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On the dominance partial ordering of Dyck paths

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Abstract

The lattice of Dyck paths with the dominance partial order is studied. The notions of filling and degree of a Dyck path are introduced, studied and used for the evaluation of the Möbius function and its powers. The relation between the symmetric group endowed with the weak Bruhat order and the set of Dyck paths is studied.

1 Introduction

A wide range of articles dealing with lattices of combinatorial objects appear frequently in the literature, e.g., lattices of integer partitions, [4, 20], permutations [11, 21] and noncrossing partitions [8, 17].

In this paper, the lattice of Dyck paths with the dominance partial order and some of its relations to other combinatorial structures are studied.

In Section 2, some basic definitions and notations referring to the set $\mathcal{D}_n$ of Dyck paths of semilength n, are given.

In Section 3, the rank of a Dyck path is studied and the rank-generating function is exhibited.

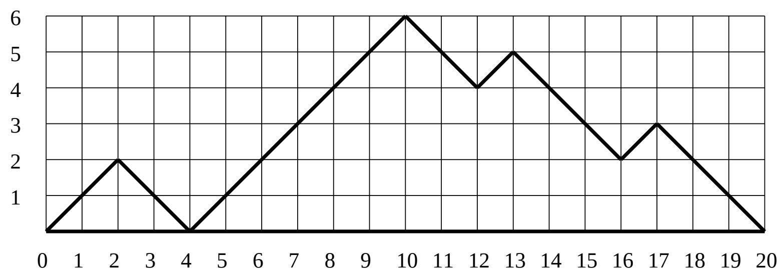
In Section 4, the notions of filling and degree of a Dyck path are introduced and studied. Various enumeration results are presented. These notions are used in Section 5 for the evaluation of the Möbius function and its powers. In this context, a new appearance of the Fibonacci numbers (A000045 of [18]) occurs.

Finally, in Section 6 the relation between the symmetric group S_n endowed with the weak Bruhat order and $\mathcal{D}_n$ is studied. More precisely, a partition of S_n into C_n (Catalan, ([A000108](A000108) of [18]) classes is constructed, satisfying an ordering condition, and the cardinal numbers of its members are evaluated.

2 Preliminaries

A *Dyck path* of semilength n is a path in the first quadrant, which begins at the origin, ends at $(2n, 0)$, and consists of steps $(1, 1)$ and $(1, -1)$, called *rise* and *fall* respectively.

It is clear that each Dyck path is coded by a word $u \in \{a, \bar{a}\}^*$, called *Dyck word*, so that every rise (resp. fall) corresponds to the letter a (resp. $\bar{a}$); see Fig. 1.



$a\ a\ \bar{a}\ \bar{a}\ a\ a\ a\ a\ a\ a\ \bar{a}\ \bar{a}\ a\ \bar{a}\ \bar{a}\ \bar{a}\ a\ \bar{a}\ \bar{a}\ \bar{a}$

Figure 1: A Dyck path and its corresponding word

Throughout this paper we will denote with D the set of all Dyck paths (or equivalently Dyck words). Furthermore, the subset of D that contains the paths u of semilength $l(u) = n$ is denoted by D_n.

We denote with ϵ the empty path. Every $u \in D \setminus \{\epsilon\}$ can be uniquely decomposed in the form $u = aw\bar{a}v$, $w, v \in D$, which is called the *first return decomposition* of u.

Every Dyck path u that meets the x-axis only at its endpoints (i.e., $u = aw\bar{a}$, with $w \in D$) is called *prime*. Every Dyck path can be uniquely decomposed into prime paths [15].

It is well known that Dyck paths are enumerated by the Catalan numbers, with generating function $C(x)$, which satisfies the relation $xC^2(x) = C(x) - 1$.

For a parameter q defined on D, we will denote with F_q the generating function of D according to the parameters l, q, i.e.,

$$F_q(x, y) = \sum_{u \in D} x^{l(u)} y^{q(u)}.$$

A point of a Dyck path is called *peak* (resp. *valley*) if it is preceded by a rise (resp. fall) and followed by a fall (resp. rise). A point of a Dyck path is called *double rise* (resp. *double fall*) if it is preceded and followed by a rise (resp. fall).

A convenient way to represent a Dyck word is by using dominating sequences. A sequence $d = (d_i)_{i \in [n]}$ of non-negative integers is called *dominating* if it satisfies the following two conditions:

(i) $\sum_{i=1}^{n} d_i = n$,

(ii) $\sum_{i=1}^{\nu} d_i \geq \nu$, for every $\nu \in [n]$.

It is well known that every non-empty Dyck word u is uniquely represented by a dominating sequence $d(u) = (d_i)_{i\in l(u)}$ where d_1 is the number of a's before the 1st occurrence of $\bar{a}$ in u, and d_i is the number of a's between the $(i-1)^{\text{th}}$ and the i^{th} occurrence of $\bar{a}$ in u, where $i \in [2, l(u)]$. For example, the word $u = aa\bar{a}\bar{a}a\bar{a}aa\bar{a}a\bar{a}\bar{a}$ is represented by the sequence $d(u) = 2, 0, 1, 2, 1, 0$.

A sequence $d = (d_i)_{i\in[n]}$ *dominates* another sequence $d' = (d'_i)_{i\in[n]}$ iff $\sum_{i=1}^{\nu} d_i \geq \sum_{i=1}^{\nu} d'_i$ for every $\nu \in [n]$. In this case we say that d' is *dominated* by d. Notice that every dominating sequence dominates the constant sequence with elements equal to 1.

The *dominance partial order "$\preceq$"* on D is defined as follows:

$u \preceq w$ iff $l(u) = l(w)$ and the sequence $d(u)$ is dominated by the sequence $d(w)$,

i.e., the path w lies above (in the broad sense) the path u. If $u \preceq w$ and $u \neq w$, we will write $u \prec w$.

In the sequel, we will denote $(D, \preceq)$ (resp. $(D_n, \preceq)$) with $\mathcal{D}$ (resp. $\mathcal{D}_n$).

Narayana and Fulton [12], using an equivalent language, proved that the set of Grand-Dyck paths of fixed length n (i.e., paths defined similarly to Dyck paths but allowed to go below the x-axis, e.g., see [16]) endowed with the dominance order is a distributive lattice. Kreweras [13, 14] has done further work in this context. Ferrari and Pinzani [9] have recently presented a more general approach to this subject.

