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A SEQUENCE OF BINOMIAL COEFFICIENTS RELATED TO LUCAS AND FIBONACCI NUMBERS

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Abstract. Let $L(n,k) = \frac{n}{n-k}\binom{n-k}{k}$. We prove that all the zeros of the polynomial $L_n(x) = \sum_{k\geq 0} L(n,k)x^k$ are real. The sequence $L(n,k)$ is thus strictly log-concave, and hence unimodal with at most two consecutive maxima. We determine those integers where the maximum is reached. In the last section we prove that $L(n,k)$ satisfies a central limit theorem as well as a local limit theorem.

1. Introduction

A positive real sequence $(a_k)_{k=0}^n$ is said to be *unimodal* if there exist integers $k_0, k_1, 0 \leq k_0 \leq k_1 \leq n$ such that

$$a_0 \leq a_1 \leq \cdots \leq a_{k_0} = a_{k_0+1} = \cdots = a_{k_1} \geq a_{k_1+1} \geq \cdots \geq a_n.$$

The integers l, $k_0 \leq l \leq k_1$ are called the *modes* of the sequence. If $k_0 < k_1$ then $(a_k)_{k=0}^n$ is said to have a *plateau* of $k_1 - k_0 + 1$ elements; if $k_0 = k_1$ then it is said to have a *peak*. A real sequence is said to be *logarithmically concave* (log-concave for short) if

$$a_k^2 \geq a_{k-1}a_{k+1}, \quad 1 \leq k \leq n-1 \qquad (1)$$

If the inequalities in (1) are strict, then $(a_k)_{k=0}^n$ is said to be *strictly log-concave* (SLC for short). A sequence is said to be have *no internal zeros* if $i < j$, $a_i \neq 0$ and $a_j \neq 0$, then $a_k \neq 0$ for $i \leq k \leq j$. A log-concave sequence with no internal zeros is obviously unimodal, and if it is SLC, then it has at most two consecutive modes. The following result is sometimes useful in proving log-concavity. For a proof of this theorem, see Hardy and Littlewood [5].

Theorem 1. *(I. Newton) Let $(a_k)_{k=0}^n$ be a real sequence. Assume that the polynomial $P(x) = \sum_{k=0}^n a_k x^k$ has only real zeros. Then*

$$a_k^2 \geq \frac{n-k+1}{n-k} \cdot \frac{k+1}{k} a_{k+1}a_{k-1}, \quad 1 \leq k \leq n-1. \qquad (2)$$

If the sequence $(a_k)_{k=0}^n$ is positive and satisfies the hypothesis of the previous theorem, then it is SLC. The two possible values of the modes are given by the next theorem.

Theorem 2. *Let $(a_k)_{k=0}^n$ be a real sequence satisfying the hypothesis of the previous theorem. Then every mode of the sequence $(a_k)_{k=0}^n$ satisfies*

$$\left\lfloor \frac{\sum\limits_{k=1}^{n} k a_k}{\sum\limits_{k=0}^{n} a_k} \right\rfloor \leq k_0 \leq \left\lceil \frac{\sum\limits_{k=0}^{n} k a_k}{\sum\limits_{k=0}^{n} a_k} \right\rceil ,$$

where $\lfloor x \rfloor$ and $\lceil x \rceil$ are respectively the floor and the ceiling of x.

For a proof of this theorem, see Benoumhani [2, 3].

Let $g(n,k) = \binom{n-k}{k}$. This sequence was been investigated by S. Tanny and M. Zuker [8]; they proved that it is SLC, and determined its modes. If r_n is the smallest mode of $g(n,k)$, then

$$r_n = \left\lceil \frac{5n - 3 - \sqrt{5n^2 + 10n + 9}}{10} \right\rceil . \qquad (3)$$

They proved that there are infinitely many integers where a double maximum occurs. The integers where this happen are given by: $n_j = F_{4j} - 1$, where F_k is the k^{th} Fibonacci number. The smallest mode corresponding to n_j is given by $r_j = \frac{1}{5}(L_{4j-1} - 4)$, where L_j is the j'th Lucas number.

