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On the structure of transitively differential algebras

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Abstract. We study finite-dimensional Lie algebras of polynomial vector fields in n variables that contain the vector fields $\frac{\partial}{\partial x_i}$ $(i = 1, \dots, n)$ and $x_1\frac{\partial}{\partial x_1} + \cdots + x_n\frac{\partial}{\partial x_n}$. We derive some general results on the structure of such Lie algebras, and provide the complete classification in the cases $n = 2$ and $n = 3$. Finally we describe a certain construction in high dimensions.

1. Introduction

It was Lie [5] who started to study the classification problem for finite-dimensional Lie algebras $\mathfrak{L}$ of vector fields in $\mathbb{C}^n$ up to local diffeomorphisms. His results concern the cases $n = 1$, $n = 2$ and $n = 3$ (this last case is not treated completely). He split the study of these algebras in the case of primitive algebras (the corresponding local transformation group has no invariant foliation) and imprimitive ones. The lists for the imprimitive algebras in [5] are, in a sense, without structure: it is impossible to predict what (classes of) algebras one will encounter. The mathematical solution for this kind of problems is adding a structure: one can study for instance simple algebras or primitive algebras. For sure it is very difficult to obtain general results for the general classification problem. Also here we add some structure by assuming that

(O) the coefficients of $X \in \mathfrak{L}$ are polynomials;

(A) $\mathfrak{L}$ contains the translations $\frac{\partial}{\partial x_i}$ $(i = 1, \dots, n)$ ("$\mathfrak{L}$ acts transitive");

(B) $\mathfrak{L}$ contains the Euler vector field $E = \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i}$ ("$\mathfrak{L}$ is graded").

In fact, due to the presence of E in $\mathfrak{L}$, we may assume that the coefficients P_i of $X = \sum P_i(x)\frac{\partial}{\partial x_i} \in \mathfrak{L}$ are all homogeneous polynomials of the same degree k; this follows by considering the commutator $[E, X]$.

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Note that in any finite-dimensional Lie algebra $\mathfrak{g}$ of C^∞-vector fields in $\mathbb{C}^n$ we have a natural filtering by the minimal degree of the Taylor series (around 0) of the coefficients. The associated graded Lie algebra will be of the type we consider in case $\mathfrak{g}$ (or rather G) acts transitively (around 0). This explains a part of the interest in these algebras. For further motivations, we refer to [1].

Our aim in this paper is to present some general results on the structure of Lie algebras $\mathfrak{L}$ with the structure above. As background we have the classification in a special subcase, namely when $\mathfrak{L}$ is multigraded. Such $\mathfrak{L}$ is described (almost completely) by a diagram made from the integers a_{ij} with

$$a_{ij} = \max\{\alpha | x_i^\alpha \partial_{x_j} \in \mathfrak{L}\}.$$

We discuss this case in section 2..

It turns out that there are many cases in which $\mathfrak{L}$ is not multigraded. However assuming that $\mathfrak{L}$ is maximal (so that $\mathfrak{L}$ becomes a so called transitively differential algebra of certain order), most examples drop out; it is a non-trivial task to distinguish the remaining ones.

We describe the contents of the paper. In section 2. we discuss multigraded algebras, and in section 3. we explain how the elements from $\mathfrak{L}$ can be seen as multilinear mappings. In section 4. we give some general results on the Lie-theoretical structure of $\mathfrak{L}$. Our main result is that the radical is contained in a multigraded Lie algebra. Next we apply these structural results to the cases $n = 2$ and $n = 3$ (section 5.); we obtain a complete classification of transitively differential algebras here. In section 6. we describe an elegant construction in higher dimensions. We end with a discussion in section 7..

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2. Multigraded algebras

In this section we discuss the definition of (essential) multigraded algebras and describe some results obtained in [4].

Throughout this paper $\mathfrak{L}$ will denote a finite-dimensional Lie algebra of polynomial vector fields on $\mathbb{C}^n$, satisfying the requirements (A) and (B) in section 1.. For any $X \in \mathfrak{L}$, we see that all its homogeneous components are in $\mathfrak{L}$. If

$$X = \sum_{i=1}^{n} P_i(x) \frac{\partial}{\partial x_i}$$

where all P_i are homogeneous polynomials of degree k, then

$$[E, X] = (k-1) \cdot X. \tag{1}$$

We call k the *order* of X. The maximal order ν among $X \in \mathfrak{L}$ is called the *order* ν of $\mathfrak{L}$, $\mathrm{ord}(\mathfrak{L})$; in this case $\mathfrak{L}$ is said to belong to the *class* $\mathcal{D}^\nu(n)$ (or $\mathcal{D}^\nu$ if it is clear what n is).

It follows easily from (1) that $\mathfrak{L}$ is $\mathbb{Z}$-graded:

$$\mathfrak{L}_d = \{X \in \mathfrak{L} \mid \mathrm{ord}(X) = d+1\}$$

with

$$\mathfrak{L} = \bigoplus_{d=-1}^{\nu-1} \mathfrak{L}_d \quad \text{and} \quad [\mathfrak{L}_{d_1}, \mathfrak{L}_{d_2}] \subset \mathfrak{L}_{d_1+d_2}.$$

In particular we have $\mathfrak{L}_{-1} = \langle \partial_1, \ldots, \partial_n \rangle$. From now on we write $\partial_i = \dfrac{\partial}{\partial x_i}$ for notational convenience.

A transitively differential algebra of order ν is a maximal Lie algebra in $\mathcal{D}^\nu$. To be more explicit

Definition 2.1. Suppose $\mathfrak{L}$ belongs to $\mathcal{D}^\nu$ and $\mathfrak{L}$ is maximal in $\mathcal{D}^\nu$, which means that $\mathfrak{L}' \in \mathcal{D}^\nu$ and $\mathfrak{L}' \supset \mathfrak{L}$ implies $\mathfrak{L}' = \mathfrak{L}$, then $\mathfrak{L}$ is called a *transitively differential algebra of order* ν.

All $\mathfrak{L}$ in the class $\mathcal{D}^\nu$ are $\mathbb{Z}$-graded; sometimes will also encounter $\mathfrak{L}$ that are even $\mathbb{Z}^n$-graded. Remember that $\mathbb{C}[x_1, \ldots, x_n]$ has a $\mathbb{Z}^n$-graded, mdeg, by

$$\text{mdeg}(x^\alpha) = \alpha$$

Correspondingly, the polynomial vector fields, being realized as special elements of End$(\mathbb{C}[x_1, \ldots, x_n])$, attain a $\mathbb{Z}^n$-grading by

$$\text{mdeg}(x^\alpha \partial_j) = (\alpha_1, \ldots, \alpha_{j-1}, \alpha_j - 1, \alpha_{j+1}, \ldots, \alpha_n). \tag{2}$$

We say that $\mathfrak{L}$ is *multigraded* if it has a basis of homogeneous elements in the $\mathbb{Z}^n$-grading (2). Equivalently $\mathfrak{L}$ has a basis of simultaneous eigenvectors for the operators ad $x_1\partial_1$, ad $x_2\partial_2, \ldots$, ad $x_n\partial_n$. Hence we have a decomposition

$$\mathfrak{L} = \bigoplus_{\alpha \in \mathbb{Z}^n} \mathfrak{L}_\alpha \quad ; \quad [\mathfrak{L}_\alpha, \mathfrak{L}_\beta] \subset \mathfrak{L}_{\alpha+\beta}$$

where for $\alpha = (\alpha_1, \ldots, \alpha_n)$ we have

$$\{\mathfrak{L}_\alpha = \{X \in \mathfrak{L} \mid [x_i\partial_i, X] = \alpha_i X \quad \text{for} \quad i = 1 \ldots n\}$$

Consequently any multigraded transitively differential algebra contains the elements $x_1\partial_1, \ldots, x_n\partial_n$.