It is easy to see that $\mathcal{D}_n$ is a sublattice of the lattice of all Grand-Dyck paths of semilength n, such that if $d(w) = (d'_i)_{i\in[n]}$, $d(v) = (d''_i)_{i\in[n]}$ we have:
$d(w \vee v) = (d_i)_{i\in[n]}$, where

$$d_1 = \max\{d'_1, d''_1\} \text{ and } d_i = \max\{\sum_{j=1}^{i} d'_j, \sum_{j=1}^{i} d''_j\} - \max\{\sum_{j=1}^{i-1} d'_j, \sum_{j=1}^{i-1} d''_j\} \text{ for every } i \in [2, n]$$

and $d(w \wedge v) = (d_i)_{i\in[n]}$, where

$$d_1 = \min\{d'_1, d''_1\} \text{ and } d_i = \min\{\sum_{j=1}^{i} d'_j, \sum_{j=1}^{i} d''_j\} - \min\{\sum_{j=1}^{i-1} d'_j, \sum_{j=1}^{i-1} d''_j\} \text{ for every } i \in [2, n].$$

We note that the least (resp. greatest) element of $\mathcal{D}_n$ is $0_n = (a\bar{a})^n$ (resp. $1_n = a^n\bar{a}^n$). Finally, for every $n \geq 2$, 1_n covers one and only one element of $\mathcal{D}_n$, namely $a^{n-1}\bar{a}a\bar{a}^{n-1}$.

The Hasse diagram of $\mathcal{D}_4$ coded by dominating sequences is given in Fig. 2.

All unexplained notations and definitions for posets and Dyck paths can be found in [20] and [5], respectively.

3 Chains and Ranks

In this section we first evaluate the length of maximal chains in intervals of $\mathcal{D}_n$. For this, notice first that for $u, w \in \mathcal{D}_n$ with $d(u) = (d_i)_{i\in[n]}$ and $d(w) = (d'_i)_{i\in[n]}$, w covers u iff there exists $j \in [n-1]$ such that $d'_i = d_i$ for every $i \neq j, j+1$ and $d'_j = d_j + 1$, $d'_{j+1} = d_{j+1} - 1$.

From the above observation, we deduce that every path w which covers the path u is obtained by turning a valley (x, y) of u into the peak $(x, y+2)$.

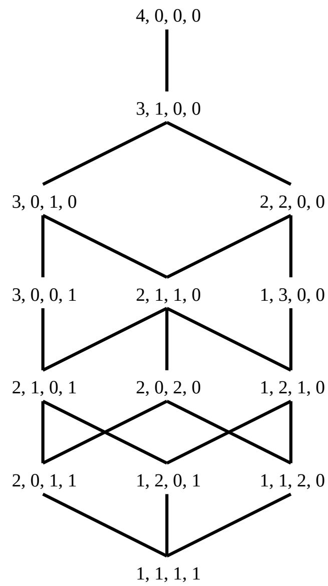


Figure 2: The Hasse diagram of $\mathcal{D}_4$

Proposition 3.1. *Let $u, w \in \mathcal{D}_n$ with $u \preceq w$ and $d(u) = (d_i)_{i\in[n]}$, $d(w) = (d'_i)_{i\in[n]}$. Then, the length of every maximal chain $\mathcal{C}$ of $[u, w]$ is equal to*

$$\sum_{i=1}^{n} i(d_i - d'_i).$$

Proof. We will use induction with respect to the length k of $\mathcal{C}$. If $k = 1$ then w covers u; so there exists $j \in [n-1]$ with $d_i = d'_i$ for every $i \in [n] \setminus \{j, j+1\}$ and $d'_j = d_j + 1$, $d'_{j+1} = d_{j+1} - 1$.

Hence

$$\begin{aligned}\sum_{i=1}^{n} i(d_i - d'_i) &= j(d_j - d'_j) + (j+1)(d_{j+1} - d'_{j+1}) \\ &= j(-1) + (j+1)1 = 1.\end{aligned}$$

Since in this case the only maximal chain of $[u, w]$ is $\{u, w\}$ and has length 1, the result holds when $k = 1$.

Suppose now that the result holds for every maximal chain of $[u, w]$ of length k, for every $u, w \in \mathcal{D}_n$ with $u \prec w$. We will prove that the result also holds for any maximal chain $\mathcal{C}$ of $[u, w]$ of length $k + 1$.

If v is the predecessor of w, then $C \setminus \{w\}$ is a maximal chain of $[u, v]$ of length k; so by the induction hypothesis we have

$$k = \sum_{i=1}^{n} i(d_i - d''_i)$$

where $d(v) = (d''_i)_{i\in[n]}$.

Since $\sum_{i=1}^{n} i(d''_i - d'_i) = 1$, we obtain that $\sum_{i=1}^{n} i(d_i - d'_i) = k + 1$. $\square$

If we apply the previous proposition for $u = 0_n$ and $w = 1_n$ we obtain that every maximal chain in $\mathcal{D}_n$ has length equal to $n(n-1)/2$. Thus, the lattice $\mathcal{D}_n$ is graded of rank $\binom{n}{2}$.

In the sequel, we investigate the parameter "rank" of $\mathcal{D}$ defined as follows : $\rho(\epsilon) = 0$ and for $u \neq \epsilon$, $\rho(u)$ is the rank of u in $\mathcal{D}_{l(u)}$.

By Proposition 3.1, it follows easily that if $u \neq \epsilon$ with $d(u) = (d_i)_{i \in l(u)}$, then

$$\rho(u) = l(u)(l(u)+1)/2 - \sum_{i=1}^{l(u)} i d_i. \tag{1}$$

Ferrari and Pinzani [9] give an equivalent expression of the above relation through the notion of the area of a Dyck path. Using relation (1) (or alternatively its equivalent relation in [9]), we can deduce that the parameter "rank" satisfies the following properties:

(i) $\rho(wv) = \rho(w) + \rho(v)$,

(ii) $\rho(aw\bar{a}) = \rho(w) + l(w)$, for every $w, v \in \mathcal{D}$.

Taking into account the above properties and the fact that every Dyck path is either empty or of the form $aw\bar{a}v$, where $w, v \in D$, we can easily deduce the following relation.

$$F_\rho(x,y) = 1 + xF_\rho(xy,y)F_\rho(x,y). \tag{2}$$

Furthermore, if $f_n(y) = \sum_{k=0}^{\binom{n}{2}} a_{n,k} y^k$, where $a_{n,k}$ denotes the number of elements of $\mathcal{D}_n$ of rank k, then by relation (2) we have

$$\sum_{n=0}^{\infty} f_n(y)x^n = 1 + x\left(\sum_{n=0}^{\infty} f_n(y)y^n x^n\right)\left(\sum_{n=0}^{\infty} f_n(y)x^n\right)$$

or, equivalently,

$$\sum_{n=0}^{\infty} f_{n+1}(y)x^n = \sum_{n=0}^{\infty}\left(\sum_{\nu=0}^{n} f_\nu(y) f_{n-\nu}(y) y^\nu\right) x^n.$$

Hence we obtain the following result.