In this paper we consider the sequence $L(n,k) = \frac{n}{n-k}\binom{n-k}{k}$, $0 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor$, $n \geq 1$. It is known that $L(n,k)$ counts the number of ways of choosing k points, no two consecutive, from a collection of n points arranged in a circle; see Stanley [7, p. 73, Lemma 2.3.4] and Sloane [6, A034807].

In Section 2, for the sake of completeness, we prove that all zeros of the polynomials $P_n(x) = \sum\limits_{k \geq 0} g(n,k) x^k$ are real. The explicit formula for $P_n(x)$ allows us to derive some identities. Also it enables us to rediscover a result of S. Tanny and M. Zuker. In the third section, we consider the polynomials $L_n(x) = \sum\limits_{k \geq 0} L(n,k) x^k$. We prove that all zeros of $L_n(x)$ are real and negative. In this case, too, the explicit formula for $L_n(x)$ gives some identities. The SLC of the sequence $L(n,k)$ is deduced from the fact that $L_n(x)$ has real zeros. We determine the modes, and the integers n where $L(n,k)$ has a double maximum. In the last section we prove that the sequence $L(n,k)$ is asymptotically normal, and satisfies a local limit theorem on R.

2. The polynomials $P_n(x)$

It is well known that the sequence $g(n,k) = \binom{n-k}{k}$, $0 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor$, is related to the Fibonacci numbers by the relation $\sum\limits_{k \geq 0} \binom{n-k}{k} = F_{n+1}$. Recall that the sequence (F_n) is defined as follows:

$$F_n = F_{n-1} + F_{n-2}, \; n \geq 2,$$

with $F_0 = 0$, $F_1 = 1$. Also we have the explicit formula

$$F_n = \frac{1}{\sqrt{5}} \left(\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right).$$

It is straightforward to see that $P_n(x)$ satisfies the recursion

$$P_n(x) = P_{n-1}(x) + xP_{n-2}(x), \qquad (4)$$

with initial conditions $P_0(x) = P_1(x) = 1$. Using the relation (4) we prove

Proposition 3. *For all $n \geq 0$, all zeros of the polynomials $P_n(x)$ are real. More precisely, we have*

$$P_n(x) = \frac{1}{\sqrt{4x+1}}\left(\left(\frac{1+\sqrt{4x+1}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{4x+1}}{2}\right)^{n+1}\right). \qquad (5)$$

Proof. Write the relation (4) in matrix form, as follows: $\begin{pmatrix} P_n(x) \\ P_{n-1}(x) \end{pmatrix} = \begin{pmatrix} 1 & x \\ 1 & 0 \end{pmatrix}\begin{pmatrix} P_{n-1}(x) \\ P_{n-2}(x) \end{pmatrix}$. We deduce

$$\begin{pmatrix} P_n(x) \\ P_{n-1}(x) \end{pmatrix} = \begin{pmatrix} 1 & x \\ 1 & 0 \end{pmatrix}^{n-1}\begin{pmatrix} P_1(x) \\ P_0(x) \end{pmatrix} = \begin{pmatrix} 1 & x \\ 1 & 0 \end{pmatrix}^{n-1}\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

The eigenvalues of the matrix $A = \begin{pmatrix} 1 & x \\ 1 & 0 \end{pmatrix}$ are

$$\lambda_1 = \frac{1+\sqrt{4x+1}}{2}\ ,\ \lambda_2 = \frac{1-\sqrt{4x+1}}{2},$$

and two eigenvectors of A are $V_1 = \begin{pmatrix} \lambda_1 \\ 1 \end{pmatrix}$ and $V_2 = \begin{pmatrix} \lambda_2 \\ 1 \end{pmatrix}$. Now the matrix A may be written

$$\begin{pmatrix} 1 & x \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}\begin{pmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{pmatrix}^{-1}.$$