Let us discuss the structure of a multigraded Lie algebra $\mathfrak{L}$ in the class $\mathcal{D}^\nu$. We assume that the elements $x_1\partial_1, \ldots x_n\partial_n$ are in $\mathfrak{L}$; this can be done without restriction. According to [4] the remaining homogeneous elements $X \in \mathfrak{L}$ fall in two classes (up to a multiple):

$$X = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \partial_j \quad \text{with } \alpha_j = 0 \tag{3}$$

or ord$(X) = 2$ and

$$X = X_j = \sum_{i=1}^n a_{ji} x_i x_j \partial_i \quad \text{with } a_{jj} = 1. \tag{4}$$

If the element X_j is in $\mathfrak{L}$ then we call j a special point. There is a remarkable relation between the coefficients a_{ij} appearing in (4) and the maximal power α such that $x_i^\alpha \partial_j \in \mathfrak{L}$. Indeed, investigations in [4] reveal that $A = (a_{ij})$ is the matrix given by

$$a_{ij} = \max\{\alpha | x_i^\alpha \partial_j \in \mathfrak{L}\} \tag{5}$$

There are two characteristic conditions for A and the special points:

(I) $a_{ij}a_{jk} \leq a_{ik}$ for all $i, j, k = 1..n$;

(II) j is special and $a_{ij} \neq 0$ for an i implies $a_{ij} = a_{ji} = 1$ and moreover i is special as well.

Clearly, for any multigraded $\mathfrak{L}$ in the class $\mathcal{D}^\nu$ we can define the matrix A by formula (5), and mark the special points $\mathcal{S} \subset \{1, 2, \ldots, n\}$. A natural question is whether A and the special points $\mathcal{S}$ determine $\mathfrak{L}$. This is not the case, but it turns out that there is a unique maximal algebra $\mathfrak{L}(A, \mathcal{S})$ with matrix A and special points $\mathcal{S}$, see [4], section V. This provides a rather detailed description of all multigraded algebras.

Example 2.2. Let us take $n = 4$, $A = \begin{pmatrix} 1 & 3 & 11 & 11 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$ and $\mathcal{S} = \{1\}$. Then

$$\begin{aligned} \mathfrak{L}(A, \mathcal{S}) = \langle &\partial_1, x_i\partial_i \ (i = 1..4), x_3\partial_4, x_4\partial_3, x_1^2\partial_1 + 3x_1x_2\partial_2 + 11x_1x_3\partial_4 + 11x_1x_4\partial_4, \\ &x_1^k\partial_2 \ (k \leq 3), x_1^k x_2^\ell \partial_i \ (k + 3\ell \leq 11, \ell \leq 2, i = 3, 4)\rangle \end{aligned} \tag{6}$$

Now $\mathfrak{L}(A, \mathcal{S})$ contains multigraded subalgebras with same matrix A and the same special point 1. An example is

$$\begin{aligned} \mathfrak{L} = \langle &\partial_1, x_i\partial_i \ (i = 1..4), x_3\partial_4, x_4\partial_3, x_1^2\partial_1 + 3x_1x_2\partial_2 + 11x_1x_3\partial_4 + 11x_1x_4\partial_4, \\ &x_1^k\partial_2 \ (k \leq 3), x_1^k\partial_i \ (k \leq 11, i = 3, 4), x_1^k x_2\partial_i \ (k \leq 3, i = 3, 4), x_2^2\partial_i \ (i = 3, 4)\rangle \end{aligned}$$

In fact $\mathfrak{L}$ is the minimal multigraded Lie algebra with matrix A and special point 1.

If we define $A' = \begin{pmatrix} 1 & 3 & 11 & 11 \\ 0 & 1 & 3 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$ then $\mathfrak{L}(A, \mathcal{S}) \subset \mathfrak{L}(A', \mathcal{S})$. Hence $\mathfrak{L}(A, \mathcal{S})$ is not a transitively differential algebra of order 11, but $\mathfrak{L}(A', \mathcal{S})$ is.

One could hope that any multigraded transitively differential algebra $\mathfrak{L}$ is of the form $\mathfrak{L}(A, \mathcal{S})$. This is not the case, though $\mathfrak{L}$ is contained in $\mathfrak{L}(A, \mathcal{S})$, for certain A and $\mathcal{S}$. Possibly $\mathfrak{L}(A, \mathcal{S})$ is of higher order than $\mathfrak{L}$. The simplest example where this happens is $n = 3$ and

$$\mathfrak{L} = \langle \partial_i, x_j\partial_j, x_2\partial_3, x_2^2\partial_3, x_2^2\partial_2 + 2x_2x_3\partial_3, x_1\partial_3, x_1^2\partial_3, x_1x_2\partial_3, x_1x_2^2\partial_3\rangle.$$

Any $\mathfrak{L}(A, \mathcal{S}) \supset \mathfrak{L}$ contains the element $x_1^2x_2^2\partial_3$, which is of order 4, while $\mathrm{ord}(\mathfrak{L}) = 3$.

There is a convenient way to construct multigraded Lie algebras. Suppose we give all variables $x_1, x_2, \dots, x_n$ a degree:

$$\deg(x_1) = d_1;\ \deg(x_2) = d_2;\ \dots; \deg(x_n) = d_n;$$

with $d_i > 0$ for all i. This induces a degree on the vector field terms by

$$\deg(x_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n} \partial_j) = \alpha_1 d_1 + \alpha_2 d_2 + \dots + \alpha_n d_n - d_j. \tag{7}$$

Now suppose that $\mathfrak{L}$ is the Lie algebra with basis consisting of all vector field terms of degree 0 or less. By our assumption that $d_i > 0$ we have that $\mathfrak{L}$ is indeed finite-dimensional. Moreover it is clear that ∂_i and the Euler vector field E are contained in $\mathfrak{L}$. Hence $\mathfrak{L}$ belong to $\mathcal{D}^\nu$ for $\nu = \lfloor \max\{d_1, \dots, d_n\} / \min\{d_1, \dots, d_n\} \rfloor$.

Let us return to Lie algebras $\mathfrak{L}$ in the class $\mathcal{D}^\nu$, not necessarily multigraded. Let $L : \mathbb{C}^n \to \mathbb{C}^n$ be a linear, invertible transformation. This transformation induces a transformation on the polynomial vector fields (as represented in some coordinate system), preserving the order. In particular a Lie algebra in the class $\mathcal{D}^\nu$ is mapped to another one in $\mathcal{D}^\nu$; these two will be called *equivalent*. Note that the multigrading of an element is *not* preserved.

We will call $\mathfrak{L}$ *essentially multigraded* if $\mathfrak{L}$ is equivalent to a multigraded Lie algebra; this means that coordinates can be chosen such that $\mathfrak{L}$ is multigraded with respect to these coordinates.

To end this section, we state a lemma that gives a way to test if $\mathfrak{L}$ is possibly essential multigraded, without performing a linear transformation. For $X = \sum_{i=1}^{n} P_i \partial_i \in \mathfrak{L}$, we can form the Jacobian $J(X)$ with

$$J_{ij}(X) = \frac{\partial P_i}{\partial x_j}.$$

Lemma 2.3. *Let $\mathfrak{L}$ be in class $\mathcal{D}^\nu$, with $\nu \geq 3$ and $\dim(\mathfrak{L}) = d$. If $\mathfrak{L}$ is essentially multigraded, then $\mathfrak{L}$ has a basis $\{X_1, X_2, \dots, X_d\}$ such that rank $J(X) = 1$ for all $X \in \{X_1, X_2, \dots, X_d\}$ with $\mathrm{ord}(X) \geq 3$.*

Proof. The rank of $J(X)$ is independent of the (linear) coordinates chosen. Hence, if $\mathfrak{L}$ is essentially multigraded, we can choose coordinates such that $\mathfrak{L}$ is multigraded. But for multigraded Lie algebras there is a homogeneous basis $\{X_1, X_2, \dots, X_d\}$ such that the elements of order 3 and higher have the form $X = x_1^{\alpha_1} \dots x_n^{\alpha_n} \partial_j$, see formula (3). Those X obviously have rank $J(X) = 1$. ■

