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A Note on the Postage Stamp Problem

Amitabha Tripathi
Department of Mathematics
Indian Institute of Technology
Hauz Khas
New Delhi - 110016
India
atripath@maths.iitd.ac.in

Abstract

Let h, k be fixed positive integers, and let A be any set of positive integers. Let $hA := \{a_1 + a_2 + \cdots + a_r : a_i \in A, r \leq h\}$ denote the set of all integers representable as a sum of no more than h elements of A, and let $n(h, A)$ denote the largest integer n such that $\{1, 2, \ldots, n\} \subseteq hA$. Let $n(h, k) = \max_A n(h, A)$, where the maximum is taken over all sets A with k elements. The purpose of this note is to determine $n(h, A)$ when the elements of A are in arithmetic progression. In particular, we determine the value of $n(h, 2)$.

1 Introduction

A set $A = \{a_1 < a_2 < \cdots < a_k\}$ is called an *h-basis* for a positive integer n if each of $1, 2, \ldots, n$ is expressible as a sum of *at most* h (not necessarily distinct) elements of A. In order that A be an h-basis for n, it is necessary that $a_1 = 1$. For fixed positive integers h and k, let $n(h, k)$ denote the largest integer for which an h-basis of k elements exists. The problem of determining $n(h, k)$ is apparently due to Rohrbach [1], and has been studied often. A large and extensive bibliography can be found in a paper of Alter and Barnett [2]. The *Postage Stamp Problem* derives its name from the situation where we require the largest integer $n = n(h, k)$ such that all stamp values from 1 to n may be made up from a collection of k integer-valued stamp denominations with the restriction that an envelope that can have no more than h stamps, repetitions being allowed. An additional related problem is to determine all sets with k elements that form an h-basis for $n(h, k)$. We call such a set an *extremal h-basis*.

It is easy to see that $n(1, k) = k$ with unique extremal basis $\{1, 2, \ldots, k\}$ and that $n(h, 1) = h$ with unique extremal basis $\{1\}$. The result $n(h, 2) = \lfloor (h^2 + 6h + 1)/4 \rfloor$ with unique extremal basis $\{1,(h+3)/2\}$ for odd h and $\{1,(h+2)/2\}$ and $\{1,(h+4)/2\}$ for even h has been rediscovered several times, for instance by Stöhr [3, 4] and by Stanton, Bate and Mullin [5]. No other closed-form formula is known for any other pair (h,k) where one of h,k is fixed. In addition, $n(h,k)$ is known for several pairs (h,k); see [2]. Asymptotic bounds for $n(h,k)$ are due to Rohrbach [1], while bounds for $n(h,3)$ and $n(2,k)$ are due to Hofmeister [6], and due to Rohrbach [1], Klotz [7], Moser [8] and others, respectively.

Let h,k be fixed positive integers, and let A be any set of positive integers. Let

$$hA := \{a_1+a_2+\cdots+a_r : a_i \in A, r \le h\}$$

denote the set of all integers representable as a sum of no more than h elements of A, and let $n(h,A)$ denote the largest integer n such that $\{1,2,\ldots,n\} \subseteq hA$. Thus $n(h,k) = \max_A n(h,A)$, where the maximum is taken over all sets A with k elements. The purpose of this note is to determine $n(h,A)$ when the elements of A are in arithmetic progression. In particular, this easily gives the value of $n(h,2)$.

2 Main Result

Throughout this section, h,k,d are fixed positive integers. Let

$$A = \{1, 1+d, 1+2d, \ldots, 1+(k-1)d\}$$

be a k-term arithmetic progression. In order that $n \in hA$, it is necessary and sufficient that the equation

$$x_0 + (1+d)x_1 + (1+2d)x_2 + \cdots + (1+(k-1)d)x_{k-1} = \sum_{i=0}^{k-1} x_i + \left(\sum_{i=0}^{k-1} ix_i\right) d = n \quad (1)$$

has a solution, with $x_i \in \mathbb{N} \cup 0$ for all i and $\sum_{i=0}^{k-1} x_i \le h$.