From this, we obtain

$$\begin{aligned}\begin{pmatrix} 1 & x \\ 1 & 0 \end{pmatrix}^{n-1} &= \frac{1}{\lambda_1-\lambda_2}\begin{pmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \lambda_1^{n-1} & 0 \\ 0 & \lambda_2^{n-1} \end{pmatrix}\begin{pmatrix} 1 & -\lambda_2 \\ -1 & \lambda_1 \end{pmatrix} \\ &= \frac{1}{\lambda_1-\lambda_2}\begin{pmatrix} \lambda_1^n - \lambda_2^n & -\lambda_1^n\lambda_2 + \lambda_1\lambda_2^n \\ \lambda_1^{n-1} - \lambda_2^{n-1} & -\lambda_1^{n-1}\lambda_2 + \lambda_1\lambda_2^{n-1} \end{pmatrix}.\end{aligned}$$

The vector $\begin{pmatrix} P_n(x) \\ P_{n-1}(x) \end{pmatrix}$ is now

$$\begin{pmatrix} P_n(x) \\ P_{n-1}(x) \end{pmatrix} = \frac{1}{\lambda_1-\lambda_2}\begin{pmatrix} \lambda_1^n - \lambda_2^n & -\lambda_1^n\lambda_2 + \lambda_1\lambda_2^n \\ \lambda_1^{n-1} - \lambda_2^{n-1} & -\lambda_1^{n-1}\lambda_2 + \lambda_1\lambda_2^{n-1} \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

So,

$$P_n(x) = \frac{1}{\lambda_1-\lambda_2}\left(\lambda_1^n - \lambda_2^n - \lambda_1^n\lambda_2 + \lambda_1\lambda_2^n\right).$$

Since $\lambda_1 + \lambda_2 = 1$ and $\lambda_1 - \lambda_2 = \sqrt{4x+1}$, we finally obtain

$$P_n(x) = \frac{1}{\sqrt{4x+1}}\left(\left(\frac{1+\sqrt{4x+1}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{4x+1}}{2}\right)^{n+1}\right).$$

This is the desired result.

For the roots of $P_n(x)$, we have

$$P_n(x) = 0 \Longleftrightarrow \left(\frac{1+\sqrt{4x+1}}{1-\sqrt{4x+1}}\right)^{n+1} = 1 \Longleftrightarrow \left(\frac{1+\sqrt{4x+1}}{1-\sqrt{4x+1}}\right) = \varepsilon_k, 1 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor$$

where the ε_k are the $(n+1)^{th}$ roots of unity. Thus,

$$P_n(x) = 0 \Longleftrightarrow \sqrt{4x+1} = \frac{\varepsilon_k - 1}{\varepsilon_k + 1} \Longleftrightarrow 4x = -1 + \left(\frac{\varepsilon_k - 1}{\varepsilon_k + 1}\right)^2.$$

Furthermore, we obtain $P_n(x) = 0 \Longleftrightarrow x = -\frac{1}{4}\left(1 + \tan^2\left(\frac{k\pi}{n+1}\right)\right), 1 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor$. This proves that the roots of $P_n(x)$ are real and negative. □

Remark. In the sequel, we need Lucas numbers. Let us recall their definition:

$$L_n = L_{n-1} + L_{n-2}, \quad L_0 = 2, \, L_1 = 1.$$

It is not hard to see that

$$L_n = \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n \quad \text{and} \quad L_n = F_n + F_{n-2},$$

holds.

Corollary 4. *We have the following identities:*

1. $\sum\limits_{n\geq 0} P_n(x) z^n = \frac{1}{1-z-xz^2}$.

2. $\sum\limits_{k\geq 0} (-1)^k \binom{n-k}{k} = \begin{cases} 0, \; if \; n = 6k+2, \; 6k+5; \\ 1, \; if \; n = 6k, \; 6k+1; \\ -1, \; if \; n = 6k+3, \; 6k+4. \end{cases}$

3. $\sum\limits_{k\geq 0} k\binom{n-k}{k} = \sum\limits_{k=0}^{n-2} F_k F_{n-k-2} = \frac{(n+1)L_n - 2F_n}{5} = \frac{(n-1)F_n + (n+1)F_{n-2}}{5}$.