3. $\mathfrak{L}$ and multilinear mappings

A very useful description of vector fields is in term of multilinear mappings, see for example [2]. Let $X \in \mathfrak{L}$, $\mathrm{ord}(X) = k$, and denote $U_{-1} = \langle \partial_1, \dots, \partial_n \rangle$. We associate to X the k-linear map $A : U_{-1} \times U_{-1} \times \dots \times U_{-1} \to U_{-1}$ by

$$A(v_1, v_2, \dots, v_k) = \frac{1}{k!}[v_1, [v_2, \dots, [v_k, X] \dots]]$$

for $v_1, v_2, \ldots, v_k \in U_{-1}$. We will write $A = \varphi_k(X)$. Using the Jacobi identity one easily checks that A is symmetric. Therefore A is completely determined by its diagonal $A(x, \ldots, x)$, $x \in U_{-1}$. If $A = \varphi_k(X)$ $(k \geq 1)$ and $B = \varphi_\ell(Y)$ $(\ell \geq 1)$, then $C = \varphi_{k+\ell-1}([X, Y])$ satisfies

$$C(x, \ldots, x) = \ell B(A(x, \ldots, x), x, \ldots, x) - kA(B(x, \ldots, x), x, \ldots, x).$$

If $\operatorname{ord}(X) = -1$, then we put $v = X \in U_{-1}$ and we have

$$C(x, \ldots, x) = kB(v, x, \ldots, x)$$

Definition 3.1. Let V be a linear subspace of U_{-1}, and $\mathfrak{L}$ in $\mathcal{D}^\nu$.

(a) We call V a *reducing subspace* of $\mathfrak{L}$ if for all $k \geq 0$ and all $X \in \mathfrak{L}_{k-1}$, $A = \varphi_k(X)$ satisfies

$$A(v, x, \ldots, x) \in V$$

for all $v \in V$ and $x \in U_{-1}$.

(b) We call $\mathfrak{L}$ *reducible* if there exists a non-zero reducing subspace V for $\mathfrak{L}$, with $V \neq U_{-1}$. If such V does not exist, $\mathfrak{L}$ is called *irreducible.*

Lemma 3.2. *Let V be a reducing subspace of $\mathfrak{L}$ and define*

$$I_{V,k-1} = \{X \in \mathfrak{L}_{k-1} \mid \textit{ for } A = \varphi_k(X) : A(x, \ldots, x) \in V \textit{ for all } x \in U_{-1}\} \qquad (8)$$

and $I_V = \bigoplus_{k=0}^{\nu} I_{V,k-1}$. *Then : I_V is an ideal in $\mathfrak{L}$.*

Proof. Let us give a proof of this useful lemma[1]. Remember $A(v, x, \ldots, x)$ is (up to a factor k) simply the commutator of A of order k and $v \in U_{-1}$. Fix coordinates $\{\partial_1, \ldots, \partial_n\}$ of U_{-1} such that $\langle \partial_{r+1}, \ldots, \partial_n \rangle = V$. Since V is a reducing subspace, any $X \in \mathfrak{L}$ is of the form

$$X = \sum_{i=1}^{r} P_i(x_1, \ldots, x_r)\partial_i + \sum_{i=r+1}^{n} P_i(x_1, \ldots, x_n)\partial_i.$$

Define $\psi : \mathfrak{L} \to \bar{\mathfrak{L}}$ by

$$\psi(X) = \sum_{i=1}^{r} P_i(x_1, \ldots, x_r)\partial_i.$$

One easily checks that ψ is a Lie algebra morphism. Hence $I_V = \ker(\psi)$ is an ideal. ∎

[1] $\mathfrak{L}/I_V$ ("die verkürzte Gruppe") appears already in [5] as an important computational technique.

Example 3.3. Let us look at $\mathfrak{L}(A,\mathcal{S})$ define in (2.2). We see two reducing subspaces, $W_1 = \langle \partial_3, \partial_4 \rangle$ and $W_2 = \langle \partial_2, \partial_3, \partial_4 \rangle$. Lemma 3.2 tells that I_{W_2} is an ideal. Clearly $\mathfrak{L}(A,\mathcal{S})/I_{W_2} \cong sl_2(\mathbb{C})$. The ideal I_{W_1} contains (but is not equal to) $\langle x_3\partial_3 - x_4\partial_4, x_4\partial_3, x_3\partial_4 \rangle \cong sl_2(\mathbb{C})$.

Since $V \subset I_V$ we conclude that any simple $\mathfrak{L}$ is irreducible. The converse is also true for $\mathrm{ord}(\mathfrak{L}) \neq 1$; this follows from proposition 4.5 below (if $\mathfrak{N} \neq \{0\}$ then one finds $\mathfrak{N} = U_{-1}$, so $\mathrm{ord}(\mathfrak{L}) \leq 1$). A counter example for $\mathrm{ord}(\mathfrak{L}) = 1$ is the transitively differential algebra $\mathfrak{L}$ of order 1, $\mathfrak{L} = U_{-1} \oplus U_0$, of all polynomial vector fields of order 0 and 1.

Let us say some words on the case that $\mathfrak{L}$ is simple. This can only happen if $\mathrm{ord}(\mathfrak{L}) = 2$. Take any $X \in \mathfrak{L}$ of order 2. Then $A = \varphi_2(X)$ is a bilinear map $A : U_{-1} \times U_{-1} \to U_{-1}$. So A defines a multiplication $*$ on U_{-1}: $v * w = A(v,w)$ for $v, w \in U_{-1}$. This multiplication is commutative:

$$v * w = w * v \quad \Leftrightarrow \quad [v,[w,X]] = [w,[v,X]]$$

for all $v, w \in U_{-1}$, and moreover

$$(v * w) * v^2 = v * (w * v^2).$$

Commutative algebras with this property are called *Jordan algebras*; so there is a relation between algebras $\mathfrak{L}$ in $\mathcal{D}^2$ and Jordan algebras. This relation can be exploited to obtain a full classification of all simple Lie algebras; however it turns out that one has to define Jordan triples, a generalization of Jordan algebras. This way one gets a 1-1 correspondence between simple Lie algebras and simple Jordan triples; we refer to [3] for more information.

4. The Lie-theoretic structure of $\mathfrak{L}$

Let $\mathfrak{L}$ be a Lie algebra in $\mathcal{D}^\nu$. We first study the radical $\mathfrak{R}$ of $\mathfrak{L}$ (the maximal solvable ideal) and the nilradical $\mathfrak{N}$ of $\mathfrak{L}$ (the maximal nilpotent ideal). Of course, $\mathfrak{N} \subset \mathfrak{R}$; according to [6], § 5, N° 3, we have that

$$\mathfrak{N} = \{X \in \mathfrak{R} | \mathrm{ad}\, X \text{ is nilpotent}\}.$$

Consequently if $X \in \mathfrak{R}$ and $\mathrm{ord}(X) \neq 1$ then $X \in \mathfrak{N}$.

Proposition 4.1. *Let $\mathfrak{L}$ be in $\mathcal{D}^\nu$ with $\nu \neq 2$ and $\mathfrak{N}$ the nilradical of $\mathfrak{L}$. Then $\mathfrak{N} \neq 0$.*

Proof. If $\nu < 2$ then $U_{-1} \subset \mathfrak{N}$. Hence assume $\nu > 2$. Let $K(X,Y)$ be the Killing form of $X, Y \in \mathfrak{L}$. We show that if $\mathrm{ord}(X) = k > 2$ then $K(X,Y) = 0$ for all $Y \in \mathfrak{L}$, hence $X \in \mathfrak{N}$.

