gamma, L_\alpha^\beta] \\ &= \frac{1}{2} \sum x_\beta \left(c(\gamma, \alpha_s + \beta - \gamma) n_{\beta-\gamma} n_\beta^{-1} L_\alpha^{\beta-\gamma} + [e_\gamma, L_\alpha^\beta] \right). \end{aligned}$$

On the other hand $[e_\gamma, X_\alpha] = c(\gamma, \alpha_s + \alpha) X_{\alpha+\gamma}$. Comparing the coefficents of x_β we see that for all $\alpha, \beta \in \hat{\mathcal{R}}$ and $\gamma \in \mathring{\mathcal{R}}$ holds

$$c(\gamma, \alpha_s + \beta - \gamma) n_{\beta-\gamma} n_\beta^{-1} L_\alpha^{\beta-\gamma} + [e_\gamma, L_\alpha^\beta] = c(\gamma, \alpha_s + \alpha) L_\gamma^\beta \tag{2}$$

Note that this is a relation between elements in $\mathfrak{g}_0$, containing only linear combinations and commutators. Consequently it holds in all representations of $\mathfrak{g}_0$ as well.

4. Extensions of simple algebras of depth 1

We proceed to investigate how the semi-simple algebra $\mathfrak{g}$ of depth 1, as considered in section 3. above, can be extended to vector fields in more variables. So we assume that $\mathfrak{g}$ is realized in terms of vector fields in $m = \dim \mathfrak{g}_{-1}$ variables, which we call $x_1, x_2, \ldots, x_m$, and we try to reconstruct the original algebra $\bar{\mathfrak{g}} \subset \mathfrak{L}$ in n variables, such that $\mathfrak{g} = \bar{\mathfrak{g}}/I_V$. For convenience we call $\bar{\mathfrak{g}}$ an extension of

$\mathfrak{g}$; we stress that $\bar{\mathfrak{g}}$ and $\mathfrak{g}$ are isomorphic Lie algebras. The remaining $n-m$ variables, we will call $y_1, y_2, \ldots, y_{n-m}$. As to the existence of $\bar{\mathfrak{g}}$, the Levi-Malcev theorem tells that $\mathfrak{L}$ contains a subalgebra $\bar{\mathfrak{g}}$ such that $\bar{\mathfrak{g}}/I_V$ and $\mathfrak{g}$ are isomorphic. In case that all ideals in $\mathfrak{L}$ are $\mathbb{Z}$-graded, it is possible to choose $\bar{\mathfrak{g}}$ to be $\mathbb{Z}$-graded as well[2]. Hence $\bar{\mathfrak{g}} = \bar{\mathfrak{g}}_{-1} \oplus \bar{\mathfrak{g}}_0 \oplus \bar{\mathfrak{g}}_1$. Let us now study $\bar{\mathfrak{g}}$ step by step.

First $g = \partial_{x_i} \in \mathfrak{g}_{-1}$ corresponds to $\bar{g} \in \bar{\mathfrak{g}}_{-1}$ of the form

$$\bar{g} = \partial_{x_i} + \sum_{j=1}^{n-m} a_{ij} \partial_{y_j}.$$

Introducing new coordinates in $\mathbb{C}^n$ by

$$\begin{cases} x_j = x'_j \\ y_j = y'_j + \sum a_{ij} x'_i \end{cases}$$

we obtain $\bar{g} = \partial_{x'_i}$. Hence we can assume that $\partial_{x_i} \in \bar{\mathfrak{g}}$ (dropping ').

Next we turn to $g \in \mathfrak{g}_0$. For $g = \sum c_{ij} x_i \partial_{x_j}$ we have

$$\bar{g} = g = \sum_{i,j \le m} c_{ij} x_i \partial_{x_j} + \sum_{j=1}^{n-m} f_j(x,y) \partial_{y_j}.$$

Now $[\partial_{x_i}, \bar{g}] \in \bar{\mathfrak{g}}$, and also

$$[\partial_{x_i}, \bar{g}] = \sum_j c_{ij} \partial_{x_j} + \sum_j \frac{\partial f_j}{\partial x_i} \partial_{y_j}.$$

This implies that $\dfrac{\partial f_j}{\partial x_i} = 0$ for all i,j. Hence

$$\bar{g} = \sum_{i,j \le m} c_{ij} x_i \partial_{x_j} + \sum_{i,j \le n-m} d_{ij} y_i \partial_{y_j}.$$

In particular, we have for $e_\alpha \in \mathfrak{g}_0$ that

$$\bar{e}_\alpha = e_\alpha + E_\alpha \text{ with } E_\alpha = \sum d_{ij} y_i \partial_{y_j}$$

and similarly

$$\bar{h}_\alpha = h_\alpha + H_\alpha.$$

Note that $[e_\alpha, E_\beta] = [e_\alpha, H_\gamma] = [h_\gamma, E_\alpha] = [h_\gamma, H_\epsilon] = 0$ for all $\alpha, \beta \in \mathring{\mathcal{R}}$ and all $\gamma, \epsilon \in \mathcal{R}$. Because $\mathfrak{g}_0$ and $\bar{\mathfrak{g}}_0$ are isomorphic, we obtain that $\{E_\alpha, H_\alpha\}$ form a representation of $\mathfrak{g}_0$, realized by vector fields in the variables $y_1, \ldots, y_{n-m}$.

Finally we arrive at $X_\alpha \in \mathfrak{g}_1$. Similar to lemma 3.6 we need that

$$\bar{X}_\alpha = \frac{1}{2} \sum x_\beta L_\alpha^\beta + \sum x_\beta (\bar{L}_\alpha^\beta - L_\alpha^\beta) + Q_\alpha,$$

[2]One can follow the proof of the Levi-Malcev theorem step by step, and finds that all occurring spaces can be chosen to be graded.

where

$$\bar{L}_\alpha^\beta = \begin{cases} c(-\alpha_s - \beta, \alpha_s + \alpha)\bar{e}_{\alpha-\beta} & (\alpha \neq \beta) \\ -\bar{h}_{\alpha_s+\alpha} & (\alpha = \beta), \end{cases}$$

and L_α^β is as in lemma 3.6, and

$$Q_\alpha = \sum_{i,j,k} c_{ijk}^\alpha y_i y_j \partial_{y_k}.$$

We still need to check the commutators $[\bar{\mathfrak{g}}_0, \bar{\mathfrak{g}}_1]$ and $[\bar{\mathfrak{g}}_1, \bar{\mathfrak{g}}_1]$. For $\bar{e}_\gamma \in \bar{\mathfrak{g}}_0$ and $\bar{X}_\alpha$, using remark 3.5, we have

$$\begin{gathered}[]
[\bar{e}_\gamma, \bar{X}_\alpha] = [e_\gamma + E_\gamma, \frac{1}{2}\sum x_\beta L_\alpha^\beta + + \sum x_\beta(\bar{L}_\alpha^\beta - L_\alpha^\beta) + Q_\alpha] \\
= \sum [e_\gamma, X_\alpha] + \sum [e_\gamma, x_\beta](\bar{L}_\alpha^\beta - L_\alpha^\beta) + \sum x_\beta [E_\gamma, \bar{L}_\alpha^\beta - L_\alpha^\beta] + [E_\gamma, Q_\alpha] \\
= c(\gamma, \alpha_s + \alpha) X_{\alpha+\gamma} + [E_\gamma, Q_\alpha] \\
+ \sum c(\gamma, \alpha_s + \beta) n_\beta n_{\beta+\gamma}^{-1} x_{\beta+\gamma}(\bar{L}_\alpha^\beta - L_\alpha^\beta) + \sum x_\beta [E_\gamma, \bar{L}_\alpha^\beta - L_\alpha^\beta] \\
= c(\gamma, \alpha_s + \alpha) X_{\alpha+\gamma} + [E_\gamma, Q_\alpha] \\
+ \sum x_\beta \left(c(\gamma, \alpha_s + \beta - \gamma) n_{\beta-\gamma} n_\beta^{-1} (\bar{L}_\alpha^{\beta-\gamma} - L_\alpha^{\beta-\gamma}) + [E_\gamma, \bar{L}_\alpha^\beta - L_\alpha^\beta] \right)
\end{gathered}$$