4. $(n+1)L_n - 2F_n = (n-1)F_n + (n+1)F_{n-2} \equiv 0 \pmod 5$.

5. $\sum\limits_{k\geq 0} (-1)^k k\binom{n-k}{k} = \begin{cases} \frac{2}{3}n, & \text{if } n = 6k; \\ \frac{n-1}{3}, & \text{if } n = 6k+1; \\ -\frac{n+1}{3}, & \text{if } n = 6k+2; \\ -\frac{2n}{3}, & \text{if } n = 6k+3; \\ -\frac{(n-1)}{3}, & \text{if } n = 6k+4; \\ \frac{n+1}{3}, & \text{if } n = 6k+5. \end{cases}$

Proof. The first is known and easy to establish using (4). For (2), put $x = -1$ in (5). For the third, differentiate the generating function of $P_n(x)$ with respect to x, and compare the coefficients, and then put $x = 1$. Relation 4 is immediate from 3. For the last one, put $x = -1$ in the derivative of $P_n(x)$. □

According to Theorem 2, every mode r_n of the sequence $\binom{n-k}{k}$ satisfies the relation

$$\left\lfloor \frac{\sum_{k=1}^{n} k\binom{n-k}{k}}{F_n} \right\rfloor \leq r_n \leq \left\lceil \frac{\sum_{k=0}^{n} k\binom{n-k}{k}}{F_n} \right\rceil .$$

S. Tanny and M. Zuker gave an exact formula for r_n, but this is somewhat opaque. So they used another method to give a more explicit one; but it is less precise. Namely, they proved that $r_n = \left\lfloor \frac{n}{2}\left(1-\frac{\sqrt{5}}{5}\right) \right\rfloor$ or $r_n = \left\lceil \frac{n}{2}\left(1-\frac{\sqrt{5}}{5}\right) \right\rceil$. We give another proof of this result.

Proposition 5. *(S. Tanny, M. Zuker [8])*
The modes of the sequences $\binom{n-k}{k}$ *are given by* $r_n = \left\lfloor \frac{n}{2}\left(1-\frac{\sqrt{5}}{5}\right) \right\rfloor$ *or* $r_n = \left\lceil \frac{n}{2}\left(1-\frac{\sqrt{5}}{5}\right) \right\rceil$.

Proof. Since all zeros of the polynomial $P_n(x)$ are real, it suffices to compute $\frac{\sum_{k=1}^{n} k\binom{n-k}{k}}{} = \frac{\sum_{k=1}^{n} k\binom{n-k}{k}}{F_n}$. The last corollary gives

$$\mu_n = \frac{\sum_{k=1}^{n} k\binom{n-k}{k}}{F_n} = \frac{(n+1)L_n - 2F_n}{5F_n} = \frac{(n+1)L_n}{5F_n} - \frac{2}{5}.$$

Using the explicit formula for the Lucas and Fibonacci numbers; we obtain

$$\mu_n = \frac{(n+1)}{2}\left(1-\frac{\sqrt{5}}{5}\right)\frac{1+a^n}{1-a^{n+1}},\ a = -\frac{3-\sqrt{5}}{2}.$$

Now consider the sequence

$$\mu_n = \frac{(n+1)}{2}\left(1-\frac{\sqrt{5}}{5}\right)\frac{1+a^n}{1-a^{n+1}} - \frac{2}{5} = \frac{(n+1)}{2}\left(1-\frac{\sqrt{5}}{5}\right)A_n - \frac{2}{5},$$

where

$$A_n = \frac{1+a^n}{1-a^{n+1}}.$$

Also, observe that for every n we have

$$A_{2n+1} < 1 < A_{2n}.$$

So

$$\mu_{2n} = \frac{2n+1}{2}\left(1-\frac{\sqrt{5}}{5}\right)A_{2n} - \frac{2}{5} \geq \frac{2n+1}{2}\left(1-\frac{\sqrt{5}}{5}\right) - \frac{2}{5},$$

and

$$\mu_{2n+1} = \frac{2n+2}{2}\left(1-\frac{\sqrt{5}}{5}\right)A_{2n+1} - \frac{2}{5} \leq \frac{2n+2}{2}\left(1-\frac{\sqrt{5}}{5}\right) - \frac{2}{5}.$$