If $\mathrm{ord}(Y) = \ell$, then

$$(k-1)K(X,Y) = K([E,X],Y) = -K(X,[E,Y]) = -(\ell-1)K(X,Y).$$

Hence $(k+\ell-2)K(X,Y) = 0$. Since $k \geq 3$ and $\ell \geq 0$, we have $K(X,Y) = 0$. ■

It follows that any semi-simple $\mathfrak{L}$ in $\mathcal{D}^\nu$ is of order 2.

Example 4.2. Consider the Lie algebra $\mathfrak{L}(A,\mathcal{S})$ in (2.2). The elements of the form $x_1^k\partial_2$, $x_1^k x_2^\ell\partial_3$, $x_1^k x_2^\ell\partial_4$ generate the ideal $\mathfrak{N}$. In view of example 3.3 we see that $\mathfrak{R} = \langle\mathfrak{N}, x_2\partial_2, x_3\partial_3 + x_4\partial_4\rangle$.

The following notion is useful for us.

Definition 4.3. Let $S \subset \mathfrak{L}$ be a linear subspace. We call S *layered* if $[v,s] \in S$ for all $v \in U_{-1}$ and $s \in S$.

In particular $\mathfrak{L}$ itself is layered, as $U_{-1} = \mathfrak{L}_{-1}$. We note that also any ideal $I \subset \mathfrak{L}$ is layered. Moreover we have

Lemma 4.4. *Let $Z(I)$ be the center of an ideal I in $\mathfrak{L}$. Then $Z(I)$ is layered.*

Proof. Choose $x \in I, z \in Z(I)$ and $v \in U_{-1}$ arbitrary. We have to prove that $[x,[v,z]] = 0$. This follows directly from the Jacobi identity:

$$[x,[v,z]] = [[x,v],z] + [v,[x,z]] = 0 + 0 = 0. \qquad \blacksquare$$

Of special interest is the case $I = \mathfrak{N}$.

Proposition 4.5. *Let $W = Z(\mathfrak{N}) \cap U_{-1}$ be as above. Then W is a reducing subspace.*

Proof. Let $X \in \mathfrak{L}_{k-1}$ and $A = \varphi_k(X)$. We have to prove that for $w \in W$, $A(w,x,\ldots,x) \in W$ for all $x \in U_{-1}$. Fix an $\bar{x} \in U_{-1}$ and define $\bar{A} \in \varphi_0(\mathfrak{L}_0)$ by

$$\bar{A}(v) = A(v,\bar{x},\ldots,\bar{x})$$

Take $Y \in \mathfrak{N} \cap \mathfrak{L}_{p-1}$ and $B = \varphi_p(Y)$. Then

$$[\bar{A},B](x,\ldots,x) = pB(\bar{A}(x),x,\ldots,x) - \bar{A}(B(x,\ldots,x))$$

and by commutation with w we get

$$\begin{aligned} 0 =& [\bar{A},B](w,x,\ldots,x) = pB(\bar{A}(w),x,\ldots,x) \\ &+ p(p-1)B(\bar{A}(x),w,x,\ldots,x) - \bar{A}(B(w,\ldots,x)) \end{aligned}$$

Now clearly the last two terms of the right-hand side are 0. Hence also the first term:

$$B(\bar{A}(w),x,\ldots,x) = 0$$

for all $x \in U_{-1}$ and $B = \varphi_p(Y)$. Hence $[\bar{A}(w),Y] = 0$ for all $Y \in \mathfrak{N}$, or $\bar{A}(w) \in W$. So we obtain

$$A(w,\bar{x},\ldots,\bar{x}) \in W$$

for all $\bar{x} \in U_{-1}$. $\blacksquare$

Let us summarize our results. If $\mathfrak{R} \neq 0$, then also $\mathfrak{N} \neq 0$ and $V = \mathfrak{R} \cap U_{-1} = \mathfrak{N} \cap U_{-1}$. This space V is a reducing subspace, as is easily checked. It is possible that $V = U_{-1}$, for example with $n = 2$ one can take

$$\mathfrak{L} = \langle x_1^k \partial_2 \ (k \leq \nu), \partial_1 \rangle$$

Moreover if $\mathfrak{R} \neq 0$ then also $W = Z(\mathfrak{N}) \cap U_{-1} \neq 0$. This W is also a reducing subspace. Inductively, using lemma 3.2, we have a flag of subspaces $0 = W_0 \subset W_1 \subset W_2 \subset \cdots \subset W_s = V$ in V such that any $X \in \mathfrak{N}$ satisfies

$$[W_{i+1}, X] \in W_i \qquad \text{for all } i = 0, \ldots, s-1. \tag{9}$$

If $W = U_{-1}$ then $\mathfrak{N} = U_{-1}$ and hence $\mathfrak{L} \subset U_{-1} \oplus U_0$, where U_0 denotes the space of *all* vector fields of order 0.

Next we study the semi-simple part of $\mathfrak{L}$.

Let $\mathfrak{L}$ be of class $\mathcal{D}^\nu$ and $\mathfrak{R}$ its radical. Then $\mathfrak{L}/\mathfrak{R}$ is semi-simple; there exists a subalgebra $\mathfrak{S} \subset \mathfrak{L}$ such that $\mathfrak{S} \cong \mathfrak{L}/\mathfrak{R}$. Such $\mathfrak{S}$ is called a Levi factor. $\mathfrak{S}$ is the direct sum of (say d) simple Lie algebras $\mathfrak{S}_1, \ldots, \mathfrak{S}_d$. Set, as before $V = \mathfrak{R} \cap U_{-1}$, and $I_V \supset \mathfrak{R}$ the corresponding ideal. It is possible that I_V contains some of the factors $\mathfrak{S}_1, \ldots, \mathfrak{S}_d$. However we have

Proposition 4.6. *Let $\mathfrak{S}_i \subset I_V$, then $\mathfrak{S}_i \subset U_0$.*

Proof. Consider the Killing form K on I_V. For $X \in V$, $K(X,Y) = 0$ for all $Y \in I_V$, as $V \subset \mathfrak{N}$. Using the proof of proposition 4.1, it follows for $X \in I_V$ and $\mathrm{ord}(X) \neq 1$, that $K(X,Y) = 0$ for all $Y \in I_V$. So the semi-simple part of I_V is contained in the part of order 1. ∎

Using all results till now we are able to formulate a proposition on the structure of $\mathfrak{L}$ in case that $V = U_{-1}$.

Proposition 4.7. *Suppose $\mathfrak{L}$ is a Lie algebra in the class $\mathcal{D}^\nu$ with radical $\mathfrak{R}$ such that $\mathfrak{R} \cap U_{-1} = U_{-1}$. Then $\mathfrak{L}$ is a subalgebra of a multigraded $\bar{\mathfrak{L}}$ of class $\mathcal{D}^{\bar\nu}$ with $\bar\nu \geq \nu$.*

Proof. By proposition 4.5 we have a flag

$$U_{-1} = W_r \supset W_{r-1} \supset \cdots \supset W_1 \supset W_0 = \{0\}$$

with W_r/W_{r-1} a reducing subspace for $\mathfrak{L}/I_{W_{r-1}}$. In particular we have that *all* elements of $\mathfrak{L}$ have a common "triangular" form: if $x^{(s)} = (x_1^{(s)}, x_2^{(s)}, \ldots, x_{r_s}^{(s)})$ are coordinates for W_s/W_{s-1} then $X \in \mathfrak{L}$ takes the form

$$X = P_r(x^{(r)})\partial_{x^{(r)}} + P_{r-1}(x^{(r-1)}, x^{(r)})\partial_{x^{(r-1)}} + \cdots + P_n(x^{(1)}, x^{(2)}, \ldots, x^{(r)})\partial_{x^{(1)}}$$

where we used some obvious vector notations. If $X \in \mathfrak{N}$ then X even takes the strictly triangular form ($\mathrm{ord}(X) \geq 1$):

$$X = P_{r-1}(x^{(r)})\partial_{x^{(r-1)}} + \cdots + P_n(x^{(2)}, \ldots, x^{(r)})\partial_{x^{(1)}}$$