Proposition 3.2. *The rank-generating function of $\mathcal{D}_n$ is given by the following recursive formula*

$$f_{n+1}(y) = \sum_{\nu=0}^{n} f_\nu(y) f_{n-\nu}(y) y^\nu,$$

where $f_0(y) = 1$.

4 Fillings and Degrees

In this section we first introduce and study the notion of the filling of a Dyck path.

The *filling* $\widetilde{u}$ of $u \in \mathcal{D} \setminus \{\epsilon\}$ is defined to be the Dyck path obtained by turning each valley (x, y) of u into the peak $(x, y+2)$. The filling of the empty path is assumed to be the empty path.

For example, if $u = aa\bar{a}\bar{a}aa\bar{a}a\bar{a}\bar{a}$ then $\widetilde{u} = aa\bar{a}a\bar{a}aa\bar{a}\bar{a}\bar{a}$.

The main properties of the filling are given in the following proposition, which is easy to prove.

Proposition 4.1. *For a non-empty Dyck path u we have :*
i) $l(\widetilde{u}) = l(u)$, $u \preceq \widetilde{u}$ and $u = \widetilde{u}$ if $u = 1_{l(u)}$.
ii) The length $l(u, \widetilde{u})$ of the interval $[u, \widetilde{u}]$ is equal to the number val(u) of all valleys of u.
iii) The cardinal number of the interval $[u, \widetilde{u}]$ is equal to $2^{\mathrm{val(u)}}$.
iv) If $u \preceq v$, then $\widetilde{u} \preceq \widetilde{v}$.

We note that since the parameter val defined by the number of valleys follows the Narayana distribution [5], we deduce that the number $a_{n,k}$ of all $u \in \mathcal{D}_n$ such that the interval $[u, \widetilde{u}]$ contains exactly k elements is equal to

$$a_{n,k} = \begin{cases} \frac{1}{n}\binom{n}{\lambda}\binom{n}{\lambda-1}, & \text{if } k = 2^\lambda,\ \lambda \in \mathbb{N}^*; \\ 0, & \text{if } k \neq 2^\lambda,\ \lambda \in \mathbb{N}^*. \end{cases}$$

Next, we consider the Dyck paths which are fillings of Dyck paths. For this, we need the following characterization.

Proposition 4.2. *A Dyck path u is a filling of a Dyck path iff u is prime and avoids $\bar{a}\bar{a}aa$.*

Proof. If $u = \widetilde{v}$ for some $v \in \mathcal{D}$, then clearly u has no valleys on level zero and so u is prime. Furthermore, if u contains a $\bar{a}\bar{a}aa$, then there exists a valley (x, y) of u such that the points $(x-1, y+1)$ and $(x+1, y+1)$ are a double fall and a double rise of u, respectively. It follows that the points (x, y), $(x-1, y+1)$ and $(x+1, y+1)$ remain unchanged during the generation of u from v, so that (x, y) is also a valley of v. This contradicts the definition of the filling, since $(x, y+2)$ is not a peak of u.

Conversely, assume that u is prime and avoids $\bar{a}\bar{a}aa$. Let v be the path that we obtain from u by turning each peak (x, y) of u into the valley $(x, y-2)$. Clearly, since u is prime it has no low peaks, hence v never crosses the x-axis and so $v \in \mathcal{D}$. In order to show that $\widetilde{v} = u$, it is enough to show that every valley of v is generated by a peak of u according to the above procedure. Indeed, if this is not true and (x, y) is a valley of v that is not generated in such a way, then (x, y) must be a valley of u, too. Since u avoids $\bar{a}\bar{a}aa$, it follows that at least one of the points $(x-1, y+1)$ and $(x+1, y+1)$ is a peak of u and hence a valley of v, which contradicts the fact that (x, y) is a valley of v. $\square$

We remark that the Dyck path v constructed by turning each peak of a Dyck path u into a valley as in the converse of the above proof, is the least Dyck path with the property $\widetilde{v} = u$. In this case, we say that v is the *antifilling* of u.

In the following, we study the set $\widetilde{\mathcal{D}}$ of all Dyck paths that are fillings of Dyck paths.

Proposition 4.3. *The generating function F of $\widetilde{\mathcal{D}}$ according to semilength satisfies the equation*

$$F(x) = 1 + xF^2(x) - x^2F(x). \tag{3}$$

Furthermore, the number a_n of all Dyck paths of semilength n that are fillings of Dyck paths is given by the formula

$$a_n = \sum_{k=[n/2]}^{n} \frac{(-1)^{n-k}}{k} \binom{k}{n-k} \binom{3k-n}{k-1}, \text{ for } n \geq 2 \tag{4}$$

whereas $a_0 = a_1 = 1$.

Proof. Let $\mathcal{A}$ be the set of all Dyck paths that avoid $\bar{a}\bar{a}aa$ and $A(x)$ its generating function according to semilength.

Clearly, by the previous proposition, every non-empty element of $\widetilde{\mathcal{D}}$ can be written uniquely in the form $aw\bar{a}$ where $w \in \mathcal{A}$, so that $F(x) = 1 + xA(x)$.

Furthermore, we can easily check that every non-empty element $u \in \mathcal{A}$ can be written uniquely in one of the following forms: $u = a\bar{a}v$, $u = aw\bar{a}$ or $u = aw\bar{a}a\bar{a}v$ where $w, v \in \mathcal{A}$ and $w \neq \epsilon$.

Thus, we have:

$$A(x) = 1 + xA(x) + x(A(x) - 1) + x^2(A(x) - 1)A(x).$$

So

$$x^2A^2(x) - (x-1)^2A(x) + 1 - x = 0$$

and hence

$$xF^2(x) - (1 + x^2)F(x) + 1 = 0$$

which gives formula (3).

For the proof of formula (4) we consider the generating function $F(x, y)$ satisfying the equation

$$F(x, y) = 1 + y(xF^2(x, y) - x^2F(x, y)). \tag{5}$$

If we set $P(\lambda) = x\lambda^2 - x^2\lambda$, then $F(x, y) = 1 + yP(F(x, y))$; so applying the Lagrange inversion formula with respect to the variable y, in the form given by Deutsch [5], we obtain that

$$[y^k]F = \frac{1}{k}[\lambda^{k-1}](P(1 + \lambda))^k \tag{6}$$

for every $k \in \mathbb{N}^*$.

