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A multidimensional version of a result of Davenport-Erdős

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Abstract

Davenport and Erdős showed that the distribution of values of sums of the form

$$S_h(x) = \sum_{m=x+1}^{x+h} \left(\frac{m}{p}\right),$$

where p is a prime and $\left(\frac{m}{p}\right)$ is the Legendre symbol, is normal as $h, p \to \infty$ such that $\frac{\log h}{\log p} \to 0$. We prove a similar result for sums of the form

$$S_h(x_1, \ldots, x_n) = \sum_{z_1=x_1+1}^{x_1+h} \cdots \sum_{z_n=x_n+1}^{x_n+h} \left(\frac{z_1 + \cdots + z_n}{p}\right).$$

1. Introduction

Given a prime number p, an integer x and a positive integer h, we consider the sum

$$S_h(x) = \sum_{m=x+1}^{x+h} \left(\frac{m}{p}\right),$$

where here and in what follows $\left(\frac{m}{p}\right)$ denotes the Legendre symbol. The expected value of such a sum is $\sqrt{h}$. If p is much larger than h, it is a very difficult problem to show that there is any cancellation in an individual sum $S_h(x)$ as above. The classical inequality of Pólya-Vinogradov (see [8], [10]) shows that $S_h(x) = O(\sqrt{p}\log p)$, and assuming the Generalized Riemann Hypothesis, Montgomery and Vaughan [7] proved that $S_h(x) = O(\sqrt{p}\log\log p)$. The results of Burgess [2] provide cancellation in $S_h(x)$ for smaller values of h, as small as $p^{1/4}$. One does expect to have cancellation in $S_h(x)$ for $h > p^\epsilon$, for fixed $\epsilon > 0$ and p large. This would imply the well-known hypothesis of Vinogradov that the smallest positive quadratic nonresidue mod p is $< p^\epsilon$, for any fixed $\epsilon > 0$ and p large enough in terms of ϵ. We mention that Ankeny [1] showed that assuming the Generalized Riemann Hypothesis, the smallest positive quadratic nonresidue mod p is $O(\log^2 p)$. It is much easier to obtain cancellation, even square root cancellation, if one averages $S_h(x)$ over x. In fact, Davenport and Erdős [5] entirely solved the problem of the distribution of values of $S_h(x)$, $0 \le x < p$, as $h, p \to \infty$ such that $\frac{\log h}{\log p} \to 0$. Under these growth conditions they showed that the distribution becomes normal. Precisely, they proved that

$$\frac{1}{p}M_p(\lambda) \to \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\lambda} e^{-\frac{1}{2}t^2}dt, \quad \text{as } p \to \infty,$$

where $M_p(\lambda)$ is the number of integers x, $0 \le x < p$, satisfying $S_h(x) \le \lambda h^{\frac{1}{2}}$.

For a fixed $n \ge 2$, we consider multidimensional sums of the form

$$S_h(x_1, \dots, x_n) = \sum_{z_1=x_1+1}^{x_1+h} \cdots \sum_{z_n=x_n+1}^{x_n+h} \left(\frac{z_1 + \cdots + z_n}{p}\right), \tag{1.1}$$

where p is a prime number, $x_1, \dots, x_n$ are integer numbers, and h is a positive integer. Upper bounds for individual sums of this type have been provided by Chung [3]. In this paper we investigate the distribution of values of these sums, and obtain a result similar to that of Davenport and Erdős. Let

$$c_n := \int_0^n f(t)^2 dt, \tag{1.2}$$

where $f(t)$ is the volume of the region in $\mathbb{R}^{n-1}$ defined by

$$\{(a_1, \dots, a_{n-1}) \in \mathbb{R}^{n-1} : 0 < a_i \le 1,\ i = 1, \dots, n-1;\ t - 1 \le a_1 + \cdots + a_{n-1} < t\}.$$

We will see that this constant c_n naturally appears as a normalizing factor in our distribution result below. Let $M_{n,p}(\lambda)$ be the number of lattice points $(x_1, \dots, x_n)$ with $0 \le x_1, \dots, x_n < p$, such that

$$S_h(x_1, \dots, x_n) \le \lambda c_n^{\frac{1}{2}} h^{n-\frac{1}{2}}.$$

Then we show that as $h, p \to \infty$ such that $\frac{\log h}{\log p} \to 0$, one has

$$\frac{1}{p^n}M_{n,p}(\lambda) \to \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\lambda} e^{-\frac{t^2}{2}}dt.$$

2. Estimating the moments

We now proceed to estimate higher moments of our sums $S_h(x_1, \ldots, x_n)$.