Thus

$$\frac{2n+1}{2}\left(1-\frac{\sqrt{5}}{5}\right) - \frac{2}{5} \leq \mu_{2n} \leq \mu_{2n+1} \leq \frac{2n+2}{2}\left(1-\frac{\sqrt{5}}{5}\right) - \frac{2}{5}.$$

We deduce that for every $n \geq 2$,

$$\frac{n}{2}\left(1-\frac{\sqrt{5}}{5}\right) - \frac{2}{5} \leq \mu_n \leq \frac{n+2}{2}\left(1-\frac{\sqrt{5}}{5}\right) - \frac{2}{5}.$$

Since the difference between the two bounds is $\left(1-\frac{\sqrt{5}}{5}\right) < 1$; there is a unique integer r_n in the interval $\left(\frac{n}{2}\left(1-\frac{\sqrt{5}}{5}\right) - \frac{2}{5}, \frac{n+2}{2}\left(1-\frac{\sqrt{5}}{5}\right) - \frac{2}{5}\right)$ and of course $r_n = \left\lfloor \frac{n}{2}\left(1-\frac{\sqrt{5}}{5}\right) \right\rfloor$ or $r_n = \left\lceil \frac{n}{2}\left(1-\frac{\sqrt{5}}{5}\right) \right\rceil$. □

3. The polynomials $L_n(x)$

In this section, we consider the sequence $L(n,k) = \frac{n}{n-k}\binom{n-k}{k}$. We prove that all zeros of the polynomials $L_n(x) = \sum\limits_{k\geq 0} L(n,k)x^k$ are real.

Proposition 6. *For all $n \geq 2$, all zeros of the polynomials $L_n(x)$ are real. We have* $L_n(x) = \left(\frac{1+\sqrt{4x+1}}{2}\right)^n + \left(\frac{1-\sqrt{4x+1}}{2}\right)^n$. (6).

Proof. Since the polynomials satisfy the recursion

$$L_n(x) = L_{n-1}(x) + xL_{n-2}(x);$$

with $L_0 = 2$, $L_1 = 1$, the proof is exactly the same as for $P_n(x)$.

Corollary 7. *We have the following identities:*

1. $\sum\limits_{n\geq 0} L_n(x)z^n = \frac{2-z}{1-z-xz^2}$.
2. $\sum\limits_{k\geq 0} \frac{n}{n-k}\binom{n-k}{k} = L_n$.
3. $\sum\limits_{k\geq 0}(-1)^k \frac{n}{n-k}\binom{n-k}{k} = \begin{cases} 1, \; if \;\; n = 6k+1 \;\; or \;\; 6k+5; \\ -1, \; if \;\; n = 6k+2 \;\; or \;\; 6k+4; \\ 2, \; if \;\; n = 6k\,; \\ -2, \; if \;\; n = 6k+3. \end{cases}$
4. $\sum\limits_{k=0}^{n-1} L_k F_{n-k-1} = nF_n$.

Proof. Relation (1) is immediate, for the second one, it suffices to put $x = 1$ in (6). For the third one, put $x = -1$ again in (6). The last one is obtained by differentiating the generating function of $L_n(x)$ with respect to x and then equating the coefficients of z^n in both sides. □

Since all zeros of the polynomials $L_n(x)$ are real, it follows that the sequence $L(n,k)$ is SLC. We follow S. Tanny and M. Zuker to give the modes.