Furthermore, we have

$$\begin{aligned}(P(1+\lambda))^k &= x^k \sum_{\nu=0}^{k} \sum_{\rho=0}^{k} \binom{k}{\nu}\binom{k}{\rho} \lambda^{\nu}(\lambda - x)^{\rho} \\ &= \sum_{\nu=0}^{k} \sum_{\rho=0}^{k} \sum_{i=0}^{\rho} (-1)^{\rho-i} \binom{k}{\nu}\binom{k}{\rho}\binom{\rho}{i} \lambda^{\nu+i} x^{k+\rho-i}.\end{aligned}$$

Then, for $i = k - \nu - 1$, from relation (6) we obtain that

$$[y^k]F = \frac{1}{k} \sum_{\rho=0}^{k} \sum_{\nu=0}^{k-1} (-1)^{\rho-k+\nu+1} \binom{k}{\nu} \binom{k}{\rho} \binom{\rho}{k-\nu-1} x^{\rho+\nu+1}.$$

Thus,

$$\begin{aligned} F(x,y) &= 1 + \sum_{k=1}^{\infty} \sum_{\rho=0}^{k} \sum_{\nu=0}^{k-1} (-1)^{\rho-k+\nu+1} \frac{1}{k} \binom{k}{\nu} \binom{k}{\rho} \binom{\rho}{k-\nu-1} x^{\rho+\nu+1} y^k \\ &= 1 + xy + \sum_{n=2}^{\infty} \sum_{k=[n/2]}^{n} \sum_{\nu=0}^{k-1} (-1)^{n-k} \frac{1}{k} \binom{k}{\nu} \binom{k}{n-\nu-1} \binom{n-\nu-1}{k-\nu-1} x^n y^k \\ &= 1 + xy + \sum_{n=2}^{\infty} \sum_{k=[n/2]}^{n} (-1)^{n-k} \frac{1}{k} \binom{k}{n-k} \binom{3k-n}{k-1} x^n y^k. \end{aligned}$$

Thus, since by relations (3) and (5) we have $F(x) = F(x,1)$, it follows that

$$a_0 = a_1 = 1 \text{ and } a_n = \sum_{k=[n/2]}^{n} (-1)^{n-k} \frac{1}{k} \binom{k}{n-k} \binom{3k-n}{k-1}, \text{ for } n \geq 2.$$

□

By formula (4) by obtain the sequence 1,1,1,2,5,13,35,97,275,. . . , which also counts the number of Dyck paths that avoid $aa\bar{a}\bar{a}$ (A086581 of [18]). This can also be seen by proving that the set of all antifillings coincides with the set of all Dyck paths that avoid $aa\bar{a}\bar{a}$.

The rest of this section deals with the notion of the degree of a Dyck path. For this, we define the i^{th} filling $u^{(i)}$ of a non-empty Dyck path u recursively, as follows:

$$u^{(0)} = u \text{ and } u^{(i)} = \widetilde{u^{(i-1)}}, \text{ for } i \geq 1.$$

We define the *degree* $\delta(u)$ of $u \in \mathcal{D} \setminus \{\epsilon\}$ to be the least non-negative integer such that $u^{(\delta(u))} = 1_{l(u)}$. The degree of the empty path is assumed to be equal to zero. For example, if $u = aa\bar{a}\bar{a}aa\bar{a}a\bar{a}\bar{a}$ then $\delta(u) = 4$.

The main properties of the degree are given in the following result.

Proposition 4.4. *For every non-empty Dyck path u we have:*
i) $\delta(\widetilde{u}) = \delta(u) - 1$, for every $u \neq 1_{l(u)}$.
ii) $\delta(au\bar{a}) = \delta(u)$.
iii) $0 \leq \delta(u) \leq l(u) - 1$, and $\delta(u) = l(u) - 1$ iff u is non-prime.
iv) If $u \prec v$, then $\delta(v) \leq \delta(u)$.

Proof. i) is obvious, whereas ii) is based on the equality $(aw\bar{a})^{(i)} = aw^{(i)}\bar{a}$, for every $i \in \mathbb{N}$.

For the proof of iii) we first show by induction with respect to the semilength of u that if u is non-prime then $\delta(u) = l(u) - 1$.

Indeed, if $l(u) = 2$, then $u = a\bar{a}a\bar{a}$ and $\delta(u) = 1 = l(u) - 1$. Assume that the result holds for every non-prime Dyck path with semilength equal to $n - 1$, where $n \geq 3$, and let u be a non-prime path of semilength n. Then, using the prime decomposition of u, there exists a finite sequence $(w_i)_{i\in[k]}$, $k \geq 2$, of Dyck words such that

$$u = aw_1\bar{a}aw_2\bar{a}a\cdots\bar{a}aw_{k-1}\bar{a}aw_k\bar{a}.$$

It follows easily that

$$\widetilde{u} = a\widetilde{w}_i a\bar{a}\widetilde{w}_2 a\bar{a}\cdots a\bar{a}\widetilde{w}_{k-1}a\bar{a}\widetilde{w}_k\bar{a}.$$

If we set

$$z = \widetilde{w}_i a\bar{a}\widetilde{w}_2 a\bar{a}\cdots a\bar{a}\widetilde{w}_{k-1}a\bar{a}\widetilde{w}_k,$$

then $z \in \mathcal{D}$, $l(z) = n-1$, $\widetilde{u} = az\bar{a}$ and z is non-prime. Thus, by the induction hypothesis we have $\delta(z) = l(z) - 1$ and so

$$\delta(u) = \delta(\widetilde{u}) + 1 = \delta(z) + 1 = l(z) = l(u) - 1.$$

Next, we note that $0 \leq \delta(u)$ and $\delta(u) = 0$ iff $u = 1_{l(u)}$. Furthermore, if $u \neq 1_{l(u)}$, there exists $\nu \in \mathbb{N}$ such that $u = a^\nu w\bar{a}^\nu$, where w is a non-prime Dyck path.

Then, by ii) we deduce that

$$\delta(u) = \delta(w) = l(w) - 1 \leq l(u) - 1.$$

It remains to check that $\delta(u) < l(u) - 1$ for every prime Dyck path u. Indeed, $u = aw\bar{a}$ with $w \in \mathcal{D}$, so that $\delta(u) = \delta(w) \leq l(w) - 1 = l(u) - 2$.

Finally, we show iv) using induction with respect to the semilength of u. It is clear that the result is true when $l(u) = 1$. Assuming that the result holds for Dyck paths of semilength $n-1$, where $n \geq 2$, we will show that if $u, v \in \mathcal{D}_n$ with $u \prec v$ then $\delta(v) \leq \delta(u)$. Clearly, by iii), it is enough to restrict ourselves to the case where u is a prime word. Thus, if $u = au'\bar{a}$ where $u' \in \mathcal{D}_{n-1}$, it follows easily that $v = av'\bar{a}$, where $v' \in \mathcal{D}_{n-1}$ and $u' \prec v'$. From the induction hypothesis it follows that $\delta(v) = \delta(v') \leq \delta(u') = \delta(u)$. □

We conclude this section with the following result.