Lemma 1. *Let p be a prime number and let h and r be positive integers. Then*

$$\sum_{x_1,\ldots,x_n (\mathrm{mod}\ p)} S_h^{2r}(x_1, \ldots, x_n) = 1 \cdot 3 \cdots (2r-3)(2r-1)$$
$$\cdot \left(c_n h^{2n-1} + O_{n,r}(h^{2n-2})\right)^r \left(p^n + O_r(p^{n-1})\right) + O_r\left(h^{2nr} p^{n-\frac{1}{2}}\right), \quad (2.1)$$

and

$$\sum_{x_1,\ldots,x_n (\mathrm{mod}\ p)} S_h^{2r-1}(x_1, \ldots, x_n) = O_r\left(h^{n(2r-1)} p^{n-\frac{1}{2}}\right). \quad (2.2)$$

Proof. Consider first the case when the exponent is $2r$. We have

$$S_h(x_1, \ldots, x_n) = \sum_{a_1=1}^{h} \cdots \sum_{a_n=1}^{h} \left(\frac{x_1 + \cdots + x_n + a_1 + \cdots + a_n}{p}\right).$$

Therefore

$$S_h^{2r}(x_1, \ldots, x_n) =$$
$$\sum_{a_{1,1}=1}^{h} \cdots \sum_{a_{n,1}=1}^{h} \cdots \sum_{a_{1,2r}=1}^{h} \cdots \sum_{a_{n,2r}=1}^{h} \left(\frac{(x_1+\cdots+x_n+a_{1,1}+\cdots+a_{n,1})\cdots(x_1+\cdots+x_n+a_{1,2r}+\cdots+a_{n,2r})}{p}\right)$$

and so

$$\sum_{x_1,\ldots,x_n (\mathrm{mod}\ p)} S_h^{2r}(x_1, \ldots, , x_n)$$
$$= \sum_{\substack{a_{i,j}=1 \\ 1\le i\le n \\ 1\le j\le 2r}}^{h} \sum_{x_1,\ldots,x_n (\mathrm{mod}\ p)} \left(\frac{(x_1+\cdots+x_n+a_{1,1}+\cdots+a_{n,1})\cdots(x_1+\cdots+x_n+a_{1,2r}+\cdots+a_{n,2r})}{p}\right).$$

Divide the sets of n-tuples $\{(a_{1,i}, \ldots, a_{n,i}) : i = 1, \ldots, 2r\}$ into two types. If there exists an i such that the number of $j \in \{1, \ldots, 2r\}$ for which $a_{1,i} + \cdots + a_{n,i} = a_{1,j} + \cdots + a_{n,j}$ is odd, we say that it is of type 1. The others will be of type 2. First consider the sum of terms of type 1. Since for each fixed $x_2, \ldots, x_n$, the product $(x_1 + \cdots + x_n + a_{1,1} + \cdots + a_{n,1}) \cdots (x_1 + \cdots + x_n + a_{1,2r} + \cdots + a_{n,2r})$, as a polynomial in x_1, is not congruent mod p to the square of another polynomial, by Weil's bounds [11] we have

$$\sum_{x_2,\ldots,x_n (\mathrm{mod}\ p)} \sum_{x_1 (\mathrm{mod}\ p)} \left(\frac{(x_1+\cdots+x_n+a_{1,1}+\cdots+a_{n,1})\cdots(x_1+\cdots+x_n+a_{1,2r}+\cdots+a_{n,2r})}{p}\right)$$
$$= \sum_{x_2,\cdots,x_n (\mathrm{mod}\ p)} O_r(p^{1/2}) = O_r(p^{n-\frac{1}{2}}).$$