Using equation (2) the last sum can be rewritten, and we obtain

$$[\bar{e}_\gamma, \bar{X}_\alpha] = c(\gamma, \alpha_s + \alpha) \left(X_{\alpha+\gamma} + \sum x_\beta (\bar{L}_{\alpha+\gamma}^\beta - L_{\alpha+\gamma}^\beta) \right) + [E_\gamma, Q_\alpha].$$

Hence if (and only if) $[E_\gamma, Q_\alpha] = c(\gamma, \alpha_s + \alpha) Q_{\alpha+\gamma}$, we obtain

$$[\bar{e}_\gamma, \bar{X}_\alpha] = c(\gamma, \alpha_s + \alpha) \bar{X}_{\alpha+\gamma}.$$

For $[\bar{h}_\gamma, \bar{X}_\alpha]$ we have the same reasoning.

Finally we need to consider $[\bar{X}_\alpha, \bar{X}_\beta] = 0$: clearly we need $[Q_\alpha, Q_\beta] = 0$. All other terms (if present) involve some x_γ; hence its suffices to prove that $[\partial_\gamma, [\bar{X}_\alpha, \bar{X}_\beta]] = 0$. Now

$$[\partial_\gamma, [\bar{X}_\alpha, \bar{X}_\beta]] = [[\partial_\gamma, \bar{X}_\alpha], \bar{X}_\beta] + [\bar{X}_\alpha, [\partial_\gamma, \bar{X}_\beta]] = [\bar{L}_\alpha^\gamma, \bar{X}_\beta] + [\bar{X}_\alpha, \bar{L}_\beta^\gamma].$$

These commutators are commutators of $\bar{\mathfrak{g}}_0$ and $\bar{\mathfrak{g}}_1$, which we just considered. Hence we obtain

$$[\partial_\gamma, [\bar{X}_\alpha, \bar{X}_\beta]] = \overline{[L_\alpha^\gamma, X_\beta] + [X_\alpha, L_\beta^\gamma]}.$$

But this is 0, since

$$[L_\alpha^\gamma, X_\beta] + [X_\alpha, L_\beta^\gamma] = [\partial_\gamma, [X_\alpha, X_\beta]] = 0.$$

Combining all results we obtain

Proposition 4.1. *Suppose $\bar{\mathfrak{g}} \subset \mathcal{W}_n$ is a graded extension of a semi-simple $\mathfrak{g} \subset \mathcal{W}_m$ of depth 1, with notations as before. Then, up to a linear change of coordinates, $\bar{\mathfrak{g}}$ is of the form*

- *for degree -1: $\bar{\partial}_\alpha = \partial_\alpha$ for all $\alpha \in \hat{\mathcal{R}}$;*

- *for degree 0:*

$$\begin{cases} \bar{e}_\alpha = e_\alpha + E_\alpha \\ \bar{h}_\alpha = h_\alpha + H_\alpha \end{cases}$$

with $\{E_\alpha, H_\alpha\}$ *a representation of* $\mathfrak{g}_0$*;*

- *for degree 1:*

$$\bar{X}_\alpha = \frac{1}{2}\sum x_\beta L^\beta_\alpha + \sum x_\beta(\bar{L}^\beta_\alpha - L^\beta_\alpha) + Q_\alpha$$

with $[Q_\alpha, Q_\beta] = 0$*, and* $\bar{X}_\alpha \mapsto Q_\alpha$ *is a morphism of* $\bar{\mathfrak{g}}_0$*-modules.*

5. $\mathfrak{L}$ as a $\bar{\mathfrak{g}}$-module; no trivial modules in V

Now we know the form of $\bar{\mathfrak{g}}$, we study $\mathfrak{L}$, or rather I_V, as a $\bar{\mathfrak{g}}$-module. As before, we write

$$\bar{\mathfrak{g}} = \bar{\mathfrak{g}}^{(1)} \times \bar{\mathfrak{g}}^{(2)} \times \cdots \times \bar{\mathfrak{g}}^{(r)},$$

with each $\bar{\mathfrak{g}}^{(i)}$ simple of depth 1. The special simple root vector in $\bar{\mathfrak{g}}^{(i)}$ of degree 1, we denote by $e^{(i)}_s$. In each $\bar{\mathfrak{g}}^{(i)}$ we have a partial Euler operator $\bar{E}^{(i)}$, satisfying $[\bar{E}^{(i)}, \bar{\mathfrak{g}}_0] = 0$ and $[\bar{E}^{(i)}, X_0] = 1$, where $X_0 = e^{(i)}_s \in \bar{\mathfrak{g}}^{(i)}$. If $\alpha^{(i)}_1, \ldots, \alpha^{(i)}_{\ell_i}$ are the simple roots of $\bar{\mathfrak{g}}^{(i)}$ with corresponding Cartan elements $\bar{h}^{(i)}_1, \ldots, \bar{h}^{(i)}_{\ell_i}$, then

$$\bar{E}^{(i)} = \sum_{j=1}^{\ell_i} c_j \bar{h}^{(i)}_j,$$

The coefficients $c = (c_1, \ldots, c_{\ell_i})$ are given below for the different simple algebras of depth 1.

Algebra	the vector $c = (c_1, \ldots, c_\ell)$
$A_{\ell;s}$	$\frac{1}{\ell+1}(\ell+1-s, 2(\ell+1-s), \ldots, s(\ell+1-s), s(\ell-s), \ldots, 2s, s)$
B_ℓ	$(1, 1, \ldots, 1)$
C_ℓ	$(1, 2, \ldots, \ell-1, \frac{1}{2}\ell)$
$D_{\ell;1}$	$(1, 1, \ldots, 1, \frac{1}{2}, \frac{1}{2})$
$D_{\ell;\ell}$	$\frac{1}{2}(1, 2, \ldots, \ell-2, \frac{1}{2}\ell, \frac{1}{2}\ell)$
E_6	$\frac{1}{3}(4, 3, 5, 6, 4, 2)$
E_7	$\frac{1}{2}(2, 3, 4, 6, 5, 4, 3)$

Table 2. The Euler operator expressed in simple Cartan elements

Essential in the sequel is that in all cases $c_j > 0$, for all $j = 1, \ldots, \ell_i$.

As $\bar{\mathfrak{g}}$ is semi-simple, the representation theory of (finite-dimensional) representations is well-established. In our case the representation to consider is I_V. We will study this representation by its lowest weight vectors. We say that v is x-independent if v is independent of all x_α $(\alpha \in \hat{\mathcal{R}})$.

