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Baron Münchhausen's Sequence

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Abstract

We investigate a coin-weighing puzzle that appeared in the all-Russian Mathematics Olympiad in 2000. The methods of analysis differ from classical coin-weighing puzzles. We generalize the puzzle by varying the number of participating coins, and deduce a complete solution. Perhaps surprisingly, the objective can be achieved in no more than two weighings regardless of the number of coins involved.

1 Introduction

The following coin-weighing puzzle, due to Alexander Shapovalov, appeared in the Regional round of the all-Russian math Olympiad in 2000 [2].

> Eight coins weighing $1, 2, \ldots, 8$ grams are given, but which weighs how much is unknown. Baron Münchhausen claims he knows which coin is which; and offers to prove himself right by conducting one weighing on a balance scale, so as to unequivocally demonstrate the weight of at least one of the coins. Is this possible, or is he exaggerating?

We chose to investigate this puzzle partly because classical coin-weighing puzzles [1] tend to ask the person designing the weighings to discover something they do not know, whereas here the party designing the weighings knows everything, and is trying to use the balance scale to convince someone else of something. There is therefore a difference of method: when the weigher is an investigator, they typically find themselves playing a minimax game against fate: they must construct experiments all of whose possible outcomes are as equally likely as possible, in order to learn the most they can even in the worst case. The Baron, however, knows everything, so he has the liberty to construct weighings whose results look very surprising from the audience's perspective.

We invite the reader to experience the enjoyment of solving this puzzle for themselves before proceeding; we will spoil it completely on page 3.

1.1 The Sequence

We will generalize this puzzle to n coins that weigh $1, 2, \ldots, n$ grams. We are interested in the minimum number of weighings on a balance scale that the Baron needs in order to convince his audience about the weight of at least one of those coins. It turns out that the answer is never more than two; over the course of the paper, we will prove this, and determine in closed from which n require two weighings, and which can be done in just one.

1.2 The Roadmap

In Section 2 we define the Baron's sequence again and show some of the flavor of this problem by calculating the first few terms. In Section 3 we prove the easy but perhaps surprising observation that this sequence is bounded; in fact that no term of this sequence can exceed three. That theorem opens the door to a complete description of all the terms of Baron Münchhausen's sequence, which we begin in Section 4 by explicitly finding the terms that are equal to one [8].

Discriminating between two weighings sufficing and three being necessary is harder. The remainder of the paper is dedicated to proving that three is not a tight upper bound; namely that the Baron can always demonstrate the weight of at least one coin among any n in at most two weighings. Section 5 serves as a signpost by restating the theorem, and Section 6 briefly introduces some notation we will use subsequently. The actual proof is sufficiently involved that we break it down into a separate Section 7 for preliminaries, and then give the proof itself in Section 8.

Finally, we close with Section 9 for some generalizations and ideas for future research.

2 Baron Münchhausen's Sequence

Baron Münchhausen's sequence $a(n)$ is defined as follows:

> Let n coins weighing $1, 2, \ldots, n$ grams be given. Suppose Baron Münchhausen knows which coin weighs how much, but his audience does not. Then $a(n)$ is the minimum number of weighings the Baron must conduct on a balance scale, so as to unequivocally demonstrate the weight of at least one of the coins.

The original Olympiad puzzle asks whether $a(8) = 1$.

2.1 Examples

The indices 1 through 8 are instructive because they turn out to exhibit all the ways that the Baron can convince his audience in one weighing. Baron Münchhausen's sequence begins with 0, 1, 1, 1, 2, 1, 1, 1:

- $a(1) = 0$ vacuously.
- $a(2) = 1$ because the weighing $1 < 2$ uniquely determines both coin weights.
- $a(3) = 1$ because $1 + 2 = 3$ uniquely determines the 3-gram coin.
- $a(4) = 1$ because $1 + 2 < 4$ uniquely determines both the 3- and 4-gram coins.
- $a(6) = 1$ because $1 + 2 + 3 = 6$ uniquely determines the 6-gram coin.
- $a(7) = 1$ because $1 + 2 + 3 < 7$ uniquely determines the 7-gram coin.