We inductively give degrees to the variables in $x^{(r)}, x^{(r-1)}$ down to $x^{(1)}$, cf. (7). First we put $\deg(x_i^{(r)}) = 1$. Then look at all terms in $P_{r-1}(x^{(r-1)}, x^{(r)})$ that are independent of $x^{(r-1)}$. Suppose d_{r-1} is the maximal degree (which in this case coincides with the maximal polynomial degree), then we put $\deg(x_i^{(r-1)}) = d_{r-1}$. Now look at all terms in P_{r-2} independent of $x^{(r-2)}$. Let d_{r-2} be the maximal degree. Then we put $\deg(x_i^{(r-2)}) = d_{r-2}$. This way we continue, and obtain that all these terms have degree 0 or less. It remains to consider the "diagonal terms" in P_i. We know from $\mathfrak{R} \cap U_{-1} = U_{-1}$ and proposition 4.6 that the diagonal terms are of order 1, hence of the form $x^{(i)}\partial_{x^{(i)}}$. Consequently these terms have degree 0. Hence $\mathfrak{L}$ is a subalgebra of the multigraded Lie algebra $\mathfrak{L}'$ consisting of all terms of degree 0 or less in the grading constructed above. ∎

It remains to study the situation in which $V \neq U_{-1}$. In this case we can consider $\mathfrak{L}' = \mathfrak{L}/I_V$. This is a Lie algebra of vector fields on $\mathbb{C}^{n'}$ with $n' = n - \dim(V)$. Note that the Euler field is $E' = E \bmod I_V$. Moreover $\mathfrak{L}'$ belongs to $\mathcal{D}^2(n')$, as $\mathfrak{L}'$ is semi-simple. Let $\mathfrak{S}'_1, \ldots, \mathfrak{S}'_{d'}$ be the simple Lie algebras that constitute $\mathfrak{L}'$. Thank to the presence of E' it is immediate that also $\mathfrak{S}'_1, \ldots, \mathfrak{S}'_{d'}$ are graded. In particular if we denote $U'_{-1} = U_{-1}/V$ we have

$$U'_{-1} = \bigoplus_{i=1}^{d'} \mathfrak{S}'_i \cap U'_{-1}$$

Clearly for any $\mathfrak{S}'_i$ we have $\mathfrak{S}'_i \cap U'_{-1} \neq \{0\}$. Let us put $n_i = \dim(\mathfrak{S}'_i \cap U'_{-1})$, so that $n_1 + \cdots + n_d + \dim(V) = n$. With this definition we have

Proposition 4.8. *Let $\mathfrak{L}' = \mathfrak{S}'_1 \oplus \cdots \oplus \mathfrak{S}'_{d'}$ be as above. Then the simple summand $\mathfrak{S}'_i$ is in $\mathcal{D}^2(n_i)$.*

Combining the propositions 4.6 and 4.8, we see that $\mathfrak{L}$ contains two types of simple subalgebras $\mathfrak{S}$; either $\mathfrak{S}$ is in U_0 or $\mathfrak{S}$ is in $U_{-1} \oplus U_0 \oplus U_1$; see also example 3.3.

5. Transitive Differential algebras in low dimensions

We will discuss the structure of Lie algebras in the class $\mathcal{D}^\nu(n)$ for $n = 2$ and $n = 3$. Apart from being interesting in its own right, we think that this is a good demonstration of the theorems from section 4..

We first consider $n = 2$. In section 4. we introduced the spaces $V = \mathfrak{N} \cap U_{-1}$ and $W = Z(\mathfrak{N}) \cap U_{-1}$. If $\dim V = 2$, we have that $\mathfrak{L}$ is a subalgebra of a multigraded one (proposition 4.7). Hence we see (cf. [4]) that $\mathfrak{L}$ is subalgebra of

$$\langle \partial_x, \partial_y, x\partial_y, \ldots, x^\nu\partial_y, x\partial_x, y\partial_y \rangle \qquad (\nu \geq 2)$$

or $\mathfrak{L}$ is contained in $U_{-1} \oplus U_0$.

At the other extreme, we have that $V = \{0\}$, which means that $\mathfrak{L}$ is semi-simple. This gives two (multigraded) possibilities, namely

$$\mathfrak{L} = \langle y^2\partial_y, x^2\partial_x, y\partial_y, x\partial_x, \partial_y, \partial_x \rangle \cong sl_2 \oplus sl_2$$

or

$$\mathfrak{L} = \langle y^2\partial_y + xy\partial_x, x^2\partial_x + xy\partial_y, y\partial_y, x\partial_y, y\partial_x, x\partial_x, \partial_x, \partial_y \rangle \cong sl_3$$

Finally we have the case that $\dim V = \dim W = 1$. Now $\mathfrak{L} = sl_2 \ltimes \mathfrak{N}$ and $X \in \mathfrak{N}$ is of the form

$$X = x^k \partial_y \quad (k \leq \nu)$$

while $\mathfrak{L}$ contains an element Y of the form

$$Y = x^2 \partial_x + (\alpha x^2 + \beta xy + \gamma y^2)\partial_y$$

Now $[\partial_y, Y] \in \mathfrak{N}$ implies that $\gamma = 0$. Moreover $[Y, x^\nu \partial_y] = 0$ implies $\beta = \nu$. For $\nu \leq 1$ it follows that one can take $\alpha = 0$, while for $\nu \geq 2$ we have $x^2\partial_y \in \mathfrak{L}$, hence we can assume that $\alpha = 0$. All together we find that $\mathfrak{L}$ is a subalgebra of

$$\langle x^\nu \partial_y, x^{\nu-1}\partial_y, \ldots, \partial_y, x^2\partial_x + \nu xy\partial_y, x\partial_x, y\partial_y, \partial_x \rangle$$

Now we turn to the case $n = 3$. Now not all $\mathfrak{L}$ in $\mathcal{D}^\nu$ are multigraded. A simple counterexample is the smallest Lie algebra in $\mathcal{D}^\nu$ containing the element $P(x_1, x_2)\partial_{x_3}$ for a homogeneous polynomial P of order ν. The construction of a transitively differential algebra that is not (essentially) multigraded is not easy. We discuss this problem by considering different cases for the dimensions of $W = Z(\mathfrak{N}) \cap U_{-1}$ and $V = \mathfrak{N} \cap U_{-1}$. We will not discuss $\dim W = 0$; in this case $\mathfrak{L}$ is a direct sum of simple Lie algebras in $\mathcal{D}^2$, see proposition 4.8.

If $\dim W = 3$, we have that $\mathfrak{L} \subset U_{-1} \oplus U_0$, hence by maximality, $\mathfrak{L} = U_{-1} \oplus U_0$.

If $\dim W = 2$, we can assume that $W = \langle \partial_y, \partial_z \rangle$ and $\mathfrak{N}$ contains $\langle x^k \partial_z, x^\ell \partial_y \rangle$ for $k \leq \kappa$, $\ell \leq \lambda$ (and possibly $\partial_x \in \mathfrak{N}$). We can assume that $\kappa \geq \lambda$ and $\kappa \geq 1$. The only possible element of order 2 can be put in the form

$$X = x^2\partial_x + \lambda xy\partial_y + (\kappa xz + \gamma xy)\partial_z.$$

A straightforward calculation (omitted here) yields that $\mathfrak{L}$ is not maximal in case that $\dim W = 2$.

We end up with the most difficult case, $\dim W = 1$. This case we split in two subcases, namely $V = W$, and $\dim V \geq 2$. In the first case, the nilradical has only elements of the form $X = P(x, y)\partial_z$. Let κ be the maximal k such that $x^k \partial_x \in \mathfrak{N}$ and λ the maximal ℓ such that $y^\ell \partial_z \in \mathfrak{N}$. We know that $\bar{\mathfrak{L}} = \mathfrak{L}/I_V$ is semi-simple, and according to the beginning of this section, we have only two possibilities: $\bar{\mathfrak{L}} = sl_2 \oplus sl_2$ or $\bar{\mathfrak{L}} = sl_3$.