Lemma 5.1. *For* $v \in I_V$ *the following statements are equivalent:*

- v *is a lowest weight vector for* $\bar{\mathfrak{g}}$*;*
- v *is an* x*-independent lowest weight vector for* $\bar{\mathfrak{g}}_0$*.*

Proof. First suppose v is lowest weight vector for $\bar{\mathfrak{g}}$. Since $\partial_\alpha \in \bar{\mathfrak{g}}_{-1}$, we have $[\partial_\alpha, v] = 0$, hence v is independent of all x_α. Moreover v is a lowest weight vector for $\bar{\mathfrak{g}}_0 \subset \bar{\mathfrak{g}}$ as well. The converse is similar. ∎

This lemma is important for us, as it turns the study of lowest weight vectors of $\bar{\mathfrak{g}}$ to the study of x-independent lowest weight vectors of $\bar{\mathfrak{g}}_0$. Note that

$$\bar{\mathfrak{g}}_0 = \mathcal{D}\bar{\mathfrak{g}}_0 \times \mathbb{C}^r,$$

where $\mathcal{D}\bar{\mathfrak{g}}_0 = [\bar{\mathfrak{g}}_0, \bar{\mathfrak{g}}_0]$ and $\mathbb{C}^r$ is the r-dimensional center of $\bar{\mathfrak{g}}_0$ spanned by the r partial Euler operators in $\bar{\mathfrak{g}}_0$. These Euler operators belong to the (chosen) Cartan subalgebra of $\bar{\mathfrak{g}}$, and therefore acts diagonizably on I_V.

For the time being we only consider $v \in \mathcal{W}_n$, independent of x, and study the $\bar{\mathfrak{g}}_0$-action on it. We will denote

$$\mathcal{W}[y] = \{v \in \mathcal{W}_n | v = \sum_{i=1}^{n-m} P_i(y)\partial_{y_i}\}.$$

If $v \in I_V \cap \mathcal{W}[y]$ then, due to the complete reducibility of $\bar{\mathfrak{g}}_0$-modules and lemma 5.1, we see that v is in the $\bar{\mathfrak{g}}_0$-module, generated by all lowest weight vectors of $\bar{\mathfrak{g}}$. Note, that since $\bar{\mathfrak{g}}_0$ does not increase the (polynomial) degree of the vector field that it acts upon, any $v \in \mathcal{W}[y]$ generates a finite-dimensional $\bar{\mathfrak{g}}_0$-module. However, a priori, the $\bar{\mathfrak{g}}$-module generated by v can be infinite-dimensional. Let us investigate this situation in more detail.

First consider the space $V = \langle \partial_{y_1}, \partial_{y_2}, \ldots, \partial_{y_{n-m}} \rangle$, which is a $\bar{\mathfrak{g}}_0$-submodule of $\mathfrak{L}$. In general $\bar{\mathfrak{g}}_0$ does not act irreducibly on V. Correspondingly, we split V,

$$V = \bigoplus_i V_{(i)},$$

where each $V_{(i)}$ is an irreducible module. We choose coordinates in $V_{(i)}$ such that the Cartan elements $\bar{h}_\alpha$ act diagonally, i.e. in these coordinates,

$$\bar{h}_\alpha = \sum_{i=1}^{m} a_i x_i \partial_{x_i} + \sum_{j=1}^{n-m} b_j y_j \partial_{y_j}.$$

From [10] we have some more knowledge on the modules in V. We denote by $\mathfrak{N}$ the largest nilpotent ideal in $\mathfrak{L}$. Remember that

$$\mathfrak{N} = \{X \in \mathfrak{R} | \text{ ad } X \text{ is nilpotent}\}$$

In [10] it is proved that there exists a flag $0 = W_{t+1} \subset W_t \subset \cdots \subset W_1 = V$ of subspaces in V such that $[\mathfrak{N}, I_{W_{i+1}}] \subset I_{W_i}$ and moreover $[\mathfrak{L}, I_{W_i}] \subset I_{W_i}$. For a subspace $W \subset V$, we define

$$I_W = \{X \in \mathfrak{L} | X = \sum_{i=1}^{k} P_i(x, y, z)\partial_{y_i}\}.$$

Note that I_{W_i} is an ideal in $\mathfrak{L}$, not merely a linear subspace. In particular W_i is a $\bar{\mathfrak{g}}_0$-submodule of V. Hence any W_i is a direct sum of the irreducible $\bar{\mathfrak{g}}_0$-modules $V_{(j)}$. We choose coordinates $\partial_{y_j^{(i)}}$ $(i = 1, \ldots, t;\ j = 1, \ldots, t_i = \dim W_i - \dim W_{i-1}$ in V such that $W_{(i)} = \langle \partial_{y_1^{(i)}}, \ldots, \partial_{y_{t_i}^{(i)}} \rangle$ is a $\bar{\mathfrak{g}}_0$-submodule of V, and $W_k = \oplus_{i \geq k} W_{(i)}$. In these coordinates, the relation $[\mathfrak{N}, I_{W_{i+1}}] \subset I_{W_i}$ turns into:

Any $X \in \mathfrak{N}$ is of the form

$$X = \sum_{i,j} P_{ij}(x, y^{(1)}, \ldots, y^{(i-1)})\partial_{y_j^{(i)}}, \tag{3}$$

while $[\mathfrak{L}, I_{W_i}] \subset I_{W_i}$ tells that in any term $P(x,y)\partial_{y_j^{(i)}}$, occurring in $X \in \mathfrak{L}$ we have

$$P(x,y) = P(x, y^{(1)}, \ldots, y^{(i)}) \tag{4}$$

In fact for terms in Q_α we can do slightly better.

Lemma 5.2. *Let Q_α be as in proposition 4.1, and suppose $y_i y_j \partial_{y_k}$ is a term in Q_α. Then if $y_i \in W^*_{(i_1)}$ and $y_j \in W^*_{(i_2)}$, then $\partial_{y_k} \in W_{(i_3)}$ with $i_3 > i_1$ and $i_3 > i_2$.*

Proof. Since $\partial_{y_i} \in \mathfrak{N}$ for all $i = 1, \ldots, n-m$, we obtain $[\partial_{y_i}, Q_\alpha] \in \mathfrak{N}$. The lemma now follows from (3). ∎

Corollary 5.3. *Suppose V is irreducible as a $\bar{\mathfrak{g}}_0$-module. Then $Q_\alpha = 0$ for all $\alpha \in \hat{\mathcal{R}}$.*
The situation above holds in particular if $\dim V = 1$.

Throughout the remainder of this section, we assume that V contains no *trivial* $\bar{\mathfrak{g}}_0$-modules. The case that V contains some trivial $\bar{\mathfrak{g}}_0$ modules is postponed to section 6.. If V contains no trivial submodules, then any y_i occurs in some $\bar{h}_j$, because $[h_j, v] = 0$ for all j implies $[X, v] = 0$ for all $X \in \bar{\mathfrak{g}}_0$: $v \in V$ generates a finite-dimensional $\bar{\mathfrak{g}}_0$-module inside V, with all weights 0.