So the sum of terms of type 1 is $O_r\left(h^{2nr} p^{n-\frac{1}{2}}\right)$. Now consider the sum of terms of type 2. Since the polynomial $(x_1 + \cdots + x_n + a_{1,1} + \cdots + a_{n,1}) \cdots (x_1 + \cdots + x_n + a_{1,2r} + \cdots + a_{n,2r})$ is a perfect square in this case, the Legendre symbol is 1, except for those values of $x_1, \ldots, x_n$

for which this product vanishes mod p. Since the product has at most r distinct factors, for any values of $x_2, \ldots, x_n$ there are at most r values of x_1 for which the product vanishes mod p. Thus the sum over $x_1, \ldots, x_n$ is at most p^n, and at least $(p-r)p^{n-1}$. Hence the contribution of terms of type 2 is

$$F(h,n,r)\left(p^n + O_r(p^{n-1})\right),$$

where $F(h,n,r)$ is the number of sets $\{(a_{1,i}, \ldots, a_{n,i}) : i = 1, \ldots, 2r\}$ yielding multinomials of type 2, i.e., sets for which each value of $a_{1,i}+\cdots+a_{n,i}$ occurs an even number of times, as i runs over the set $\{1, 2, \ldots, 2r\}$. For any integer m with $n \le m \le nh$, let $N_m(h,n)$ be the number of n-tuples $(a_{1,i}, \ldots, a_{n,i})$ for which $1 \le a_{1,i}, \ldots, a_{n,i} \le h$ and $a_{1,i} + \cdots + a_{n,i} = m$. Then the number of pairs of n-tuples $(a_{1,i}, \ldots, a_{n,i})$, $(a_{1,j}, \ldots, a_{n,j})$, with $a_{1,i}+\cdots+a_{n,i} = a_{1,j}+\cdots+a_{n,j}$, is $\sum_m (N_m(h,n))^2$. In what follows we write simply N_m instead of $N_m(h,n)$. The number of ways to choose r such pairs of n-tuples (not necessarily distinct) is $(\sum_m N_m^2)^r$, and the number of ways to arrange these pairs in $2r$ places is $(2r-1)(2r-3)\cdots 3\cdot 1$. Hence,

$$F(h,n,r) \le 1\cdot 3\cdots(2r-3)(2r-1)\left(\sum_m N_m^2\right)^r.$$

On the other hand, the number of ways of choosing r pairs of distinct sums is at least

$$\left(\sum_m N_m^2\right)\left(\sum_m N_m^2 - \max_m\{N_m^2\}\right)\cdots\left(\sum_m N_m^2 - (r-1)\max_m\{N_m^2\}\right)$$
$$\ge \left(\sum_m N_m^2 - r\max_m\{N_m^2\}\right)^r,$$

and the number of different ways to arrange them in $2r$ places is $(2r-1)(2r-3)\cdots 3\cdot 1$. Thus

$$1\cdot 3\cdots(2r-3)(2r-1)\left(\sum_m N_m^2 - r\max_m N_m^2\right)^r \le F(h,n,r)$$
$$\le 1\cdot 3\cdots(2r-3)(2r-1)\left(\sum_m N_m^2\right)^r.$$

Next, we estimate the number $N_m(h,n) = N_m$. It is clear that for any m with $0 < m \le nh$, N_m is the number of lattice points in the region R_m in $\mathbb{R}^{n-1}$ given by

$$R_m := \begin{cases} 0 < a_i \le h, & \text{for } i = 1, \ldots, n-1; \\ m-h \le a_1 + \cdots + a_{n-1} < m. \end{cases}$$

We send the region R_m to the unit cube in $\mathbb{R}^{n-1}$ via the map $\mathbf{x} \mapsto \frac{\mathbf{x}}{h}$. Then we have

$$\overline{R_m} := \begin{cases} 0 < a_i \le 1, & \text{for } i = 1, \ldots, n-1; \\ \frac{m}{h} - 1 \le a_1 + \cdots + a_{n-1} < \frac{m}{h}. \end{cases}$$

By the Lipschitz principle [4] we know that

$$N_m = vol(R_m) + O_n(h^{n-2}) = h^{n-1} vol(\overline{R}_m) + O_n(h^{n-2}).$$

With f defined as in the Introduction, we may write $vol(\overline{R}_m) = f(\frac{m}{h})$. Then

$$\begin{aligned}\sum_{0<m\le nh} N_m^2 &= \sum_{0<m\le nh} h^{2n-2}\left(f\left(\frac{m}{h}\right)\right)^2 + \sum_{0<m\le nh} O_n(h^{2n-3}) \\ &= h^{2n-1}\sum_{0<m\le nh}\left(f\left(\frac{m}{h}\right)\right)^2\frac{1}{h} + O_n(h^{2n-2}) \\ &= h^{2n-1}\int_0^n (f(t))^2\,dt + O_n(h^{2n-2}) \\ &= h^{2n-1}c_n + O_n(h^{2n-2}), \quad \text{as } h\to\infty.\end{aligned}$$

Hence

$$F(h,n,r) = 1\cdot 3\cdots(2r-3)(2r-1)\left(c_n h^{2n-1} + O_{n,r}(h^{2n-2})\right)^r,$$

and (2.1) follows. It is clear that (2.2) holds, since there are no sets of type 2 in this case. This completes the proof of the lemma.