If $\bar{\mathfrak{L}} = sl_2 \oplus sl_2$, then $\mathfrak{L}$ contains

$$Y_1 = x^2\partial_x + Q_1(x, y, z)\partial_z \text{ and } Y_2 = y^2\partial_y + Q_2(x, y, z)\partial_z$$

for some quadratic polynomials Q_1 and Q_2.
As $[Y_1, x^\kappa \partial_z] = 0$ and $[Y_2, y^\lambda \partial_z] = 0$ and $[Y_1, Y_2] = 0$, we find

$$Q_1(x, y, z) = \kappa xz + \alpha_1 x^2 + \alpha_2 xy \text{ and } Q_2(x, y, z) = \lambda yz + \beta_2 xy + \beta_3 y^2.$$

for some α_i and β_i. We may assume that these α_i and β_i are 0 in case $\kappa \geq 2$ and $\lambda \geq 2$. By direct calculations one can show that this holds in the remaining cases $\kappa \leq 1$ or $\lambda \leq 1$ as well. Hence $\mathfrak{L}$ has order $\nu = \kappa + \lambda$ and basis

$$\{x^k y^\ell \partial_z \ (k \leq \kappa, \ell \leq \lambda), z\partial_z, x^2\partial_x + \kappa xz\partial_z, x\partial_x, \partial_x, y^2\partial_y + \lambda yz\partial_z, y\partial_y, \partial_y\}$$

Next we consider the case that $\bar{\mathfrak{L}} = sl_3$. Now $\mathfrak{L}$ contains elements of the form

$$Y_1 = x^2\partial_x + xy\partial_y + Q_1(x,y,z)\partial_z \text{ and } Y_2 = xy\partial_x + y^2\partial_y + Q_2(x,y,z)\partial_z.$$

By the same method as in the previous case, we obtain $\kappa = \lambda$ and that we can put $\alpha_i = 0$ and $\beta_i = 0$ in all cases. Now $\mathfrak{L}$ has order $\nu = \kappa$ and a basis for $\mathfrak{L}$ is

$$\{x^k y^\ell \partial_z \ (k+\ell \le \kappa), z\partial_z, x^2\partial_x + xy\partial_y + \kappa xz\partial_z,$$
$$x\partial_x, y\partial_x, \partial_x, xy\partial_x + y^2\partial_y + \kappa yz\partial_z, x\partial_y, y\partial_y, \partial_y\}$$

Finally we arrive in the case that $\dim W = 1$ and $\dim V \ge 2$. By choosing good coordinates in U_{-1} we can assume that $\mathfrak{N}$ contains elements of the form

$$X = P(x,y)\partial_z \text{ and } Y = x^\ell \partial_y + Q(x,y)\partial_z$$

Let λ be the maximal ℓ such that $\mathfrak{N}$ contains an element $Y = x^\ell\partial_y + Q(x,y)\partial_z$ for some polynomial Q. For $\lambda = 0$ or for $\lambda = 1$ and $\kappa \ne 0$, there is no transitively differential algebra; one always can add terms such that we end up in the case with $\dim V = 1$ and the semi-simple part $\bar{\mathfrak{L}}$ being $sl_2 \oplus sl_2$ or sl_3, respectively (this again needs calculations that are omitted here). The case $\lambda = 1$ and $\kappa = 0$ is a special case in the series below. So from now on we assume $\lambda \ge 2$ and try to construct an $\mathfrak{L}$ that is not (essentially) multigraded. Let κ be the maximal k such that $x^k\partial_z \in \mathfrak{L}$. Now consider all terms that $x^a y^b \partial_z$ occurring in some $P(x,y)\partial_z \in \mathfrak{L}$, and take the P for which $a + \lambda b$ is maximal for some a, b. As P is homogeneous, b is maximal among the terms $x^a y^b$ occurring in P. By applying ad Y exactly b times see that $x^{a+\lambda b}\partial_z \in \mathfrak{L}$. By definition of κ, we have:

$$\boxed{a + \lambda b \le \kappa} \tag{10}$$

Now we can give the variable x, y, z degrees, according the scheme in the proof of proposition 4.7. We put

$$\deg(x) = 1, \quad \deg(y) = \lambda \text{ and } \deg(z) = \kappa.$$

Thanks to (10) all terms in $\mathfrak{N}$ have degree 0 or less except, possibly, terms occurring in Q. We can assume that this happens in case that $\ell = \lambda$. If all terms in Q have degree 0 or less, one can prove to end up with $\mathfrak{L}$ being

$$\langle x^a y^b \partial_z \ (a+\lambda b \le \kappa), x^\ell \partial_y \ (\ell \le \lambda), x^2\partial_x + \lambda xy\partial_y + \kappa xz\partial_z, x\partial_x, \partial_x, y\partial_y, z\partial_z\rangle$$

So assume $Y = x^\lambda\partial_y + Q(x,y)\partial_z$ and assume that $x^c y^d$ occurs in Q with $c + d\lambda > \kappa$. We take d maximal. If $d = 0$ then Q does not depend on y, and we can eliminate $Q(x) = \beta x^\lambda$ by the change of variables $y' = y - \beta z$; so the case $d = 0$ belongs to the multigraded case above. When Q depends on y, we consider

$$[Y, [\partial_x, Y]] = x^{\lambda-1}(\deg_x Q - \lambda Q)\frac{\partial Q}{\partial y}\partial_z$$

Hence $[Y, [\partial_x, Y]]$ contains the term $x^{\lambda-1+c}y^{d-1}\partial_z$. Using (10) for $a = \lambda - 1 + c$ and $b = d - 1$ gives that $\lambda - 1 + c + \lambda(d-1) \le \kappa$, or $c + \lambda b \le \kappa + 1$. Together with $c + \lambda d > \kappa$ and $c + d = \lambda$ we obtain that $d = \frac{\kappa}{\lambda - 1} - 1$. Hence necessarily

$\kappa = \alpha(\lambda-1)$ for an integer α. Consequently $d = \alpha-1$ (so $\alpha \geq 2$) and $c = \lambda+1-\alpha$. Hence also $\alpha \leq \lambda + 1$. By rescaling z we can assume that the coefficient of $x^c y^d$ in Q is 1.

We now derived more or less the structure of $\mathfrak{N}$. However, it could be that $\dim V = 2$, so that $\bar{\mathfrak{L}} = \mathfrak{L}/I_V = sl_2$. Direct calculations show that this is only possible if $\alpha = 2$. Hence for $\alpha \geq 3$ we have

$$\begin{aligned}\mathfrak{L} = \langle x^a y^b \partial_z \ (a + \lambda b \leq \kappa), (\mathrm{ad}\ \partial_x)^k \left(x^\lambda \partial_y + x^{\lambda-\alpha+1} y^{\alpha-1} \partial_z\right) \ (k \leq \lambda), \\ x\partial_x + y\partial_y + z\partial_z, \partial_x, y\partial_y + \alpha z \partial_z\rangle, \end{aligned} \tag{11}$$

In the case $\alpha \geq 3$ we see that $\mathfrak{L}$ is a subalgebra of

$$\langle x^a y^b \partial_z \ (a + \lambda b \leq \kappa + 1), x^\ell \partial_y \ (\ell \leq \lambda), x\partial_x, y\partial_y, z\partial_z, \partial_x\rangle,$$

which is multigraded, but of order $\kappa + 1$.

For $\alpha = 2$ we can add to $\mathfrak{L}$ in (11)

$$Z = x^2 \partial_x + \lambda x y \partial_y + (\kappa x z + \frac{1}{2} y^2) \partial_z. \tag{12}$$

In this case $\mathfrak{L}$ is maximal in the sense that adding any vector field will generate an infinite-dimensional Lie algebra.