Lemma 5.4. *Suppose V contains no trivial $\bar{\mathfrak{g}}_0$-modules. Then $\mathcal{W}[y]$ contains only a finite number of linearly independent $\bar{\mathfrak{g}}$-lowest weight vectors, that generate a finite-dimensional $\bar{\mathfrak{g}}$-module.*

Proof. Remember that a $\bar{\mathfrak{g}}$-lowest weight vector v is automatically in $\mathcal{W}[y]$, and if it generates a finite-dimensional module, then $h_i(v) = -\lambda_i v$ with all λ_i non-negative integers. Since the Euler operator $\bar{E}^{(j)} \in \bar{\mathfrak{g}}^{(j)}$ is a positive sum of the $h_k^{(j)}$, this implies that $\bar{E}^{(j)}(v) = -\lambda v$ with $\lambda \geq 0$ for all lowest weight vectors v that generate a finite-dimensional $\bar{\mathfrak{g}}$-module. Since $\bar{E}^{(j)}$ is central in $\bar{\mathfrak{g}}_0$, it is a scalar on the irreducible $\bar{\mathfrak{g}}_0$-submodules; in particular $\bar{E}^{(j)}$ in constant on each $V_{(i)}$. Hence $[\bar{E}^{(j)}, \partial_{y_i}] = -b_i \partial_{y_i}$, with $b_i \geq 0$ for all $i = 1, \ldots, k$. So if we define

$$\bar{E} = \sum_j \bar{E}^{(j)} = \sum_{i=1}^{m} x_i \partial_{x_i} + \sum_{i=1}^{n-m} b_i y_i \partial_{y_i},$$

we have $b_i \geq 0$ for all i. Since y_i appears in at least one $h_k^{(j)}$ (and hence in at least one $\bar{E}^{(j)}$), we even have that $b_i > 0$ for all $i = 1, \ldots, n-m$. We put $b = \max_i b_i$. Then for a monomial $y_1^{\alpha_1} y_2^{\alpha_2} \ldots y_k^{\alpha_k} \partial_{y_a}$ we obtain

$$[\bar{E}, y_1^{\alpha_1} y_2^{\alpha_2} \ldots y_k^{\alpha_k} \partial_{y_a}] = \beta y_1^{\alpha_1} y_2^{\alpha_2} \ldots y_k^{\alpha_k} \partial_{y_a}$$

with $\beta \geq \alpha_i b_i - b$ for all $i = 1, \ldots, n-m$. If the lowest weight vector $v \in \mathcal{W}[y]$ generates a finite-dimensional $\bar{\mathfrak{g}}$-module, and v contains the term $y^\alpha \partial_{y_a}$ then necessarily $\alpha_i \leq b/b_i$, for all $i = 1, \ldots, n-m$. This restrict the possibilities for v generating a finite-dimensional $\bar{\mathfrak{g}}$-module to a finite-dimensional space. ∎

As a consequence to the proof of the lemma above, we state

Corollary 5.5. *Consider the $\bar{\mathfrak{g}}$-module S, generated by 1 and the monomials x_i, for $i = 1, \dots, m$. The only finite-dimensional submodule in S is $\langle 1 \rangle$.*

Proof. The method of proof is the same as above; we have $[\bar{E}, x_1^{a_1} \dots x_m^{a_m}] = (a_1 + a_2 + \dots + a_m) x_1^{a_1} \dots x_m^{a_m}$. Hence the lowest weight vectors, except those in $\langle 1 \rangle$ all have positive weights; so all generated modules are infinite-dimensional. ■

Lemma 5.4 allows is to prove the following proposition.

Proposition 5.6. *Suppose $\mathfrak{L}$ is such that V contains no trivial $\bar{\mathfrak{g}}_0$-modules. Then $\mathfrak{L}$ is contained in a maximal graded transitive Lie algebra $\mathfrak{L}'$.*

Proof. There are three cases to consider:

(a) $V = \{0\}$, so $\mathfrak{L}$ is semi-simple. Suppose $\mathfrak{L}' \supset \mathfrak{L}$. Then we can decompose $\mathfrak{L}'$ into irreducible $\mathfrak{L}$-modules. Let M be a submodule. If $X \in M$, then by suitable differentiations, hence action with elements $\partial_{x_i} \in \mathfrak{L}$ we obtain that $\partial_{x_j} \in M$. Hence $M \subset \mathfrak{L}$, and $\mathfrak{L}' = \mathfrak{L}$. So $\mathfrak{L}$ itself is already maximal.

(b) $V \neq \{0\}$ and $V \neq U_{-1}$. We construct $\mathfrak{L}'$ by adding to $\mathfrak{L}$ *all* finite-dimensional submodules in $\mathcal{W}_n$. According to lemma 5.4 $\mathfrak{L}'$ is finite-dimensional, and taking commutators among elements in $\mathfrak{L}'$ does not take us outside $\mathfrak{L}'$. Clearly, $\mathfrak{L}'$ is maximal.

(c) $V = U_{-1}$. Then $\mathfrak{L}$ is contained in a multi-graded transitive Lie algebra $\mathfrak{L}'$, which can be chosen maximal (see [10, 9]). ■

A special situation occurs when $Q_\alpha = 0$ for all $\alpha \in \hat{\mathcal{R}}$. In this case we can characterize the finite-dimensional $\bar{\mathfrak{g}}$-modules by the weight.

Lemma 5.7. *Let $\mathfrak{L}$ be such that $[V, [V, \bar{\mathfrak{g}}]] = 0$, and suppose that $v \in \mathcal{W}[y]$ is a lowest weight vector for $\bar{\mathfrak{g}}$, satisfying $[h_i, v] = -\lambda_i v$ with all λ_i non-negative integers. Then v generates a finite-dimensional $\bar{\mathfrak{g}}$-module.*

Proof. It is enough to prove that $(\mathrm{ad}\ \bar{e}_i)^{\lambda_i+1} v = 0$, as these last relations are the defining relations for the irreducible module of lowest weight $(\lambda_1, \dots, \lambda_m)$ (see [3], §21). Due to the structure of $sl_2(\mathbb{C})$-modules this is equivalent to proving that $(\mathrm{ad}\ \bar{e}_i)^a v = 0$ for a natural number a. For $\bar{e}_i \in \bar{\mathfrak{g}}_0$ this is clear, since the $\bar{\mathfrak{g}}_0$-module generated by v is finite-dimensional. So now consider $\bar{e}_s^{(i)}$, with $\deg(\bar{e}_s) = 1$. (We use the notations from section 4., but write $\bar{X}_0 = \bar{e}_s^{(i)}$ omitting $^{(i)}$ everywhere.) This element has the form

$$\bar{X}_0 = \frac{1}{2} \sum x_\beta L_0^\beta + \sum x_\beta (\bar{L}_0^\beta - L_0^\beta).$$

Since v is x-independent, we have

$$[\bar{X}_0, v] = \sum x_\beta [\bar{L}_0^\beta, v].$$

Now $\bar{L}_0^\beta = [\partial_\beta, \bar{X}_0] \in \bar{\mathfrak{g}}_-$ for $\beta \neq 0$. Hence in $[\bar{L}_0^\beta, v]$ only the term $[\bar{L}_0^0, v] = [-\bar{h}_s, v]$ contributes. Hence

$$[\bar{X}_0, v] = x_0[-\bar{h}_s, v] = x_0\lambda_s v.$$

The next step gives, by using $[\bar{X}_0, x_0] = [\frac{1}{2}x_0\bar{h}_s, x_0] = -x_0^2$ (lemma 3.4)

$$[\bar{X}_0, [\bar{X}_0, x]] = [\bar{X}_0, x_0]\lambda_s v + x_0\lambda_s[\bar{X}_0, v] = -x_0^2\lambda_s v + x_0^2\lambda_s^2 v = (\lambda_s)(\lambda_s - 1)x_0^2 v.$$

Inductively we find

$$(\text{ad } \bar{X}_0)^a v = \prod_{i=0}^{a-1}(\lambda_s - i)x_0^a v.$$

Hence $(\text{ad } \bar{X}_0)^a v = 0$ for $a = \lambda_s + 1$. ∎

Remark 5.8. The characterization in lemma 5.7 does not hold when $Q_\alpha \neq 0$. As an example, consider in three variables $\mathfrak{L}$ with

$$\bar{\mathfrak{g}} = \langle \partial_x, 2x\partial_x + 2y_1\partial_{y_1} + 2y_2\partial_{y_2}, x^2\partial_x + 2xy_1\partial_{y_1} + 2xy_2\partial_{y_2} + y_1^2\partial_{y_2} \rangle.$$