□

3. Main results

By using the estimates for the higher moments of $S_h(x_1,\dots,x_n)$ given in Lemma 1, we show that under appropriate growth conditions on h, p, the distribution of our sums $S_h(x_1,\dots,x_n)$ is normal.

Theorem 1. *Let h be any function of p such that*

$$h\to\infty, \quad \frac{\log h}{\log p}\to 0 \quad \text{as } p\to\infty. \tag{3.1}$$

Let $M_{n,p}(\lambda)$ denote the number of lattice points $(x_1,\dots,x_n)$, $0\le x_1,\dots,x_n<p$, such that

$$S_h(x_1,\dots,x_n)\le \lambda c_n^{\frac{1}{2}} h^{n-\frac{1}{2}},$$

with $S_h(x_1,\dots,x_n)$ defined by (1.1) and c_n defined by (1.2). Then

$$\frac{1}{p^n}M_{n,p}(\lambda)\to\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\lambda} e^{-\frac{t^2}{2}}\,dt, \quad \text{as } p\to\infty.$$

Proof. We consider the sum

$$\frac{1}{p^n}\sum_{x_1,\dots,x_n \pmod p}\left(\frac{1}{c_n^{1/2}h^{n-1/2}}S_h(x_1,\dots,x_n)\right)^r. \tag{3.2}$$

It follows from the above lemma that for each fixed r and n, if r is even, then the quantity from (3.2) is

$$1\cdot 3\cdots(r-3)(r-1)\left(1+O_{n,r}\left(\frac{1}{h}\right)\right)^r\left(1+O_r\left(\frac{1}{p}\right)\right) + O_{n,r}(h^{\frac{r}{2}}p^{-\frac{1}{2}}),$$

while if r is odd, the quantity from (3.2) is $O_{n,r}(h^{\frac{r}{2}}p^{-\frac{1}{2}})$. Using (3.1), we have that for each positive integer r,

$$\frac{1}{p^n}\sum_{x_1,\dots,x_n \pmod p}\left(\frac{1}{c_n^{1/2}h^{n-1/2}}S_h(x_1,\dots,x_n)\right)^r \to \mu_r, \quad \text{as } p\to\infty, \tag{3.3}$$

where $\mu_r = \begin{cases} 1 \cdot 3 \cdots (r-1), & \text{if } r \text{ is even;} \\ 0, & \text{if } r \text{ is odd.} \end{cases}$

Let $N_{n,p}(s)$ be the number of n-tuples $(x_1, \ldots, x_n)$ with $0 \le x_i < p$, $i = 1, \ldots, n$ such that $S_h(x_1, \ldots, x_n) \le s$. Then $N_{n,p}(s)$ is a non-decreasing function of s with discontinuities at certain integral values of s. We also note that $N_{n,p}(s) = 0$ if $s < -h^n$, $N_{n,p}(s) = p^n$ if $s \ge h^n$, and $M_{n,p}(\lambda) = N_{n,p}(\lambda c_n^{\frac{1}{2}} h^{n-\frac{1}{2}})$. We write (3.3) in the form

$$\frac{1}{p^n} \sum_{s=-h^n}^{h^n} \left(\frac{s}{c_n^{\frac{1}{2}} h^{n-\frac{1}{2}}} \right)^r (N_{n,p}(s) - N_{n,p}(s-1)) \to \mu_r, \quad \text{as } p \to \infty. \tag{3.4}$$

This is similar to relation (26) of Davenport-Erdős [5]. Following their argument, if we set

$$\Phi_{n,p}(t) = \frac{1}{p^n} N_{n,p}(t c_n^{\frac{1}{2}} h^{n-\frac{1}{2}}) = \frac{1}{p^n} M_{n,p}(t),$$

and

$$\Phi(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} e^{-\frac{1}{2}u^2} du,$$

we obtain

$$\int_{-\infty}^{\infty} t^r d\Phi_{n,p}(t) \to \int_{-\infty}^{\infty} t^r d\Phi(t), \quad \text{as } p \to \infty, \tag{3.5}$$

for any fixed positive integer r, which is the analogue of relation (28) from [5]. It now remains to show that, for each real number λ,

$$\Phi_{n,p}(\lambda) \to \Phi(\lambda), \quad \text{as } p \to \infty. \tag{3.6}$$