To summarize this section we formulate the following proposition.

Proposition 5.1. *Let $\mathfrak{L}$ be a transitively differential algebra in 3 dimensions that is not semi-simple. Then either $\mathfrak{L}$ is (essentially) multigraded or $\mathfrak{L}$ belongs to the series of transitively differential algebras given in (11) and (12) with integers $\lambda \geq 1$, $2 \leq \alpha \leq \lambda + 1$ and $\kappa = \alpha(\lambda - 1)$. Moreover* $\operatorname{ord} \mathfrak{L} = \kappa$ *for $\lambda \geq 2$ and* $\operatorname{ord} \mathfrak{L} = 2$ *if $\lambda = 1$.*

6. A construction in high dimensions

We describe a construction for transitively differential algebras $\mathfrak{L}$ of order 3, which generalizes the construction in [7]. Before giving the detailed construction, let us the global structure of $\mathfrak{L}$. As $\nu = 3$, the nilradical of $\mathfrak{L}$ is non-zero; hence we have an invariant subspace $W = \langle z_1, z_2, \ldots, z_m\rangle$; $W = U_{-1} \cap Z(\mathfrak{N})$. Next we consider $\bar{\mathfrak{L}} = \mathfrak{L}/I_W$. Now $\bar{\mathfrak{L}}$ has also order 3, and we obtain by a similar procedure $\bar{W} = \langle y_1, y_2, \ldots, y_\ell\rangle$. The remaining algebra $\bar{\bar{\mathfrak{L}}} = \bar{\mathfrak{L}}/I_{\bar{W}}$ is the affine algebra $U_{-1} \oplus U_0$ in the variables $\bar{\bar{W}} = \langle x_1, x_2, \ldots, x_k\rangle$. It will turn out that ℓ is related to k by $\ell = \binom{k+1}{2}$. So $\mathfrak{L}$ only depends on the two parameters k and m, and $n = k + \ell + m$.

We now give the detailed construction of $\mathfrak{L}$, starting at order 3. First, $\mathfrak{L}$ will contain the elements

$$\boxed{x_a x_b x_c \partial_{z_d}} \qquad (a \leq b \leq c \leq k; \quad d \leq m). \tag{13}$$

Secondly we construct the vector

$$v = (x_1^2, x_1 x_2, \ldots, x_1 x_k, x_2^2, \ldots, x_k^2)$$

of all quadratic monomials in $x_1, \ldots, x_k$. Hence v is vector with $\ell = \binom{k+1}{2}$ components. Let $\partial_y = (\partial_{y_1}, \ldots, \partial_{y_\ell})$ and $v \cdot \partial_y = x_1^2 \partial_{y_1} + \cdots + x_k^2 \partial_{y_\ell}$. Then $\mathfrak{L}$ will contain

$$\boxed{x_a(v \cdot \partial_y)} \qquad (a \leq k). \tag{14}$$

Now we fixed v, we look for all vector w, quadratic in $x_1, \ldots, x_k$ satisfying

$$\frac{\partial v}{\partial x_a} \cdot w = 0 \qquad (\text{for all } a \leq k)$$

Note that this automatically implies that also $v \cdot w = 0$, by using

$$\frac{\partial v}{\partial x_a} \cdot w = 0 \ \Rightarrow \ \sum x_a \frac{\partial v}{\partial x_a} \cdot w = 0 \ \Rightarrow \ 2v \cdot w = 0. \tag{15}$$

The space of quadratic vectors has dimension $\binom{k+1}{2} \cdot \binom{k+1}{2}$. The linear constraints $\frac{\partial v}{\partial x_a} \cdot w = 0$ are independent and yield each $\binom{k+2}{3}$ equations on the coefficients. So the space of solutions w has dimension

$$r = \frac{1}{4}k^2(k+1)^2 - k\binom{k+2}{3} = \frac{1}{2}k\binom{k+1}{3}.$$

Below we will describe a construction for the space of solutions. For now let $w_1, \ldots, w_r$ be a basis for this space and $y = (y_1, \ldots, y_\ell)$. Then $\mathfrak{L}$ contains the elements

$$\boxed{(w_a \cdot y)\partial_{z_b}} \qquad (a \leq r; \quad b \leq m). \tag{16}$$

Next we consider the quadratic elements. These are simply the derivatives of the cubic ones. So the elements in (13) give

$$\boxed{x_a x_b \partial_{z_c}} \qquad (a \leq b \leq k; \quad c \leq m). \tag{17}$$

The elements in (14) yield

$$\boxed{x_a \frac{\partial v}{\partial x_b} \cdot \partial_y} \qquad (a, b \leq k) \tag{18}$$

By (15) this is exactly the linear span of the elements $[\partial_{x_b}, x_a v \cdot \partial_y]$.

Next we have the elements

$$\boxed{(\frac{\partial w_a}{\partial x_b} \cdot y)\partial_{z_c}} \qquad (a, b \leq k; c \leq m). \tag{19}$$

These elements are not linearly independent; this will be discussed at the end of the section.

Finally we arrive at the elements of order 1. First we have

$$\boxed{x_a \partial_{z_c}} \ \boxed{x_a \partial_{y_b}} \ \boxed{y_b \partial_{z_c}} \text{ and } \boxed{z_d \partial_{z_c}} \qquad (a \leq k; \quad b \leq \ell; \quad c, d \leq m). \tag{20}$$

Secondly we have the element

$$\boxed{y_1\partial_{y_1} + y_2\partial_{y_2} + \cdots + y_\ell\partial_{y_\ell}} \tag{21}$$

Finally we have elements of the form $x_a\partial_{x_b} + \ldots$. The vector fields $x_a\partial_{x_b}$ act on $v \cdot \partial_y$. This action can be view as a linear action on the space $\bar{W} = \langle \partial_{y_1}, \ldots \partial_{y_\ell} \rangle$. As such, modulo the vector field (21), it can be uniquely represented by a vector field of the form

$$\sum_{c,d} \alpha_{c,d}^{(a,b)} y_c\partial_{y_d}.$$

If we write $v = (v_1, \ldots, v_\ell)$ we have explicitly

$$x_a\partial_{x_b}(v_1\partial_{y_1} + v_2\partial_{y_2} + \cdots + v_\ell\partial_{y_\ell}) =$$
$$-v_1(\sum_d \alpha_{1,d}^{(a,b)}\partial_{y_d}) - v_2(\sum_d \alpha_{2,d}^{(a,b)}\partial_{y_d}) - \cdots - v_\ell(\sum_d \alpha_{\ell,d}^{(a,b)}\partial_{y_d})$$

By this construction we automatically have that

$$[x_a\partial_{x_b} - \sum_{c,d} \alpha_{c,d}^{(a,b)} y_c\partial_{y_d}, v \cdot \partial_y] = 0.$$

To $\mathfrak{L}$ we add the vector fields

$$\boxed{x_a\partial_{x_b} - \textstyle\sum_{c,d} \alpha_{c,d}^{(a,b)} y_c\partial_{y_d}} \qquad (a, b \leq k). \tag{22}$$

This completes the description of $\mathfrak{L}$.

A direct calculation shows that $\mathfrak{L}$ is indeed a Lie algebra. We will discuss some of the most difficult commutators. Let X be of type (14) and Y of type (16). They commute by

$$[X, Y] = [x_a(v \cdot \partial_y), (w_c \cdot y)\partial_{z_b}] = x_a(v \cdot w_c)\partial_{z_b} = 0$$

Another commutator to check is type (16) and type (22). For this we consider

$$0 = [x_a\partial_{x_b} - \sum_{c,d} \alpha_{c,d}^{(a,b)} y_c\partial_{y_d}, [(w_c \cdot y)\partial_{z_b}, v \cdot \partial_y]]$$

By the Jacobi identity we find that

$$[[x_a\partial_{x_b} - \sum_{c,d} \alpha_{c,d}^{(a,b)} y_c\partial_{y_d}, (w_c \cdot y)\partial_{z_b}], v \cdot \partial_y] = 0$$

Now the inner commutator is of the form $(f \cdot y)\partial_{z_b}$ with f quadratic in x, and moreover $f \cdot v = 0$. Similarly we find $f \cdot \frac{\partial v}{\partial x_i} = 0$ for all i. Hence we conclude that f is a linear combination of $w_1, \ldots, w_r$.