Theorem 8. *The smallest mode of the sequence $L(n,k)$ is given by*

$$k_n = \left\lceil \frac{5n-4-\sqrt{5n^2-4}}{10} \right\rceil.$$

Proof. The integer k_n satisfies

$$\begin{cases} L(n,k_n-1) < L(n,k_n) & (a) \\ L(n,k_n) \geq L(n,k_n+1) & (b) \end{cases}$$

Let

$$f(x) = 5x^2 - (5n+6)x + n^2 + 3n + 2,$$

and

$$g(x) = 5x^2 - (5n-4)x + n^2 + 2n + 1.$$

We have

$$(a) \iff f(k_n) > 0;$$
$$(b) \iff g(k_n) \leq 0.$$

The roots of the first equation are $\frac{5n+6\pm\sqrt{5n^2-4}}{10}$, and those of the second one are $\frac{5n-4\pm\sqrt{5n^2-4}}{10}$. The desired integer satisfies

$$\frac{5n-4-\sqrt{5n^2-4}}{10} \le k_n < \frac{5n+6-\sqrt{5n^2-4}}{10}.$$

Which is what we wanted. □

The previous formula for k_n is not as explicit as expected. We give a more explicit one.

Corollary 9. *The integer k_n satisfies the following*

$$k_n = \left\lfloor \frac{n}{2}\left(1-\frac{\sqrt{5}}{5}\right)\right\rfloor \text{ or } k_n = \left\lceil \frac{n}{2}\left(1-\frac{\sqrt{5}}{5}\right)\right\rceil.$$

Proof. The proof is the same as for r_n . □

In the next result, the integers n, such that the sequence $L(n,k)$ has a double maximum will be determined. Before determining these integers, we need the following lemmas:

Lemma 10. *For every $n \ge 0$, $5F_n^2 + 4(-1)^n = L_n^2$.*

Proof. This is known, and straightforward using the explicit formulas of F_n and L_n . □

Lemma 11. *For every $n \ge 0$, $5F_{4n+1} - L_{4n+1} - 4 \equiv 0 \pmod{10}$.*

Proof. Again, the explicit formulas of F_n and L_n give easily the wanted result. □

Theorem 12. *The sequence $L(n,k)$ has a double maximum if and only if $n = F_{4j+1}$, and in this case the smallest mode is given by $k_n = F_{2j}^2$.*

Proof. If l is the smallest mode of $L(n,k)$ then it satisfies

$$L(n,l) = L(n,\, l+1),$$

which is equivalent to

$$f(n,l) = 5l^2 - (5n-4)l + n^2 - 2n + 1 = 0\,. \tag{7}$$

Equation (7) has two roots in l

$$l_{1,2} = \frac{5n-4\pm\sqrt{5n^2-4}}{10}.$$

The solution greater than $\frac{n}{2}$ is rejected, since the modes of $L(n,k)$ are less than $\frac{n}{2}$. The smallest one remains, i.e.,

$$l = \frac{5n-4-\sqrt{5n^2-4}}{10}\,. \tag{8}$$

So, we are looking for all pairs of integers (n_j,k_j), $0 \le k_j \le \frac{n_j}{2}$, satisfying (7) (or (8)). We may transform (8) to an equation related to Pell's equation as in Tanny and Zuker [8], and then use some classical facts about units (invertible elements) in quadratic fields (see Cohn [4] for details). But we proceed differently: by Lemma 10, $5F_{2j+1}^2 - 4 = L_{2n+1}^2$, and by Lemma 11, $5F_{4j+1} - 4 - \sqrt{5F_{4j+1}^2 - 4} \equiv 5F_{4j+1} - 4 - L_{4j+1} \equiv 0 \pmod{10}$, that is, $k_j = \frac{55F_{4j+1}-4\pm\sqrt{5F_{4j+1}^2-4}}{10} = \frac{5F_{4j+1}-4-L_{4j+1}}{10} = F_{2j}^2 \le \frac{F_{4j+1}}{2}$. So, some of the Fibonacci numbers are certainly among the n_j. Now let $(n_0,k_0) = (1,0)$, $(n_1,k_1) = (5,1)$, $(n_2,k_2) =$

$(34, 9)$, $(n_3, k_3) = (233, 64)$, ...,with $n_j = F_{4j+1}$, $k_j = F_{2j}^2$. The following recursions are easily derived:

$$\begin{cases} n_{j+1} = 7n_j - n_{j-1}; \\ k_{j+1} = 7k_j - k_{j-1} + 2\,. \end{cases} \tag{9}$$