The assertion of (3.6) follows from the well-known fact (see [6]) in the theory of probability that if F_k and F are probability distributions with finite moments $m_{k,r}$, m_r of all orders, respectively, and if F is the unique distribution with the moments m_r such that $m_{k,r} \to m_r$ for all r as $k \to \infty$, then $F_k \to F$ as $k \to \infty$. We give the outline of the proof following the argument of Davenport-Erdős [5]. Suppose that (3.6) fails for some λ. Then we can find a subsequence $\{\Phi_{n,p'}\}$ and a $\delta > 0$ such that

$$|\Phi_{n,p'}(\lambda) - \Phi(\lambda)| \ge \delta, \quad \text{for all } p'. \tag{3.7}$$

By the two theorems of Helly (see the introduction to [9]) there exists a subsequence $\{\Phi_{n,p''}\}$ of $\{\Phi_{n,p'}\}$ which converges to a distribution Ψ at every point of continuity, and

$$\int_{-\infty}^{\infty} t^r d\Psi(t) = \lim_{p'' \to \infty} \int_{-\infty}^{\infty} t^r d\Phi_{n,p''} = \int_{-\infty}^{\infty} t^r d\Phi(t).$$

Since Φ is the only distribution with these special moments $\mu_1, \mu_2, \ldots$, we have $\Psi(t) = \Phi(t)$ for all t. This contradicts (3.7). Hence one concludes that, as $p \to \infty$,

$$\frac{1}{p^n} M_{n,p}(\lambda) = \Phi_{n,p}(\lambda) \to \Phi(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\lambda} e^{-\frac{1}{2}t^2} dt,$$

which completes the proof of the theorem. ☐

We remark that c_n can be explicitly computed for any given value of n. The following proposition provides an equivalent formulation of c_n, which allows for easier computations in higher dimensions. For any n, consider the polynomial in two variables

$$g_n(X,Y) = \sum_{l=0}^{n-1} \left(\sum_{k=0}^{l} (-1)^k \binom{n}{k} \binom{X+(l-k)Y+n-1}{n-1} \right)^2 .$$

Note that the total degree of $g_n(X,Y)$ is at most $2n-2$.

Proposition 1. *For any n,*

$$c_n = \sum_{k=0}^{2n-2} \frac{a_{n,k}}{k+1} ,$$

where $a_{n,k}$ is the coefficient of $X^k Y^{2n-2-k}$ in $g_n(X,Y)$.

Proof. We know that for fixed n and $h \to \infty$,

$$\sum_m N_m^2 = h^{2n-1} c_n + O_n(h^{2n-2}),$$

where $N_m = N_m(h,n)$ is the number of n-tuples $(a_1,\ldots,a_n)$ such that $a_1+\cdots+a_n = m$, with $1 \le a_i \le h$. Replacing m by $m' = m-n$ and each a_i by $b_i = a_i - 1$, we get $\sum_m N_m^2 = \sum_{m'} (N'_{m'})^2$, where $N'_{m'}$ is the number of n-tuples $(b_1,\ldots,b_n)$ such that $b_1+\cdots+b_n = m'$, with $0 \le b_i \le h-1$.

Now, the number of ways to obtain a sum of m' from n non-negative integers, with no restrictions, is $\binom{m'+n-1}{n-1}$. If we restrict any fixed b_i to satisfy the inequality $b_i \ge h$, then the number of ways drops to $\binom{m'-h+n-1}{n-1}$. If we restrict any two b_i, b_j to satisfy $b_i, b_j \ge h$ then we have $\binom{m'-2h+n-1}{n-1}$ ways, and so on.