Finally, we want to prove that $\mathfrak{L}$ is maximal. Therefore suppose that $\mathfrak{L}'$ is in $\mathcal{D}^3$ and contains $\mathfrak{L}$. Then $\mathfrak{L}'$ has a reducing subspace W, which must be a subspace of $\langle z_1, \ldots, z_m \rangle$. As $\mathfrak{L}$ acts irreducibly on $\langle z_1, \ldots, z_m \rangle$, we see that this space coincides with W. So no elements of $\mathfrak{L}'$ contain terms of the form $z_a\partial_{y_b}$ or $z_a\partial_{x_b}$.

Next we see that y does not appear quadratically by commuting such elements with type (14). After this it is not difficult to check maximality.

That $\mathfrak{L}$ is not essentially multigraded can be seen from the fact that the Jacobian of $x_1(v \cdot \partial_y)$ has rank greater than 1 (see lemma 2.3). Moreover by giving the variables degrees according to

$$\deg x_i = 1; \qquad \deg y_i = 2; \qquad \deg z_i = 4$$

we see that $\mathfrak{L}$ is contained in a multigraded Lie algebra of order 4.

Let us now describe the space of w-functions. For this we consider the condition $v \cdot w = 0$. This condition splits in $\binom{k+3}{4}$ constraints on the coefficient of monomials of the form

$$x_a^4; \quad x_a^3 x_b; \quad x_a^2 x_b^2; \quad x_a^2 x_b x_c; \quad x_a x_b x_c x_d.$$

Correspondingly, the solutions w naturally split up. We describe the 5 cases. Without loss of generality, we assume $(a, b, c, d) = (1, 2, 3, 4)$.

- (x_1^4). Then $w = (\alpha_1 x_1^2, 0, \dots, 0)$, and $v \cdot w = 0$ yields $\alpha_1 = 0$.
- $(x_1^3 x_2)$. Having $v = (x_1^2, x_1 x_2, \dots)$ we get $w = (\alpha_1 x_1 x_2, \alpha_2 x_1^2, 0, \dots, 0)$. Again $v \cdot w = 0$ and $\frac{\partial v}{\partial x_1} \cdot w = 0$ yield $w = 0$.
- $(x_1^2 x_2^2)$. Permute the components of v, so that $v = (x_1^2, x_1 x_2, x_2^2, \dots)$. Then $w = (\alpha_1 x_2^2, \alpha_2 x_1 x_2, \alpha_3 x_1^2, 0, \dots, 0)$. It gives one solution
$$w = (x_2^2, -2x_1 x_2, x_1^2, 0, \dots, 0). \tag{23}$$
- $(x_1^2 x_2 x_3)$. If $v = (x_1^2, x_1 x_2, x_1 x_3, x_2 x_3, \dots)$ then
$$w = (x_2 x_3, -x_1 x_3, -x_1 x_2, x_1^2, 0, \dots, 0). \tag{24}$$
- $(x_1 x_2 x_3 x_4)$. Finally the most involved case. If
$$v = (x_1 x_2, x_1 x_3, x_1 x_4, x_2 x_3, x_2 x_4, x_3 x_4, \dots)$$
then we find two linearly independent solutions:

$$\begin{aligned} w_1 &= (x_3 x_4, 0, -x_2 x_3, -x_1 x_4, 0, x_1 x_2, 0, \dots, 0) \\ \text{and } w_2 &= (0, x_2 x_4, -x_2 x_3, -x_1 x_4, x_1 x_3, 0, 0, \dots, 0) \end{aligned} \tag{25}$$

Now we can count the dimension of the w-space. For type (23) we have $\binom{k}{2}$, for type (24) we have $k\binom{k-1}{2}$ and for type (25) we have $\binom{k}{4}$ combinations, and the last one doubled. Hence we have totally

$$r = \binom{k}{2} + k\binom{k-1}{2} + 2\binom{k}{4} = \frac{1}{2}k\binom{k+1}{3}$$

independent w-functions.

Our w-space coincides with the space T_n in the paper [7]. However the dimensions do not agree. This is due to the fact that, starting from $n = 4$, the space T_n does not agree with the space of *all* symmetric operators on the set of skew-symmetric matrices.

From the explicit form of the w-functions, it is clear that the derivatives $\frac{\partial w_a}{\partial x_b}$ appearing in (19) are not linearly independent. These derivatives span exactly the space of all vectors $f = (f_1, \ldots, f_\ell)$, with f_i linear in x, such that $f \cdot v = 0$. Counting the dimension s of this space in the same way as we determined the w-space itself, we obtain[2]

$$s = 2\binom{k+1}{3}.$$

Now we can compute the dimension of $\mathfrak{L}$ depending on k and m. We present this in the following table. We divide the elements in certain classes, namely vertically by order and horizontally by appearing variables.

	$x\partial_z$	$x\partial_y$	$x\partial_x$	$(x)y\partial_z$	$y\partial_y$	$z\partial_z$
order 3	$\binom{k+2}{3}m$	k	–	rm	–	–
order 2	$\binom{k+1}{2}m$	k^2	–	sm	–	–
order 1	km	$k\ell$	k^2	ℓm	1	m^2
order 0	m	ℓ	k	–	–	–

Hence for the dimension of $\mathfrak{L}$ we obtain

$$\dim \mathfrak{L} = \frac{1}{12}\left(k^4 + 6k^3 + 17k^2 + 24k + 12m + 12\right)m + \frac{1}{2}\left(k^3 + 6k^2 + 5k + 2\right)$$

In particular we have for $k = 2$ and $m = 1$, that $\dim \mathfrak{L} = 39$, while for $k = 3$ and $m = 6$ (this is the case $n = 3$ in [7]), we have $\dim \mathfrak{L} = 325$.

7. Conclusion

We discussed Lie algebras $\mathfrak{L}$ in the class $\mathcal{D}^\nu$ in Lie-theoretic sense (section 4.) and more concretely for some specific situations in the section 5. and 6.. Our point of view is to be complementary to the cases that $\mathfrak{L}$ is semi-simple or $\mathfrak{L}$ is multigraded. Such algebras are easy to construct, but when restricting to transitively differential algebras it is harder; for $n = 3$ we find only one series of transitively differential algebras that are not multigraded.

In view of the discussions of sections 2., 4., 5. and 6. one can wonder how useful the common definition of transitively differential algebra of order ν is. For our classification problem it seems better to consider only those algebras that are maximal in the class of *all* finite-dimensional algebras. Let call such $\mathfrak{L}$ a "transitively differential algebra of order ∞". In 2. it becomes clear that a multigraded transitively differential algebra of order ∞ is of the form $\mathfrak{L}(A, \mathcal{S})$. In section 4. we saw that the radical of any $\mathfrak{L}$ is contained in a multigraded transitively differential algebra of order ∞. Exactly the same holds for the examples of section 5. given in (11) and also for the transitively differential algebras constructed in section 6.: if considered in the class $\mathcal{D}^{\nu+1}$ instead of $\mathcal{D}^\nu$ all these examples are contained in a multigraded transitively differential algebra $\mathfrak{L}(A, \mathcal{S})$ of order ∞.

[2]This is different in [7] as well.

For future research it would be interesting to concentrate on transitively differential algebras of order ∞: from the results of this paper it becomes clear that we should study a semi-direct product of a (semi-)simple algebra in class $\mathcal{D}^2$ and the radical. The representation of the first on the second should be analyzed in detail; the abstract representation theory should be connected to the concrete situation of polynomial vector fields.

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