Now, we prove that all solutions of (7) are in fact $(n_j = F_{4j+1},\, k_j = F_{2j}^2)_{j\geq 0}$. We will show that if (n_j, k_j) is a solution of (7), then

$$(n_{j+1}, k_{j+1}) = (7n_j - n_{j-1}, 7k_j - k_{j-1} + 2)$$

is another one. Indeed

$$\begin{aligned} f(n_{j+1}, k_{j+1}) &= 5k_{j+1}^2 - (5n_{j+1} - 4)k_{j+1} + n_{j+1}^2 - 2n_{j+1} + 1 \\ &= 5(7k_j - k_{j-1} + 2)^2 - (5(7n_j - n_{j-1}) - 4)(7k_j - k_{j-1} + 2) \\ &\quad +(7n_j - n_{j-1})^2 - 2(7n_j - n_{j-1}) + 1 \\ &= 0 \end{aligned}$$

since $f(n_i, k_i) = 5k_i^2 - (5n_i - 4)k_i + n_i^2 - 2n_i + 1 = 0$ for $0 \leq i \leq j$. Suppose that (n, k) is another one, $0 \leq k \leq \frac{n}{2}$; different from those (n_j, k_j). There is a unique (n_i, k_i) such that $n_i < n < n_{i+1}$. We verify easily that $f(7n - n_{i-1}, 7k - k_{i-1} + 2) = 0$. This means that $(n, k) = (n_i, k_i)$, and proves that all the solutions of (7) are given by the recursions (9).This ends the proof. □

Remarks. 1. There is a relation between the modes of the sequence $g(n, k)$ and those of $L(n, k)$. Let (m_j, r_j) be the sequence of integers such that $g(m_j, r_j) = g(m_j, r_j + 1)$. Since $m_j = F_{4j} - 1$,and $r_j = \frac{1}{5}(L_{4j-1} - 4)$, it is easy to establish (by direct calculations, or generating functions of r_j), that

$$\begin{cases} n_j = r_{j+1} - r_j; \\ k_j = m_j - 2r_j - 1. \end{cases}$$

2. Note that our relation for k_j was derived by S. Tanny and M. Zuker [9, p. 301]. There, the initial conditions for the Fibonacci numbers are: $F_0 = F_1 = 1$.

3. Using the recursions (9), we obtain the generating functions:

$$g(x) = \sum_{j=0}^{\infty} n_j x^j = \frac{1 - 2x}{1 - 7x + x^2} \quad \text{and} \quad h(x) = \sum_{j=1}^{\infty} k_j x^j = \frac{x + x^2}{(1 - x)(1 - 7x + x^2)}.$$

4. A central and a local theorem for $L(n, k)$

A positive real sequence $a(n, k)_{k=0}^n$, with $A_n = \sum_{k=0}^{n} a(n, k) \neq 0$, is said to satisfy a central limit theorem (or is *asymptotically normal*) with mean μ_n and variance σ_n^2 if

$$\lim_{n\longrightarrow+\infty} \sup_{x\in R} \left| \sum_{0\leq k\leq \mu_n + x\sigma_n} \frac{a(n, k)}{A_n} - (2\pi)^{-1/2} \int_{-\infty}^{x} e^{-\frac{t^2}{2}} dt \right| = 0.$$

The sequence satisfies a local limit theorem on $B \subseteq R$; with mean μ_n and variance σ_n^2 if

$$\lim_{n\longrightarrow+\infty} \sup_{x\in B} \left| \frac{\sigma_n a(n, \mu_n + x\sigma_n)}{A_n} - (2\pi)^{-1/2} e^{-\frac{x^2}{2}} \right| = 0.$$

Recall the following result (see Bender [1]).