Suppose $x_0, x_1, \ldots, x_{k-1}$ are nonnegative integers whose sum is at most a. Then $x_1 + 2x_2 + \cdots + (k-1)x_{k-1}$ assumes all values $0,1,\ldots,(k-1)a$ as the x_i's range over nonnegative integers whose sum does not exceed a. Indeed, to achieve the sum $q(k-1)+r$ for $0 \le q < a$ and $0 \le r < k-1$ or for $q = a$, we may choose $x_{k-1} = q$, $x_r = 0$ or 1 according as $r = 0$ or $r > 0$, and all other x_i zero. We are now in a position to state our main result.

Theorem 1 *Let h,k,d be positive integers. Then*

$$n(h, \{1, 1+d, 1+2d, \ldots, 1+(k-1)d\}) = \begin{cases} h, & \text{if } h \le d-1; \\ h+(k-1)(h+1-d)d, & \text{if } h \ge d. \end{cases}$$

Proof. We write $A = \{1, 1+d, 1+2d, \ldots, 1+(k-1)d\}$. The case $h \le d-1$ is easy to see. Henceforth, we assume $h \ge d$. Suppose $x_0, x_1, \ldots, x_{k-1}$ are chosen such that the sum in

(1) equals $n = n(h, A)$. If $\sum_{i=0}^{k-1} x_i < h$, x_0 may be incremented by 1 without violating the restriction on the sum of the x_i's, thereby achieving the sum $n(h, A)+1$. Thus $\sum_{i=0}^{k-1} x_i = h$, so that $n(h, A) \equiv h \pmod{d}$ by (1) and $m := \sum_{i=0}^{k-1} ix_i \le (k-1)h$.

Now $h + 1 + md \in hA$ if and only if (1) has a solution with $\sum_{i=0}^{k-1} x_i = h + 1 - \lambda d$ and $\sum_{i=0}^{k-1} ix_i = m + \lambda$ for some $\lambda \in \mathbb{N}$. Such a simultaneous solution exists precisely when $m+\lambda \le (h+1-\lambda d)(k-1)$, that is, when $m \le (h+1-\lambda d)(k-1)-\lambda \le (h+1-d)(k-1)-1$. Thus $h + 1 + md \notin hA$ for $m \ge (h + 1 - d)(k - 1)$, and $n(h, A) \le h + (k - 1)(h + 1 - d)d$.

It remains to show that every positive integer less than or equal to $h+(k-1)(h+1-d)d$ is an element of hA. Any such integer N can be expressed as $r + qd$, where r, q satisfy the inequalities $1 \le r \le h$ and $q \le (k - 1)r$, as follows. We choose the largest $r \equiv N \pmod{d}$ which is also less than or equal to h. Such an r is greater than or equal to $h+1-d$, so that $qd = N-r \le N-(h+1-d) \le h+(h+1-d)((k-1)d-1) < ((k-1)(h+1-d)+1)d$, and $q \le (k - 1)(h + 1 - d) \le (k - 1)r$. Thus $\sum_{i=0}^{k-1} x_i = r$ and $\sum_{i=0}^{k-1} ix_i = q$ is simultaneously solvable by the argument immediately preceding the Theorem. This completes the proof. □

Corollary 2 *For $h \ge 1$,*

$$n(h, 2) = \left\lfloor \frac{h^2 + 6h + 1}{4} \right\rfloor .$$

Moreover, the only extremal basis is $\{1, (h + 3)/2\}$ if h is odd, and $\{1, (h + 2)/2\}$ and $\{1, (h + 4)/2\}$ if h is even.

Proof. From Theorem 1,

$$n(h, 2) = h + \max_{d\ge1} (h + 1 - d)d = h + \left\lfloor \frac{(h + 1)^2}{4} \right\rfloor = \left\lfloor \frac{h^2 + 6h + 1}{4} \right\rfloor .$$

It is easy to see that the maximum is achieved at $d = (h + 1)/2$, so that there is only one extremal basis if h is odd and two such bases if h is even. □

Remark. The function $n(h, 2)$ is sequence A014616 in Sloane's table [9].

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