For $n = 5$, $a(5) > 1$ because Baron Münchhausen cannot do it in one weighing. This example is small enough to prove exhaustively: every possible outcome of every possible weighing admits of multiple assignments of weights to coins, which do not fix the weight of any one coin. For an example of the reasoning, suppose the Baron puts 1 coin on one cup and 2 coins on another, and shows that the single coin is heavier. All of $1+2 < 4$, $2+1 < 5$, and $1 + 3 < 5$ are consistent with this data, so no coin is uniquely identified. Checking all the other cases, as usual, is left to the reader.

For $n = 8$, the original Olympiad problem asks whether one weighing is enough. It is—Baron Münchhausen can convince his audience by placing all the coins weighing 1 through 5 grams on one cup of the scale, and the two coins weighing 7 and 8 grams on the other. The only way for two coins with weights from the set $1, 2, \ldots, 8$ to balance five is for the two to weigh the most they can, at $7 + 8 = 15$ grams, and for the five to weigh the least they can at $1 + 2 + 3 + 4 + 5 = 15$ grams. This arrangement leaves exactly one coin off the scale, whose weight, by elimination, must be 6 grams.

3 Three Weighings are Always Enough

Theorem 1. $a(n) \leq 3$.

Before going into the proof we would like to introduce a little notation. First, we denote the x-th triangular number $x(x+1)/2$ by T_x.

We already did this in our example weighings, but we would like to make official the fact that, when describing weighings the Baron should carry out, we will denote a coin weighing i grams with just the number i. In addition, we will use round brackets to denote one coin of the weight indicated by the expression enclosed in the brackets. We need this notation to distinguish $i+1$, which represents two coins of weight i and 1 on some cup, from $(i+1)$, which represents one coin of weight $i+1$ on some cup.

Proof. We know, since Gauss proved it in 1796 [4, 6], that any number n can be represented as a sum of not more than 3 triangular numbers. Let $n = T_i + T_j + T_k$, where $T_i \leq T_j \leq T_k$ are triangular numbers with indices i, j, and k.

Barring degenerate cases, Baron Münchhausen can display a sequence of three weighings with one coin on the right cup each:

$$\begin{aligned} 1+2+3+\cdots+k &= T_k, \\ T_k+1+2+3+\cdots+j &= (T_k+T_j), \\ (T_k+T_j)+1+2+3+\cdots+i &= n. \end{aligned}$$

The first weighing demonstrates a lower bound of T_k on the weight of the coin the Baron put on the right. Since he then reuses that coin on the left, the second weighing demonstrates a lower bound of $T_k + T_j$ on the weight of the coin that goes on the right in the second weighing. Since he then reuses *that* coin in the third weighing, the audience finds a lower bound of $T_k+T_j+T_i$ on the coin on the right hand side of the last weighing. But since there are only n coins, there is already an upper bound of n on that coin's weight, so the assumed equality $T_k + T_j + T_i = n$ determines that coin's weight completely (as well as the weights of the T_k- and $(T_k + T_j)$-gram coins).

The Baron should start with the largest triangular number to make sure that he will not need any coin to appear in the same weighing twice: since k is the largest index, the coin T_k will not be in the sequence $1, 2, \ldots, j$, and the coin $(T_k + T_j)$ will not be in the sequence $1, 2, \ldots, i$.

The degenerate cases $i = 0$, $j = 0$, $k = 1$ or $k = 0$ are all easier. □

4 When does One Weighing Suffice?

Theorem 2. *The weight of some coin can be confirmed with just one weighing if and only if all of:*

1. *one cup contains all the coins with weights from 1 to some i;*

2. *the other cup contains all the coins with weights from some j to n;*

3. *either the scale balances, or the cup containing the 1-gram coin is lighter by one gram; and*

4. *at least one cup contains exactly one coin, or exactly one coin is left off the scale.*

Why can such a weighing be convincing? In general, i will be much larger than $n-j$. The only way for so few coins to weigh as much as (or more than) so many will be for the few to be the heaviest and the many to be the lightest. We show in the proof that those are exactly the convincing weighing structures; thereafter, in Section 4.1, we discuss the circumstances under which such a weighing exists and can therefore determine the weight of a single coin.