In this case all $v \in \mathcal{W}[y]$ are $\bar{\mathfrak{g}}$-lowest weight vectors. The space of lowest weight vectors v for which $[h_1, v] = -\lambda v$ with $\lambda \geq 0$ is

$$\langle \partial_{y_1}, \partial_{y_2}, y_1\partial_{y_1}, y_1\partial_{y_2}, y_2\partial_{y_1}, y_2\partial_{y_2} \rangle,$$

while the space of lowest weight vectors generating a finite-dimensional $\bar{\mathfrak{g}}$-module is

$$\langle \partial_{y_1}, \partial_{y_2}, y_1\partial_{y_2}, y_1\partial_{y_1} + 2y_2\partial_{y_2} \rangle.$$

The graded transitive Lie algebra $\mathfrak{L}$ containing $\bar{\mathfrak{g}}$ above and $\langle \partial_{y_1}, \partial_{y_2}, y_1\partial_{y_2}, y_1 + 2y_2\partial_{y_2} \rangle$ is maximal (we added all $\bar{\mathfrak{g}}$-modules that are possible). This example appears in section 5 of [10] for the case $\lambda = 2$.

6. $\mathfrak{L}$ as a $\bar{\mathfrak{g}}_0$-module; trivial modules in V

Now we consider the case that V contains some trivial $\bar{\mathfrak{g}}_0$-modules, say $V_{(1)}, \ldots, V_{(n-m-k)}$ with coordinates $z_1, \ldots, z_{n-m-k}$. Let $y_1, \ldots, y_k$ be the remaining coordinates. From now on $\mathcal{W}[y]$ will refer to the variables $y_1, \ldots, y_k$ only. Since $[\bar{E}, y_i\partial_{z_j}] = b_i y_i \partial_{z_j}$ with $b_i > 0$ we see that terms of the form $y_i\partial_{z_j}$ are not present in I_V. Also $\bar{\mathfrak{g}}$ contains no z_i or ∂_{z_j}:

Lemma 6.1. *Let V contain $n - m - k$ trivial $\bar{\mathfrak{g}}$-modules, with coordinates $z_1, \ldots, z_{n-m-k}$. Then z_i and ∂_{z_j} do not occur in $\bar{\mathfrak{g}}$ $(i, j = 1, \ldots, n - m - k)$.*

Proof. Since, by assumption $[\partial_{z_i}, X] = 0$ for all $X \in \bar{\mathfrak{g}}_0$, the only place where z_i or ∂_{z_j} can appear is in the terms of Q_α in $X_\alpha \in \bar{\mathfrak{g}}_1$. Since all terms T in Q_α satisfy $[\bar{E}, T] = T$, we see that T contains a y_k, hence, like above, ∂_{z_j} is not possible. Hence the only possibility is $T = y_k z_i \partial_{y_j}$. But now $[\partial_{z_i}, T]$ is a term in $\mathfrak{N}$ with $\bar{E}$ eigenvalue $+1$. This is impossible. ∎

This lemma implies that $\mathfrak{L}$ with trivial modules is an extension of one without trivial modules, namely the algebra in x and y variables only. We denote this algebra by $\mathfrak{L}(x,y)$:

$$\mathfrak{L}(x,y) = \bar{\mathfrak{g}} \ltimes (I_V \cap \mathbb{C}[x_1,\dots,x_m]\mathcal{W}[y]).$$

We will study $\mathfrak{L}$ as an extension of $\mathfrak{L}(x,y)$. If we split $\mathfrak{L}(x,y)$ into irreducible $\bar{\mathfrak{g}}$-modules, we see that the z-dependence within such a module M is "constant": if $P(z)v \in \mathfrak{L}$, for $v \in M$, then also $P(z)w \in \mathfrak{L}$ for all $w \in M$. It follows that if M contains an element that doesn't act nilpotently in $\mathfrak{L}$, then $P(z)$ must be constant. This happens in particular for simple Levi subalgebras in $\mathfrak{L}$, as well as for $\mathfrak{R}/\mathfrak{N}$. Hence only $M \subset \mathfrak{N} \cap \mathfrak{L}(x,y)$ can get z-dependent coefficients.

Important is the following lemma.

Lemma 6.2. *Suppose $V \subset \mathfrak{L}$ contain the maximal trivial $\bar{\mathfrak{g}}_0$-module $\tilde{W} = \langle \partial_{z_1},\dots,\partial_{z_{n-m-k}}\rangle$ with complementary module $W = \langle \partial_{y_1},\dots,\partial_{y_k}\rangle$. Then I_W is an ideal in $\mathfrak{L}$.*

Proof. Suppose $Y \in I_W$. We need to prove that $[X,Y] \in I_W$ for all $X \in \mathfrak{L}$. There are 4 cases to consider:

(1) $\partial_{x_i} \in \bar{\mathfrak{g}}_{-1}$; $[\partial_{x_i}, Y] \in I_W$ is clear.

(2) $X \in \bar{\mathfrak{g}}_0$; since W is a $\bar{\mathfrak{g}}_0$-submodule, $[X,Y] \in I_W$.

(3) $X_\alpha \in \bar{\mathfrak{g}}_1$; since Q_α contains no terms of the form $y_i y_j \partial_{z_a}$ or $y_i z_j \partial_{z_a}$, $[X_\alpha, Y] \in I_W$.

(4) $X \in I_V$; the argument is the same is in (3).

Hence I_W is an ideal. ∎

From this lemma we can obtain a sequence of spaces W_i (and hence $W_{(j)}$) compatible with the z and y-variables. Let us describe this construction. First we consider $\mathfrak{L}(x,y) := \bar{\mathfrak{g}} \times (I_W \cap \mathcal{W}[y])$. If $Z(\mathfrak{N}')$ denotes the center of the nilradical of $\mathfrak{L}(x,y)$, then we define $W_t = Z(\mathfrak{N}') \cap W$, say $W_t = \langle y_1^{(t)},\dots,y_{i_t}^{(t)}\rangle$. Here t is the number of steps that has to be taken in the process below. By construction $[\partial_{y_i^{(t)}}, X] = 0$ for $i = 1,\dots,i_t$ and $X \in \mathfrak{N}'$; from this using lemma 6.2 one easily obtains that the variables $y_1^{(t)},\dots,y_{i_t}^{(t)}$ appear only in the coefficients of $\partial_{y_1^{(t)}},\dots,\partial_{y_{i_t}^{(t)}}$.

After this first step we continue with $\mathfrak{L}(x,y)/I_{W_s}$, and we obtain in a similar fashion W'_{t-1}, which in $\mathfrak{L}(x,y)$ lifts to $W_{t-1} = W'_{t-1} + W_t$. This process we repeat till W is exhausted; then we proceed with $\mathfrak{L}/I_W$. Hence we can choose the $\bar{\mathfrak{g}}_0$-modules $W_{(1)},\dots,W_{(t)}$ such that the first ones are subspaces of $\tilde{W} = \langle \partial_{z_1},\dots,\partial_{z_{n-m-k}}\rangle$, while the last ones are subspaces of $W = \langle \partial_{y_1},\dots,\partial_{y_k}\rangle$. We can choose the y and z coordinates such that $W_{(1)},\dots,W_{(t)}$ are spanned by monomials.