Proposition 4.5. *The number of all $u \in \mathcal{D}_n$, $n \geq 2$, with degree equal to k, where $k \in [n-1]$, is equal to $C_{k+1} - C_k$.*

Proof. Clearly, since every non-empty Dyck path u can be written uniquely in either of the forms $u = aw\bar{a}$ or $u = aw\bar{a}v$, where $w, v \in \mathcal{D}$ and $v \neq \epsilon$, applying Proposition 4.4 we obtain that

$$F_\delta(x, y) = 1 + xF_\delta(x, y) + xC(xy)(C(xy) - 1).$$

It follows that

$$\begin{aligned}
F_\delta(x, y) &= \frac{1 + y^{-1}\sum\limits_{n=1}^{\infty} C_n(xy)^n - x\sum\limits_{n=0}^{\infty} C_n(xy)^n}{1-x} \\
&= \frac{1 + \sum\limits_{n=1}^{\infty}\left(C_n - C_{n-1}\right) y^{n-1}x^n}{1-x} \\
&= \left(\sum_{n=0}^{\infty} g_n(y)x^n\right)\left(\sum_{n=0}^{\infty} x^n\right),
\end{aligned}$$

where

$$g_n(y) = \begin{cases} \left(C_n - C_{n-1}\right) y^{n-1}, & \text{if } n \geq 1; \\ 1, & \text{if } n = 0. \end{cases}$$

Therefore, we have

$$
\begin{aligned}
F_\delta(x,y) &= \sum_{n=0}^{\infty}\sum_{k=0}^{n} g_k(y)x^n \\
&= \sum_{n=0}^{\infty} x^n + \sum_{n=2}^{\infty}\sum_{k=1}^{n-1}(C_{k+1}-C_k)y^k x^n,
\end{aligned}
$$

which gives the required result. □

5 The Möbius function

In this section we study the Möbius function of $\mathcal{D}$ and its powers. We recall [20] that the Möbius function μ of a poset $(P,\preceq)$ is defined by

$$\mu(x,y) = -\sum_{x\preceq z\prec y}\mu(x,z) \text{ for } x\prec y \text{ and } \mu(x,x)=1.$$

Furthermore, the k-th power of μ, for $k\geq 2$, is defined by

$$\mu^k(x,y) = \sum_{x=x_0\preceq x_1\preceq\cdots\preceq x_k=y}\mu(x_0,x_1)\mu(x_1,x_2)\cdots\mu(x_{k-1},x_k).$$

It is known [3] that if $(P,\preceq)$ is a locally finite distributive lattice, then its Möbius function is given by the formula

$$\mu(x,y) = \begin{cases}(-1)^\nu, & \text{if } y \text{ is a join of } \nu \text{ elements covering } x;\\ 0, & \text{if } y \text{ is not a join of elements covering } x.\end{cases}$$

For the lattice of Dyck paths, we have that $v\in(u,\widetilde{u}]$ with $l(u,v)=\nu$ iff v is obtained by turning ν of the valleys of u into peaks or, equivalently, iff v is a join of ν elements of D covering u. Thus, from the above formula we obtain the following result.

Proposition 5.1. *The Möbius function of $\mathcal{D}$ is given by the formula*

$$\mu(u,v) = \begin{cases}(-1)^{l(u,v)}, & \text{if } u\preceq v\preceq\widetilde{u};\\ 0, & \text{otherwise}\end{cases}$$

for every $u,v\in\mathcal{D}$, where $l(u,v)$ denotes the length of the interval $[u,v]$.

In the sequel we study the powers of the Möbius function of $\mathcal{D}_n$. For this, we need the following result, which is an easy consequence of Proposition 5.1.

Corollary 5.2. *For every multichain $u_0\preceq u_1\preceq\cdots\preceq u_k$ in $\mathcal{D}_n$ with*

$$\mu(u_0,u_1)\mu(u_1,u_2)\cdots\mu(u_{k-1},u_k)\neq 0$$

we have $u_i\preceq\widetilde{u_{i-1}}$ and $u_i\preceq u_0^i$ for every $i\in[k]$.

Clearly, if $u \in \mathcal{D}_n$ and $k < \delta(u)$, for every multichain $u = u_0 \preceq u_1 \preceq \cdots \preceq u_k = 1_n$ we have that $u^{(k)} \prec u_k$ so that, by the above Corollary, we deduce that

$$\mu(u_0, u_1)\mu(u_1, u_2) \cdots \mu(u_{k-1}, u_k) = 0.$$

This shows that $\mu^k(u, 1_n) = 0$ for every $k < \delta(u)$.

In the following result we consider the case where $k = \delta(u)$.

Proposition 5.3. *Let $u \in \mathcal{D}_n$ and $j, \nu \in \mathbb{N}^*$ such that $u^{(j)} = 0^{\nu}_{l(u)}$. Then, we have that*

$$\mu^{\delta(u)}(u, 1_n) = (-1)^{l(u^{(j)}, 1_n)} \mu^j(u, u^{(j)}).$$

Proof. Let $u = u_0 \preceq u_1 \preceq \cdots \preceq u_{\delta(u)} = 1_n$ be a multichain of $\mathcal{D}_n$ with

$$\mu(u_0, u_1)\mu(u_1, u_2) \cdots \mu(u_{\delta(u)-1}, u_{\delta(u)}) \neq 0;$$

then, by Corollary 5.2 it follows that $u_i \preceq \widetilde{u_{i-1}}$ and $u_i \preceq u^{(i)}$ for every $i \in [\delta(u)]$. We show that $u_i = u^{(i)}$ for every $i \geq j$.

Indeed, if this is not true, let ξ be the greatest element of $[j, \delta(u)]$ such that $u_\xi \prec u^{(\xi)}$.

Clearly, since $u_{\delta(u)} = 1_{l(u)} = u^{(\delta(u))}$, we have that $\xi < \delta(u)$.

Since $u^{(\xi+1)} = u_{\xi+1} \preceq \widetilde{u_\xi} \preceq u^{(\xi+1)}$, we obtain that $u^{(\xi+1)} = \widetilde{u_\xi}$.

Furthermore, since $u^{(\xi)} = 0_n^{(\nu+\xi-j)}$, $u^{(\xi+1)} = 0_n^{(\nu+\xi-j+1)}$ and the antifilling of $0_n^{(k+1)}$ is $0_n^{(k)}$ for every $k \in [n-2]$, we obtain that $u^{(\xi)}$ is the antifilling of $u^{(\xi+1)}$, though $\widetilde{u_\xi} = u^{(\xi+1)}$ and $u_\xi \prec u^{(\xi)}$, which is a contradiction.