Theorem 13. *Let $(P_n)_{n\geq 1}$ be a sequence of real polynomials; with only real negative zeros. The sequence of the coefficients of the $(P_n)_{n\geq 1}$ satisfies a central limit theorem; with* $\mu_n = \frac{P_n''(1)}{P_n(1)}$ *and* $\sigma_n^2 = \left(\frac{P_n''(1)}{P_n(1)} + \frac{P_n'(1)}{P_n(1)} - \left(\frac{P_n'(1)}{P_n(1)}\right)^2\right)$ *provided that* $\lim\limits_{n\longrightarrow+\infty} \sigma_n^2 = +\infty$. *If, in addition, the sequence of the coefficients of each P_n is with no internal zeros; then the sequence of the coefficients satisfies a local limit theorem on R.*

The fact that the zeros of the sequence $L_n(x)$ are real implies the following result.

Theorem 14. *The sequence $(L(n,k))_{k\geq 0}$ satisfies a central limit and a local limit theorem on R with* $\mu_n = \frac{L_n''(1)}{L_n(1)} \sim \frac{n}{2}\left(1 - \frac{\sqrt{5}}{5}\right)$ *and* $\sigma_n^2 = \frac{L_n''(1)}{L_n(1)} + \frac{L_n'(1)}{L_n(1)} - \left(\frac{L_n'(1)}{L_n(1)}\right) \sim 5^{-\frac{3}{4}} n$

Proof. We have

$$\sigma_n^2 = \frac{L_n''(1)}{L_n(1)} + \frac{L_n'(1)}{L_n(1)} - \left(\frac{L_n'(1)}{L_n(1)}\right)^2 = \frac{n^2 L_{n-2} L_n - 5n^2 F_{n-1}}{5L_n^2} + \frac{3nF_{n-1} - nL_{n-2}}{5L_n}.$$

Let $\alpha = \frac{1+\sqrt{5}}{2}$, $\beta = \frac{1-\sqrt{5}}{2}$. Using the explicit formulas of L_n and F_n, we obtain

$$\sigma_n^2 = \frac{(-1)^n n^2}{\alpha^{2n} + \beta^{2n} + 2(-1)^n} + \frac{\alpha^{n-2}\left(\frac{3\sqrt{5}\alpha}{5} - 1\right)n - \beta^{n-2}\left(\frac{3\sqrt{5}\beta}{5} + 1\right)n}{5\left(\alpha^n + \beta^n\right)} \sim 5^{-\frac{3}{4}} n.$$

So, $\lim\limits_{n\longrightarrow+\infty} \sigma_n = +\infty$. The local limit theorem is then easily seen to be satisfied; since $L(n,k) \neq 0$, for $0 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor$. □

As a consequence of the local limit theorem, we have

Corollary 15. *Let* $L = \max\{L(n,k),\ 0 \leq k \leq \frac{n}{2}\}$. *Then*

$$L \sim \frac{5^{\frac{3}{4}}\left(\frac{1+\sqrt{5}}{2}\right)^n}{\sqrt{2\pi n}}.$$

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References

[1] E. A. Bender, Central and local limit theorems applied to asymptotic enumeration, *J. Combin. Theory*, Ser. A **15** (1973), 91–111.

[2] M. Benoumhani, Polynômes à racines réelles et applications combinatoires, Thèse de doctorat, Université Claude Bernard, Lyon 1, Lyon, France.1993.

[3] M. Benoumhani, Sur une propriété des polynômes à racines négatives, *J. Math. Pures Appl,* **75** (1996), 85–105.

[4] H. Cohn, *A Classical Invitation to Algebraic Numbers and Class Fields*, Springer-Verlag, 1978.

[5] G. H. Hardy, J. E. Littlewood and G. Pólya, *Inequalities*, Cambridge Univ. Press, 1956.

[6] N. Sloane, Online Encyclopedia of Integer Sequences, `www.research.att.com/~njas/sequences/index.html`.

[7] R. Stanley, *Enumerative Combinatorics*, Wadsworth & Brooks / Cole, Monterey, California 1986.

[8] S. Tanny and M. Zuker, On a unimodal sequence of binomial coefficients, *Discrete Math.* **9** (1974), 79–89.

[9] S. Tanny and M. Zuker, Analytic methods applied to a sequence of binomial coefficients, *Discrete Math.* **24** (1978), 299–310.

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