Proposition 6.3. *Let $\mathfrak{L}$ be graded transitive Lie algebra, such that V contains trivial $\bar{\mathfrak{g}}_0$-modules. Then: $\mathfrak{L}$ is not maximal.*

Proof. We construct the spaces $W_{(1)},\dots,W_{(t)}$ as described above. Consequently, any $X \in \mathfrak{N}$ is of the form

$$X = \sum_{i,j} P_{ij}(z^{(1)},\dots,z^{(i-1)}))\partial_{z_j^{(i)}} + \sum_{i,j} Q_{ij}(z, y^{(1)},\dots,y^{(i-1)})\partial_{y_j^{(i)}}. \tag{5}$$

We give to terms of the form occurring in (5) a $\mathbb{Z}$-valued degree, which we call the zdeg, such that all terms in $\mathfrak{N}$ have non-positive degree. (Remember that z-dependent terms only occur in $\mathfrak{N}$.) To start we put $\mathrm{zdeg}(z_j^{(1)}) = 1 =: d_1$ for all j. Next for $X \in \mathfrak{N}$ we write

$$X = \sum_j P_{2j}(z^{(1)})\partial_{z_j^{(2)}} + \text{ other terms}$$

Let $d_2'(X)$ be the maximum over the zdeg of the monomials occurring in $P_{2,j}$ and $d_2'' = \max_X d_2(X)$. We now put $\mathrm{zdeg}(z_j^{(2)}) = d_2 \geq d_2''$. This way we continue: for d_3 we take a (natural) number at least as big as the maximum over the zdeg of the monomials occurring in P_{3j}, and we put $\mathrm{zdeg}(z_j^{(3)}) = d_3$. In the end we gave all z and y variables a degree, which is the same (namely d_i) for variables belonging to the same module $W_{(i)}$.

By construction, all terms T occurring in $X \in \mathfrak{N}$ have $\mathrm{zdeg}(T) \leq 0$. Moreover there are only finitely many terms T with $\mathrm{zdeg}(T) \leq 0$. Thanks to (4) and recalling that $I_V \cap \mathfrak{L}_i \subset \mathfrak{N}$ for $i \neq 0$, we see that all terms occurring in I_V have $\mathrm{zdeg}(T) \leq 0$.

Now we make an extension of $\mathfrak{L}$, which we denote by $(\mathfrak{L}(x,y),d)$, where $d = (d_1,\dots,d_t)$. We add to $\mathfrak{L}(x,y)$ the following z-dependent elements

$$\langle z_1^{a_1}\dots z_{n-m-k}^{a_{n-m-k}} X | X \in \mathcal{W}[y] \cap \mathfrak{N}\rangle \oplus \langle z_1^{b_1}\dots z_{n-m-k}^{b_{n-m-k}}\partial_{z_b}\rangle \tag{6}$$

with $\sum_i a_i\,\mathrm{zdeg}(z_i) \leq -\mathrm{zdeg}(T)$ for all terms T in X, and $\sum_i b_i\,\mathrm{zdeg}(z_i) \leq \mathrm{zdeg}(z_b)$. By construction, $(\mathfrak{L}(x,y),d) \supset \mathfrak{L}$, and $(\mathfrak{L}(x,y),d)$ is a graded transitive Lie algebra. However, increasing d_t to $d_t' = d_t + 1$, while fixing $d_1,\dots,d_{t-1}$, we obtain $\mathfrak{L}'$, which properly contains $\mathfrak{L}$: $z_1^{d_t'}\partial_{y_k}$ is in $\mathfrak{L}'$, but not in $\mathfrak{L}$. Hence $\mathfrak{L}$ is not maximal. ∎

Remark 6.4. When we write $(\mathfrak{L}(x,y),d)$, we assume throughout that we have a sequence of space $W_{(i)}$ like above, $i = 1,\dots,t$, such that the first, say $\tilde{t}$ spaces consists of (all) trivial $\bar{\mathfrak{g}}_0$-modules, and $d_i = \mathrm{zdeg}(z_j^{(i)})$ for all $z_j^{(i)} \in W_{(i)}^*$. We can (and will) assume that d_i is a non-negative integer, but moreover that $d_i \geq 1$ for $i = 1,\dots,\tilde{t}$. Note that the zdegree is constant on all irreducible submodules $V_{(i)}$ in V.

The situation in lemma 6.3 is not completely satisfactory. Take an $\mathfrak{L}$ for which V contains some trivial $\bar{\mathfrak{g}}$-modules. Sometimes it is possible to construct a maximal $\mathfrak{L}'$ properly containing $\mathfrak{L}$. Let us denote V, $\bar{\mathfrak{g}}$ etc. corresponding to $\mathfrak{L}'$ by V', $\bar{\mathfrak{g}}'$ etc. From proposition 6.3 we know that V' contains no trivial $\bar{\mathfrak{g}}'$-modules: hence some variables among $z_1,\dots,z_{n-m-k}$ (say the first a) are turned into x-variables. Hence

$$\bar{\mathfrak{g}}' = \bar{\mathfrak{g}}^{(1)} \times \dots \times \bar{\mathfrak{g}}^{(r)} \times \dots \times \bar{\mathfrak{g}}^{(r')},$$

with $r' > r$. Corresponding to $\bar{\mathfrak{g}}^{(r+1)} \times \cdots \times \bar{\mathfrak{g}}^{(r')}$ we have the partial Euler operator, denoted by Z:

$$Z = \sum_{i=1}^{a} z_i \partial_{z_i} + \sum_{i=a+1}^{n-m-k} c_i z_i \partial_{z_i} + \sum_{j=1}^{k} b_j y_j \partial_{y_j}.$$

Like in lemma 5.4 we have that $c_i > 0$ and $b_j \geq 0$ for all i and j. It seems difficult to investigate whether a maximal $\mathfrak{L}'$ containing $\mathfrak{L}$ always exists. A criterion is given in the following lemma.

Lemma 6.5. *Suppose $V \subset \mathfrak{L}$ contains trivial $\bar{\mathfrak{g}}_0$-modules. If there exists an $\mathfrak{L}' = (\mathfrak{L}'(x,y), d)$ such that*

(1) $\mathfrak{L}$ is contained in $(\mathfrak{L}'(x,y), d)$;

(2) The vector field Y,

$$Y = \sum_{i,j} d_i y_j^{(i)} \partial_{y_j^{(i)}}$$

is in $\mathfrak{L}'(x,y)$,

then $\mathfrak{L}$ is contained in a maximal graded transitive Lie algebra.

Proof. We define the vector field Z,

$$Z = \sum_{i,j} d_i z_j^{(i)} \partial_{z_j^{(i)}} + \sum_{i,j} d_i y_j^{(i)} \partial_{y_j^{(i)}}.$$

By assumption (2) above, we have that $Z \in (\mathfrak{L}'(x,y), d)$. We will say that a term T in $\mathcal{W}_n$ has zdegree d if $[Z, T] = T$. Since $Z \in (\mathfrak{L}'(x,y), d)$, we can find a basis in $(\mathfrak{L}'(x,y), d)$ of homogeneous elements with respect to the zdegree. If $\{X_i\}$ is a zdegree-homogeneous basis in $\mathfrak{L}'$, then $\operatorname{zdeg}(X_i) < 0$ implies $X_i \in \mathfrak{N}'$. We now extend $(\mathfrak{L}'(x,y), d)$ to a maximal graded transitive Lie algebra. Let $z_1, \ldots z_a$ be the variables of zdegree 1. We consider the vector fields Z_i defined by

$$Z_i = z_i Z \text{ for } i = 1, \ldots, a.$$

We show that $\{X_j\} \cup \{Z_i\}$ ($j = 1, \ldots, \dim \mathfrak{L}'$ and $i = 1, \ldots, a$) form a basis of a Lie algebra, which we denote by $\mathfrak{L}''$. First direct calculation shows that $[Z, Z_i] = Z_i$ and $[Z_i, Z_j] = 0$. It remains to check $[Z_i, X_j]$. We distinguish several cases:

- $X_j = \partial_{z_c}$. Then

$$[Z_i, X_j] = [z_i Z, \partial_{z_c}] = z_i [Z, \partial_{z_c}] + [z_i, \partial_{z_c}] Z = -z_i \partial_{z_c} - Z$$

- $X_j = z_b \partial_{z_i}$. Then

$$[Z_i, X_j] = [z_i Z, z_b \partial_{z_i}] = [z_i, z_b \partial_{z_i}] Z = -Z_b$$

- $X_j \notin \mathfrak{N}'$ and X_j does not contain ∂_{z_i}. Hence $\operatorname{zdeg}(X_j) = 0$ and

$$[Z_i, X_j] = [z_i Z, X_j] = z_i [Z, X_j] + [z_i, X_j] Z = 0.$$

- The remaining case: $X_j \in \mathfrak{N}'$, $\mathrm{zdeg}(X_j) = d$ and X_j does not contain ∂_{z_i}:

$$[Z_i, X_j] = z_i[Z, X_j] + [z_i, X_j]Z = dz_i X_j.$$

If $d = 0$, then $[Z_i, X_j] = 0$, while if $d < 0$, then $z_i X_j \in \mathfrak{L}'$, as desired.

Finally we see that $\mathfrak{L}''$ contains an extra simple Levi subalgebra, isomorphic to $sl(a+1, \mathbb{C})$, with basis $\{\partial_{z_i}, z_i\partial_{z_j}\ (i \neq j), Z + z_i\partial_{z_i}, Z_i\}$ where $i, j = 1, \ldots, a$. The corresponding Euler operator is exactly Z. Since $[Z, \partial_{z_j^{(i)}}] = d_i \partial_{z_j^{(i)}}$ for all i, j and by our assumption $d_i \neq 0$ for all $i \leq \tilde{t}$ (remark 6.4), we see that in $V'' = U_{-1} \cap \mathfrak{N}''$ there are no trivial $\bar{\mathfrak{g}}_0''$-modules. By proposition 5.6, $\mathfrak{L}''$ (and hence $\mathfrak{L}$) is contained in a maximal graded transitive Lie algebra. ■

Let us demonstrate how this lemma can be used. We say that a term T,

$$T = x_1^{a_1} \ldots x_m^{a_m} y_{j_1}^{(i_1)} \ldots y_{j_b}^{(i_b)} \partial_{y_j^{(i)}}$$

has type $(i_1, \ldots, i_b; i)$. Here we can assume that $i_1 \leq i_2 \cdots \leq i_b$. We say that a vector field v has type $(i_1, \ldots, i_b; i)$ if all terms in v are of this type. Note that all y-dependent term that occur in $\bar{\mathfrak{g}}_0$ are of type $(i; i)$. Consequently, all terms in a lowest weight vector of a certain type, form by themselves also a lowest weight vector for $\bar{\mathfrak{g}}_0$. Unfortunately, the terms Q_α that appear in $X_\alpha \in \bar{\mathfrak{g}}_1$ break the decomposition in different types. But if all $Q_\alpha = 0$ then we obtain the following result.

Proposition 6.6. *Let $\mathfrak{L}$ be such that $[V, [V, \bar{\mathfrak{g}}]] = 0$. Then $\mathfrak{L}$ is contained in a maximal graded transitive Lie algebra.*

Proof. We can extend $\mathfrak{L}(x, y)$ to a maximal $\mathfrak{L}'(x, y)$ (in $\mathcal{W}_{m+k}$) by proposition 5.6. Since all terms in $\bar{\mathfrak{g}}$ preserve the type of a term ($Q_\alpha = 0$!), we can assume that the $\bar{\mathfrak{g}}$-modules in I_V are generated be lowest weight vectors of a certain type. It is possible to choose $d = (d_1, \ldots, d_t)$ such that $(\mathfrak{L}'(x, y), d)$ contains $\mathfrak{L}$. Consider

$$Y = \sum_{i,j} d_i y_j^{(i)} \partial_{y_j^{(i)}}.$$

Clearly $[Y, X] = 0$ for all vector fields X of type $(i; i)$; hence $[Y, \bar{\mathfrak{g}}] = 0$, which implies $Y \in \mathfrak{L}'(x, y)$. Applying lemma 6.5, we find that $\mathfrak{L}$ is contained in a maximal graded transitive Lie algebra. ■

7. The low dimensional cases

We now apply the results of the previous sections to describe all maximal graded transitive Lie algebras $\mathfrak{L}$ in the cases $n \leq 3$.

Let us start with $n = 1$. Since $m \geq 1$, we see that $n = m$, and hence $\mathfrak{L} = \mathfrak{g}$. So $\mathfrak{L}$ is simple, and looking at table 1. we obtain that $\mathfrak{L} = A_{1;1}$, i.e. $\mathfrak{L} = \langle \partial_x, x\partial_x, x^2\partial_x \rangle$.

For $n = 2$ we have the possibilities $m = 1$ and $m = 2$. If $m = 1$, then $\mathfrak{g} = A_{1;1}$ as above, and $\dim V = 1$. Using lemma 5.2, this implies that

$$\bar{\mathfrak{g}} = \langle \partial_x, x\partial_x + b_1 y\partial_y, x^2\partial_x + \frac{1}{2} b_1 xy\partial_y \rangle.$$

with $b_1 \geq 0$. If $b_1 \neq 0$, we can add the lowest weight vectors ∂_y and $y\partial_y$, and obtain the algebra

$$\mathfrak{L} = \langle \partial_x, x\partial_x + b_1 y\partial_y, x^2\partial_x + \frac{1}{2}b_1 xy\partial_y, y\partial_y, x^k\partial_y \ (k \leq b_1)\rangle.$$

If $b_1 = 0$ then $\mathfrak{L}$ is contained in $A_{1;1} \times A_{1;1}$, hence $\mathfrak{L}$ is not maximal. If $m = 2$ there are two possibilities: either $\mathfrak{L}$ is simple, $\mathfrak{L} = A_{2;1}$ or $\mathfrak{L}$ is semi-simple, $\mathfrak{L} = A_{1,1} \times A_{1;1}$. Note that up to now all algebras are multi-graded.

We continue with $n = 3$, hence $m = 1$ or $m = 2$ or $m = 3$. If $m = 1$ then $\mathfrak{g} = A_{1;1}$ as above. Now $\mathfrak{g}_0 = \mathbb{C}$, hence all irreducible representations are of dimension 1. Again using lemma 5.2 we obtain

$$\bar{\mathfrak{g}} = \langle \partial_x, x\partial_x + b_1 y_1\partial_{y_1} + b_2 y_2\partial_{y_2}, x^2\partial_x + \frac{1}{2}b_1 xy_1\partial_{y_1} + \frac{1}{2}b_1 xy_1\partial_{y_1} + cy_1^2\partial_{y_2}\rangle. \quad (7)$$

If $b_1 = 0$ or $b_2 = 0$, then $c = 0$ (lemma 6.1), and $\mathfrak{L}$ is not maximal (proposition 6.3), but is contained in a maximal one (proposition 6.6). So consider $c = 0$ and we can assume that $b_2 \geq b_1 > 0$. Moreover b_1 and b_2 are integers. If b is the biggest integer β with $\beta \leq b_2/b_1$ then we obtain that $\mathfrak{L}$ is generated by $\bar{\mathfrak{g}}$ and the $\bar{\mathfrak{g}}$-lowest weight vectors $y_1\partial_{y_1}, y_2\partial_{y_2}, \partial_{y_1}$ and $y_1^\beta\partial_{y_2}$ with $\beta \leq b$.