Since for each k, there are $\binom{n}{k}$ ways to choose exactly k of the b_i's to be greater than h, we obtain by the inclusion-exclusion principle,

$$N'_{m'} = \sum_{0 \le k \le m'/h} (-1)^k \binom{n}{k} \binom{m'-kh+n-1}{n-1}.$$

So we have, for $lh \le m' < (l+1)h$, $0 \le l \le n-1$,

$$N'_{m'} = \sum_{k=0}^{l} (-1)^k \binom{n}{k} \binom{m'-kh+n-1}{n-1}.$$

Replacing m' by $s+lh$, with $0 \le s \le h-1$, we get

$$N'_{s+lh} = \sum_{k=0}^{l} (-1)^k \binom{n}{k} \binom{s+(l-k)h+n-1}{n-1}.$$

Therefore

$$\sum_{m'} (N'_{m'})^2 = \sum_{s=0}^{h-1} \sum_{l=0}^{n-1} \left(\sum_{k=0}^{l} (-1)^k \binom{n}{k} \binom{s+(l-k)h+n-1}{n-1} \right)^2$$
$$= \sum_{s=0}^{h-1} g_n(s,h).$$

It follows that

$$\sum_{s=0}^{h-1} g_n(s,h) = h^{2n-1}c_n + O_n(h^{2n-2}). \tag{3.8}$$

Now, the main contribution in $g_n(s,h)$ comes from the terms where the exponents of s and h add up to $2n-2$. Since for any $0 \le k \le 2n-2$,

$$\sum_{s=0}^{h-1} s^k = \frac{1}{k+1}h^{k+1} + O_n(h^k),$$

we obtain

$$\begin{aligned}\sum_{s=0}^{h-1} g_n(s,h) &= \sum_{s=0}^{h-1}\left(\sum_{k=0}^{2n-2} a_{n,k}s^k h^{2n-2-k} + \text{lower order terms}\right)\\ &= \sum_{k=0}^{2n-2}\sum_{s=0}^{h-1} a_{n,k}s^k h^{2n-2-k} + O_n(h^{2n-2})\\ &= \sum_{k=0}^{2n-2}\frac{a_{n,k}}{k+1}h^{2n-1} + O_n(h^{2n-2}).\end{aligned}$$

By combining this with (3.8), we obtain the desired result. □

For $n = 2,3,4,5,6$, one finds that $c_2 = \frac{2}{3}$, $c_3 = \frac{11}{20}$, $c_4 = \frac{151}{315}$, $c_5 = \frac{15619}{36288}$, $c_6 = \frac{655177}{1663200}$. The numerator and the denominator of c_n grow rapidly as n increases. For instance, for $n = 10$ and $n = 25$ we have

$$c_{10} = \frac{37307713155613}{121645100408832},$$

and

$$c_{25} = \frac{675361967823236555923456864701225753248337661154331976453}{3465993527260783822633915460520201577706853740052480000000}.$$

One can also work with boxes instead of cubes, and obtain similar distribution results. For example, in dimension two, we may consider the sum

$$S_{h,k}(x,y) = \sum_{u=x+1}^{x+h}\sum_{v=y+1}^{y+k}\left(\frac{u+v}{p}\right),$$

where x, y are any integers and h, k are positive integers, with $h \ge k$, say. Then, by using the same arguments as in the proof of Theorem 1, one can prove the following result.

Theorem 2. *Let h, k be functions of p such that*

$$h \ge k, \quad \frac{h}{k} \to \alpha, \quad k \to \infty, \quad \frac{\log k}{\log p} \to 0, \quad \text{as } p \to \infty.$$

Denote $\beta = \sqrt{\alpha - \frac{1}{3}}$ and $\beta' = \sqrt{1 - \frac{1}{3\alpha}}$. Let $M_p(\lambda)$ be the number of pairs (x,y) with $0 \le x, y < p$, x, y integers, such that $S_{h,k}(x,y) \le \lambda\beta k^{\frac{3}{2}}$. Let ${M_p}'(\lambda)$ be the number of pairs (x,y) with $0 \le x,y < p$, x, y integers, such that $S_{h,k}(x,y) \le \lambda\beta' h^{\frac{1}{2}}k$. Then, as $p \to \infty$,

$$\frac{1}{p^2}M_p(\lambda) \to \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\lambda} e^{-\frac{1}{2}x^2}dx,$$

and

$$\frac{1}{p^2} M_p{}'(\lambda) \to \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\lambda} e^{-\frac{1}{2}x^2} dx.$$

We remark that when h is much larger than k, $S_{h,k}(x, y)$ is close to k times the 1-dimensional sum $S_h(x + y)$. Also, in this case α is large, β' is close to 1, and the above statement for $M_{p'}(\lambda)$ approaches the 1-dimensional result of Davenport and Erdős. Note also that in case $\alpha = 1$, we have $\beta = \sqrt{2/3} = \sqrt{c_2}$, and the statement of Theorem 2 for $M_p(\lambda)$ coincides with that of Theorem 1 for $n = 2$.

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