Thus, $u_i = u^{(i)}$ for every $i \geq j$.

It follows that

$$\mu^{\delta(u)}(u, 1_n) = \sum \mu(u_0, u_1)\mu(u_1, u_2) \cdots \mu(u_{j-1}, u^{(j)})\mu(u^{(j)}, u^{(j+1)})\mu(u^{(j+1)}, u^{(j+2)}) \cdots \mu(u^{(\delta(u)-1)}, u^{(\delta(u))})$$

where the sum is taken over all multichains $u = u_0 \preceq u_1 \preceq \cdots u_j = u^{(j)}$ of $\mathcal{D}_n$.

Since, by Proposition 5.1, we have

$$\begin{aligned}
\mu(u^{(j)}, u^{(j+1)})\mu(u^{(j+1)}, u^{(j+2)}) \cdots \mu(u^{(\delta(u)-1)}, u^{(\delta(u))}) \\
&= (-1)^{l(u^{(j)}, u^{(j+1)})}(-1)^{l(u^{(j+1)}, u^{(j+2)})} \cdots (-1)^{l(u^{(\delta(u)-1)}, u^{(\delta(u))})} \\
&= (-1)^{l(u^{(j)}, 1_n)},
\end{aligned}$$

we deduce that

$$\mu^{\delta(u)}(u, 1_n) = (-1)^{l(u^{(j)}, 1_n)} \mu^j(u, u^{(j)}).$$

□

Remark Let $\mathcal{N}$ be the set of all non-empty Dyck paths u such that $\widetilde{u} = 0^{\nu}_{l(u)}$ for some $\nu \in \mathbb{N}^*$. Then, taking $j = 1$ in the previous proposition, we obtain that

$$\begin{aligned}
\mu^{\delta(u)}(u, 1_{l(u)}) &= (-1)^{l(\widetilde{u}, 1_{l(u)})} \mu(u, \widetilde{u}) \\
&= (-1)^{l(\widetilde{u}, 1_{l(u)})}(-1)^{l(u, \widetilde{u})} \\
&= (-1)^{l(u, 1_{l(u)})}
\end{aligned}$$

for every $u \in \mathcal{N}$.

In particular if $u = 0_n$ for some $n \in \mathbb{N}^*$, we have that $\delta(u) = n - 1$ and

$$\mu^{n-1}(0_n, 1_n) = (-1)^{\binom{n}{2}}.$$

Thus, by Lemma 4.1 in [6] we deduce that the zeta polynomial of $\mathcal{D}_n$ satisfies the following formula

$$Z(\mathcal{D}_n, -k) = \begin{cases} (-1)^{\binom{n}{2}}, & \text{if } k = n-1; \\ 0, & \text{if } 1 \leq k < n-1. \end{cases}$$

We close this section by enumerating the sets $\mathcal{N} \cap \mathcal{D}_n$.

For this, we consider the set

$$\mathcal{N}_{\nu,n} = \{u \in \mathcal{D}_n : \widetilde{u} = 0_n^{\nu}\}$$

where $\nu \in \mathbb{N}^*$ and $\nu \leq n - 1$.

Clearly, for every $p \in \mathbb{N}^*$, by considering the bijection $u \to a^p u \bar{a}^p$ we can deduce that

$$|\mathcal{N}_{\nu,n}| = |\mathcal{N}_{\nu+p,n+p}| \tag{7}$$

for every $p \in \mathbb{N}^*$.

Furthermore, we have the following result.

Proposition 5.4. *For every $\nu \in \mathbb{N}^*$, the sequence $(\mathcal{N}_{\nu,n})$, $n \geq \nu + 1$ satisfies the following relation*

$$|\mathcal{N}_{\nu,n}| = F_{n-\nu+2},$$

where (F_n) denotes the Fibonacci sequence.

Proof. In view of relation (7) it is enough to show that

$$|\mathcal{N}_{1,n}| = F_{n+1}$$

for every $n \geq 2$.

Clearly, since $|\mathcal{N}_{1,2}| = 2$ and $|\mathcal{N}_{1,3}| = 3$, it is enough to show that

$$|\mathcal{N}_{1,n}| = |\mathcal{N}_{1,n-1}| + |\mathcal{N}_{1,n-2}|$$

for every $n \geq 4$.

Every element of $\mathcal{N}_{1,n}$ is obtained by turning some peaks of $\widetilde{0}_n$ into valleys. However, in this procedure for the generation of the elements of $\mathcal{N}_{1,n}$ we must turn at least one of each pair of consecutive peaks into valleys.

Thus, if A_1 (resp. A_2) is the set that consists of all elements of $\mathcal{N}_{1,n}$ that pass from the point $(2, 0)$ (resp. $(2, 2)$), then $\{A_1, A_2\}$ is a partition of $\mathcal{N}_{1,n}$.

Clearly, $A_1 = \mathcal{N}_{1,n-1}$ and since the elements of A_2 must pass from the point $(4, 0)$, we have $A_2 = \mathcal{N}_{1,n-2}$, which gives the required result. □

We note that using the previous proposition, we obtain by a simple summation that $|\mathcal{N} \cap \mathcal{D}_n| = F_{n+3} - 3$, (A006327 of [18]).

6 Dyck paths and permutations

We recall that a simple reduction of a permutation $\pi = \pi(1)\pi(2)\cdots\pi(n)$ is a permutation obtained from π by interchanging some $\pi(i)$ with $\pi(i+1)$, provided that $\pi(i) > \pi(i+1)$.

The weak Bruhat order $\ltimes$ is defined on the symmetric group S_n as follows:

$\sigma \ltimes \pi$ iff σ can be obtained from π by a sequence of simple reductions.

The poset $(S_n, \ltimes)$ is a well known distributive lattice, graded of rank $\binom{n}{2}$ and has many interesting properties [7, 19]. In the following, we examine the connection between the lattices $(S_n, \ltimes)$ and $(D_n, \preceq)$. For this, we first define the set $\mathcal{L}_n$ of all finite sequences $(A_i)_{i\in[n]}$ of pairwise disjoint subsets of $[n]$ such that $[\nu] \subseteq \bigcup\limits_{i=1}^{\nu} A_i$ for every $\nu \in [n]$ and $\bigcup\limits_{i=1}^{n} A_i = [n]$.