If $c \neq 0$, then for $X = x^2\partial_x + \frac{1}{2}b_1 xy_1\partial_{y_1} + \frac{1}{2}b_1 xy_1\partial_{y_1} + cy_1^2\partial_{y_2}$ we need

$$[x\partial_x + b_1 y_1\partial_{y_1} + b_2 y_2\partial_{y_2}, X] = X,$$

which implies that $b_1 + \frac{1}{2}b_2 = 1$, so $b_2 = 2(b_1 - 1)$. The lowest weight vectors that generate a finite-dimensional $\bar{\mathfrak{g}}$-module are given by

$$\partial_{y_1}, \partial_{y_2}, y_1\partial_{y_2}, y_1\partial_{y_1} + 2y_2\partial_{y_2},$$

compare to remark 5.8. This finishes the case $m = 1$. If $m = 2$, then either $\mathfrak{g} = A_{2;1}$ or $\mathfrak{g} = A_{1;1} \times A_{1;1}$. In both cases the lowest weight vectors in I_V are given by ∂_{y_1} and $y_1\partial_{y_1}$. If $m = 3$ then by table 1 there are 4 possibilities: $A_{3;1}$, $A_{2;1} \times A_{1;1}$, $A_{1;1} \times A_{1;1} \times A_{1;1}$ and B_2 (or the isomorphic C_2).

We will not discuss the case $n = 4$ completely. However, there is one interesting case for $\mathfrak{L}$, namely $m = 1$ and $k = 2$, i.e. V contains one trivial $\bar{\mathfrak{g}}_0$-module. Only in this case, we have a trivial $\bar{\mathfrak{g}}_0$-module in V, while $Q \neq 0$ is possible; consequently we have no proposition guaranteeing that $\mathfrak{L}$ is contained in a maximal algebra. We show that it is, nevertheless. We have that $\bar{\mathfrak{g}}$ is as given in (7) with $b_2 = 2(b_1 - 1) > 0$. Further that $\mathfrak{L}(x, y)$ is contained in the maximal algebra $\mathfrak{L}'(x, y)$ with lowest weight vectors (see above) $\partial_{y_1}, \partial_{y_2}, y_1\partial_{y_2}, y_1\partial_{y_1} + 2y_2\partial_{y_2}$. We try to apply lemma 6.5, with $d_1 = 1$ (as always), $d_2 = \delta$ and $d_3 = 2\delta$, hence $\delta y_1\partial_{y_1} + 2\delta y_2\partial_{y_2} \in \mathfrak{L}'(x, y)$, and δ is chosen big enough to ensure that $\mathfrak{L}$ is contained in $(\mathfrak{L}'(x, y), d)$. That this is possible is not obvious, since d_2 and d_3 are connected by a relation (otherwise we would increase first d_2 and then d_3 sufficiently). Let a and b be the maximal numbers such that the terms $z^a\partial_{y_1}$ and $z^b\partial_{y_2}$ appear in $\mathfrak{L}$. Now choose $\delta = \max\{a, b\}$. Then $\text{zdeg}(z^a\partial_{y_1}) = a - \delta \leq 0$ and $\text{zdeg}(z^b\partial_{y_2}) \leq b - 2\delta \leq 0$. It remains to check the term $z^c\partial_{y_1}\partial_{y_2}$ appearing in $\mathfrak{L}$. However taking the commutator of this term with ∂_{y_1} we see that $c \leq b$. Hence $\text{zdeg}(z^c\partial_{y_1}\partial_{y_2}) \leq b + \delta - 2\delta \leq 0$. Applying lemma 6.5, we see that $\mathfrak{L}$ is contained in a maximal $\mathfrak{L}''$ with $\bar{\mathfrak{g}}'' = A_{1;1} \times A_{1;1}$.

8. Conclusion

We were able to give a fairly detailed description of the structure of graded transitive Lie algebras $\mathfrak{L}$. In particular the maximal ones are described. It turns out that these maximal ones have so much structure that they can be described by general (n independent) theorems. Two questions are not answered completely satisfactory. First the problem of Q_α; can we describe in more detail when $Q_\alpha \neq 0$ is possible? Second, if $Q_\alpha \neq 0$ and V contains trivial $\bar{\mathfrak{g}}_0$-modules, we do not know whether $\mathfrak{L}$ is contained in a maximal one. Do there exist $\mathfrak{L}$ not contained in a maximal one? For such $\mathfrak{L}$, according to section 7., $n \geq 5$.

Going back to Lie's original problem, we can hope that these results will help in two directions. First the graded, but non-transitive case, and second the transitive, but not necessarily graded case. The last problem is particularly interesting, but also very difficult. The realization theorem of Guillemin and Sternberg (see [2]) states that any pair $(\mathfrak{L}; \mathfrak{L}_+)$ of abstract Lie algebras, where $\mathfrak{L}_+$ is a subalgebra in $\mathfrak{L}$ of codimension n containing no ideals of $\mathfrak{L}$, can be realized as a transitive Lie algebra of formal vector fields in n variables. Moreover this realization is unique up to a (formal) change of coordinates. In particular, if $\mathfrak{L}$ is simple, $\mathfrak{L}_+$ can be chosen arbitrary. Hence the classification of finite-dimensional transitive Lie algebras of formal vector fields entails the classification of all subalgebras of a simple Lie algebra.

Appendix – Notations

For the benefit of the reader, we provide a list with notations that are used in different sections.

Name	Description
n	dimension of the space
$\mathcal{W}_n$	Lie algebra of polynomial vector fields on $\mathbb{C}^n$
U_k	vector fields in $\mathcal{W}_n$ of degree k
$\mathfrak{L}$	graded transitive Lie algebra (subalgebra of $\mathcal{W}_n$)
E	Euler operator
$\mathfrak{R}$	radical of $\mathfrak{L}$ (largest solvable ideal in $\mathfrak{L}$)
$\mathfrak{N}$	nilradical of $\mathfrak{L}$ (largest nilpotent ideal in $\mathfrak{L}$)
V	$\mathfrak{R} \cap U_{-1}$ (constant vector fields in $\mathfrak{R}$)
I_V	$\{X \in \mathfrak{L} \| X = \sum_{i=m+1}^{n} P_i(x)\partial_{x_i}\}$
$\mathfrak{g}$	$\mathfrak{L}/I_V$ (semi-simple summands of $\mathfrak{L}$ modulo I_V)
$\bar{\mathfrak{g}}$	graded Levi subalgebra of $\mathfrak{L}$, such that $\bar{\mathfrak{g}}/I_V$ is isomorphic to $\mathfrak{g}$
m	$n - \dim V = \dim(U_{-1} \cap \bar{\mathfrak{g}})$
$\mathcal{R}$	roots of $\mathfrak{g}$ or $\bar{\mathfrak{g}}$
α_s	simple root in $\mathcal{R}$ such that $\deg e_s = 1$
$\hat{\mathcal{R}}$	all $\alpha \geq 0$ such that $\alpha + \alpha_s \in \mathcal{R}$
$\bar{E}^{(j)}$	Euler operator of j^{th} simple summand in $\bar{\mathfrak{g}}$
r	number of simple summands in $\bar{\mathfrak{g}}$
$\bar{E}$	sum of all $\bar{E}^{(j)}$
x	the variables $x_1, \ldots, x_m$ or x_α occurring in $\mathfrak{g}$
y	the variables in V^*, not in trivial modules
z	the variables in V^*, but not occurring in $\bar{\mathfrak{g}}$

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