For example, $\mathcal{L}_3 = \{(\{1\},\{2\},\{3\}),(\{1\},\{2,3\},\emptyset),(\{1,2\},\emptyset,\{3\}),$
$(\{1,2\},\{3\},\emptyset),(\{1,3\},\{2\},\emptyset),(\{1,2,3\},\emptyset,\emptyset)\}$.

We will show that the sets S_n and $\mathcal{L}_n$ can be identified.

Indeed, for $\sigma \in S_n$ and $i \in [n]$ we define A_i^σ to be the set of all elements $j \in [n]$ for which there exist exactly $i-1$ elements of σ, which are less than j and lie on the left of j in σ.

We can easily check that the sequence $(A_i^\sigma)_{i\in[n]}$ belongs to $\mathcal{L}_n$.

Conversely, if $(A_i)_{i\in[n]} \in \mathcal{L}_n$ then there exists unique $\sigma \in S_n$ such that $A_i^\sigma = A_i$ for every $i \in [n]$.

For the construction of σ we define recursively a finite sequence $(\sigma_j)_{j\in[n]}$, such that $\sigma_j \in S_j$ and for $j > 1$ with $j \in A_i$ where $i \in [n]$, σ_j is generated from σ_{j-1} by inserting j before the i^{th} element of σ_{j-1} if $i < j$, or by placing j at the end of σ_{j-1} if $i = j$. Then, for $\sigma = \sigma_n$ we have $A_i^\sigma = A_i$ for each $i \in [n]$.

From the above discussion we have the following result.

Proposition 6.1. *The mapping $\sigma \to (A_i^\sigma)_{i\in[n]}$ is a bijection between the sets S_n and $\mathcal{L}_n$.*

Remark. Using the above bijection we can characterize the set of all permutations of $S_n(312)$ (i.e. the ones avoiding the pattern 312) as follows: $\sigma \in S_n(132)$ iff for every $j, k \in [n]$ with $j \in A_i^\sigma$, $k \in A_l^\sigma$ and $i < l$ we have that $j < k$.

Indeed, assume that $\sigma \in S_n(312)$ and $j, k \in [n]$ with $j \in A_i^\sigma$, $k \in A_l^\sigma$, $i < l$ and $j > k$. Since $i < l$, we have that j lies on the left of k and there exists some element m which is less that k and lies between them in σ. Then, the triplet jmk is an appearance of the pattern 312 in σ, which is a contradiction.

Conversely, assume that the sequence $(A_i^\sigma)_{i\in[n]}$ satisfies the above condition though σ contains the pattern 312. Let jmk be the first appearance of the pattern 312 in σ, with $j \in A_i^\sigma$ and $k \in A_l^\sigma$. It follows that each element lying on the left of j in σ that is less than j is also less than k. Thus, $i < l$ although $j > k$, which is a contradiction.

From the above remark it follows that if $\sigma \in S_n(312)$, then every non-empty A_i^σ consists of consecutive integers.

Next, we consider the family of sets $(\Gamma_u)_{u\in\mathcal{D}_n}$, where

$$\Gamma_u = \{\sigma \in S_n : d(u) = (|A_i^\sigma|)_{i\in[n]}\}.$$

Proposition 6.2. *The family $(\Gamma_u)_{u\in\mathcal{D}_n}$ is a partition of S_n such that if $\sigma\in\Gamma_u$, $\pi\in\Gamma_w$ and π covers σ, then w covers u.*

Proof. We first show that $\Gamma_u \neq \emptyset$, for every $u\in\mathcal{D}_n$. Indeed, if $d(u)=(d_i)_{i\in[n]}$, set $A_1=[d_1]$ and for $i>1$, $A_i=\emptyset$ iff $d_i=0$ and $A_i=[\sum_{j=1}^{i-1} d_j+1, \sum_{j=1}^{i} d_j]$ iff $d_i\neq 0$. It follows that the sequence $(A_i)_{i\in[n]}$ belongs to $\mathcal{L}_n$ and $|A_i|=d_i$ for every $i\in[n]$, so that by Proposition 6.1 there exists $\sigma\in S_n$ such that $d(u)=(|A_i^\sigma|)_{i\in[n]}$ and $\sigma\in\Gamma_u$.

Next, if $\sigma\in S_n$, since the sequence $(A_i^\sigma)_{i\in[n]}$ belongs to $\mathcal{L}_n$, it follows that $(|A_i^\sigma|)_{i\in[n]}$ is a dominating sequence, so that there exists unique $u\in\mathcal{D}_n$ such that $d(u)=(|A_i^\sigma|)_{i\in[n]}$ and hence $\sigma\in\Gamma_u$.

This shows that $\bigcup_{u\in\mathcal{D}_n}\Gamma_u=S_n$, and since the sets Γ_u are pairwise disjoint, the family $(\Gamma_u)_{u\in\mathcal{D}_n}$ is a partition of S_n.

Finally, since π covers σ, there exists unique $k\in[n-1]$ such that

$$\sigma(j)=\pi(j) \text{ for every } j\in[n]\setminus\{k,k+1\} \text{ and } \sigma(k)=\pi(k+1)<\pi(k)=\sigma(k+1).$$

Then, if $\pi(k)\in A_\nu^\pi$ we have that

$$\pi(k)\in A_{\nu+1}^\sigma,\ A_i^\pi=A_i^\sigma \text{ for every } i\in[n]\setminus\{\nu,\nu+1\},\ A_\nu^\pi=A_\nu^\sigma\cup\{\pi(k)\},\ A_{\nu+1}^\pi=A_{\nu+1}^\sigma\setminus\{\pi(k)\}.$$

Thus, since $d(u)=(|A_i^\sigma|)_{i\in[n]}$, $d(w)=(|A_i^\pi|)_{i\in[n]}$, $|A_i^\pi|=|A_i^\sigma|$ for every $i\in[n]\setminus\{\nu,\nu+1\}$, $|A_\nu^\pi|=|A_\nu^\sigma|+1$, and $|A_{\nu+1}^\pi|=|A_{\nu+1}^\sigma|-1$, we deduce that w covers u. □

Remarks

1. By Proposition 6.2 it follows easily that if $\sigma\in\Gamma_u$ and $\pi\in\Gamma_w$ with $\sigma\ltimes\pi$, then $u\preceq w$.

2. Since $\Gamma_{0_n}=\{\hat{0}_n\}$ and $\Gamma_{1_n}=\{\hat{1}_n\}$ where $\hat{0}_n$ and $\hat{1}_n$ are the least and greatest element of S_n respectively, by Proposition 6.2 it follows that for $u\in\mathcal{D}_n$ we have

$$\rho(\sigma)=\rho(u) \text{ for every } \sigma\in\Gamma_u,$$

where ρ denotes the rank function.

3. Following the first part of proof of Proposition 6.2 we realize that the sequence $(A_i^\sigma)_{i\in[n]}$ constructed for every $u\in\mathcal{D}_n$ satisfies the conditions of the previous remark and hence $\sigma\in S_n(312)$. Since the number of permutations of $S_n(312)$ is equal to C_n, we deduce that each Γ_u contains exactly one element of $S_n(312)$. Thus, we can define a bijection f from $S_n(312)$ to D_n, such that $f(\sigma)=u$ iff $\sigma\in\Gamma_u$. This bijection has been also presented in different ways in [1], [2] and [10].

In the following, we will evaluate the cardinal number of Γ_u for a Dyck path $u\in\mathcal{D}_n$ with $n\geq 2$. For this we need to consider the Dyck path $u'\in\mathcal{D}_{n-1}$ obtained from the path $u=a^\nu\bar{a}\tau\in\mathcal{D}_n$ if we delete its first peak, i.e., the path $u'=a^{\nu-1}\tau$.

Lemma 6.3. *For every $u\in\mathcal{D}_n$ with $d(u)=(d_i)_{i\in[n]}$ and $d(u')=(d'_i)_{i\in[n-1]}$, we have*

$$d'_1=d_1+d_2-1, d'_i=d_{i+1} \text{ for every } i\in[2,n-1]$$

and

$$|\Gamma_u|=\binom{d_1+d_2-1}{d_2}|\Gamma_{u'}|. \tag{8}$$

Proof. Clearly, the proof of the first part of this lemma is evident. So we restrict ourselves to the proof of relation (8).

For $\sigma' \in \Gamma_{u'}$ and a subset $B \subseteq A_1^{\sigma'}$ with $|B| = d_2$, we consider the finite sequence $(A_i)_{i\in[n]}$ defined as follows :

$A_1 = \{1\} \cup \{x : x-1 \in A_1^{\sigma'} \setminus B\}$, $A_2 = \{x : x-1 \in B\}$ and $A_i = \{x : x-1 \in A_{i-1}^{\sigma'}\}$ for $i \geq 3$.

Then, since $(A_i^{\sigma})_{i\in[n-1]} \in \mathcal{L}_{n-1}$, by the previous construction it follows that $(A_i)_{i\in[n]} \in \mathcal{L}_n$, so that by Proposition 6.1 there exists unique $\sigma \in S_n$ such that

$A_1^{\sigma} = \{1\} \cup \{x : x-1 \in A_1^{\sigma'} \setminus B\}$, $A_2^{\sigma} = \{x : x-1 \in B\}$ and $A_i^{\sigma} = \{x : x-1 \in A_{i-1}^{\sigma'}\}$.

So $|A_1^{\sigma}| = 1 + (|A_1^{\sigma'}| \setminus |B|) = d_1$, $|A_2^{\sigma}| = |B| = d_2$ and $|A_i^{\sigma}| = |A_{i-1}^{\sigma'}| = d'_{i-1} = d_i$ for every $i \geq 3$. Thus, $d(u) = (|A_i^{\sigma}|)_{i\in[n]}$ and $\sigma \in \Gamma_u$.

Moreover, we will show that every $\sigma \in \Gamma_u$ is generated by a unique pair (σ', B) as above. Indeed, given $\sigma \in \Gamma_u$ we consider the sequence $(A'_i)_{i\in[n-1]}$ of sets in $[n-1]$ defined by

$$A'_1 = \{x : x+1 \in A_1^{\sigma} \cup A_2^{\sigma}\},\ \ A'_i = \{x : x+1 \in A_{i+1}^{\sigma}\},$$

as well as the set

$$B = \{x : x+1 \in A_2^{\sigma}\} \subseteq A'_1.$$

Then, $(A'_i) \in \mathcal{L}_{n-1}$ and so by Proposition 6.1 there exists a unique $\sigma' \in S_{n-1}$ such that $A'_i = A_i^{\sigma'}$ for every $i \in [n-1]$. It follows that $\sigma' \in \Gamma_{u'}$ and hence (σ', B) is the required pair.

Thus, since $|A_1^{\sigma'}| = d_1 + d_2 - 1$ and $|B| = d_2$, we deduce that each permutation $\sigma' \in \Gamma_{u'}$ generates exactly $\binom{d_1+d_2-1}{d_2}$ permutations $\sigma \in \Gamma_u$ according to the above procedure.

This shows that

$$|\Gamma_u| = \binom{d_1+d_2-1}{d_2} |\Gamma_{u'}|. \qquad \square$$

Proposition 6.4. *If $u \in \mathcal{D}_n$ with $d(u) = (d_i)_{i\in[n]}$, we have that*

$$|\Gamma_u| = \prod_{j=1}^{n-1} \frac{\sum_{i=1}^{j} d_i - j + 1}{d_j!}.$$

Proof. We consider the finite sequence $(u_j)_{j\in[n]}$ of Dyck paths, where $u_1 = u$ and for $j > 1$ the Dyck path u_j is obtained from u_{j-1} by deleting its first peak.

Clearly, $u_j \in \mathcal{D}_{n-j+1}$ for every $j \in [n]$; if $d(u_j) = (d_i^j)_{i\in[n-j+1]}$, by Lemma 6.3 we have that

$$d_1^j = d_1^{j-1} + d_2^{j-1} - 1, d_i^j = d_{i+1}^{j-1}$$

and

$$|\Gamma_{u_{j-1}}| = \binom{d_1^{j-1} + d_2^{j-1} - 1}{d_2^{j-1}} |\Gamma_{u_j}| \tag{9}$$

for every $j \in [2, n]$.

It is easy to check that $d_2^{j-1} = d_j$ and $d_1^{j-1} + d_2^{j-1} - 1 = \sum_{i=1}^{j} d_i - j + 1$, for every $j \in [2, n]$.

Furthermore, using the previous equalities and applying formula (9) for every $j \in [2, n]$, we obtain that

$$
\begin{aligned}
|\Gamma_u| &= \prod_{j=2}^{n} \binom{\sum_{i=1}^{j} d_i - j + 1}{d_j} |\Gamma_{u_n}| \\
&= \prod_{j=1}^{n-1} \frac{\sum_{i=1}^{j} d_i - j + 1}{d_j!}.
\end{aligned}
$$

□

We note that since the family $(\Gamma_u)_{u \in \mathcal{D}_n}$ is a partition of S_n, by the previous proposition we obtain an identity for the factorial number, i.e.,

$$
\sum_{d} \prod_{j=1}^{n-1} \frac{\sum_{i=1}^{j} d_i - j + 1}{d_j!} = n!
$$

where the sum is taken over all dominating sequences $d = (d_i)_{i \in [n]}$.

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