Proof. What does it mean for Baron Münchhausen to convince his audience of the weight k of some coin, using just one weighing? From the perspective of the audience, a weighing is a number of coins in one cup, a number of coins in the other cup, and a number of coins not on the scale, together with the result the scale shows. For the audience to be convinced of the weight of some particular coin, it must therefore be the case that all possible arrangements of coin weights consistent with that data agree on the weight k of the coin in question.

"If" direction. Suppose the Baron's audience observes a weighing whose left cup contains i coins, and whose right cup contains $n-j+1$ coins. The least that i coins can weigh is T_i. The most that $n-j+1$ coins can weigh is $T_n - T_{j-1}$. If these numbers are equal, the audience can conclude that, for the scale to balance, the left cup must have exactly the coins of weights $1 \cdots i$, and the right cup must have exactly the coins of weights $j \cdots n$. Likewise, if $T_n - T_{j-1}$ exceeds T_i by one, the same allocation is the only way for the scale to indicate that the cup with i coins is lighter. By condition *(iv)*, one of the groups of coins this weighing determines must be a singleton, so the audience knows the weight of that coin.

"Only if" direction. Our proof strategy is to look for ways to alter a given arrangement of coin weights so as to change the weight given to the coin whose weight is being demonstrated; the requirement that all such alterations are impossible yields the desired constraints on convincing weighings.

First, the coin whose weight k the Baron is trying to confirm has to be alone in its group. After that observation we divide the proof of the theorem into several cases.

Case 1. The k-gram coin is on a cup and the scale is balanced. Then by above k is alone on its cup. We want to show two things: $k = n$, and the coins on the other cup weigh 1, 2, ..., i grams for some i. For the first part, observe that if $k < n$, then the coin with weight $k+1$ must not be on the scale (otherwise it would overbalance coin k). Therefore, we can substitute coin $k+1$ for coin k, and substitute a coin one gram heavier for the heaviest coin that was on the other cup, and produce thereby a different weight arrangement with the same observable characteristics but a different weight for the coin the Baron claims has weight k.

To prove the second part, suppose the contrary. Then it is possible to substitute a coin 1 gram lighter for one of the coins on the other cup. The $(k-1)$-gram coin must not be on the scale after this substitution, because otherwise the scale would not have balanced. We can therefore also substitute $k-1$ for k, thereby restoring balance and proving the original weighing unconvincing.

Consequently, $k = n = 1 + 2 + \cdots + i = T_i$ is a triangular number.

Case 2. The k-gram coin is on the lighter cup of the scale. Then: first, $k = 1$, because otherwise we could swap k and the 1-gram coin, making the light cup lighter and the heavy cup heavier or unaffected; second, the 2-gram coin is on the heavy cup and is the only coin on the heavy cup, because otherwise we could swap k with the 2-gram coin and not change the weights by enough to affect the imbalance; and finally $n = 2$ because otherwise we could change the weighing $1 < 2$ into $2 < 3$.

Thus the theorem holds, and the only example of this case is $k = 1, n = 2$.

Case 3. The k-gram coin is on the heavier cup of the scale. Then $k = n$ and the lighter cup consists of some collection of the lightest available coins, by the same argument as Case 1. Furthermore, k must weigh exactly 1 gram more than the lighter cup, because otherwise, $k-1$ is not on the lighter cup and can be substituted for k, making the weighing unconvincing.

Consequently, $k = n = T_i + 1$ is one more than a triangular number.

Case 4. The k-gram coin is not on a cup and the scale is not balanced. Then, since k must be off the scale by itself, all the other coins must be on one cup or the other. Furthermore, all coins heavier than k must be on the heavier cup, because otherwise we could make the lighter cup even lighter by substituting k for one of those coins. Likewise, all coins lighter than k must be on the lighter cup. So the theorem holds; and furthermore, the cups must again differ in weight by exactly 1 gram, because otherwise we could swap k with either $k-1$ or $k+1$ without changing the weights enough to affect the result on the scale.

Consequently, the weight of the lighter cup is $k(k-1)/2$, and the weight of the heavier cup is $k(k-1)/2+1$. Thus the total weight of all the coins is $n(n+1)/2 = k(k-1)/2 + k + (k(k-1)/2+1) = k^2+1$. In other words, case 4 is possible iff n is the index of a triangular number that is one greater than a square.

Case 5. The k-gram coin is not on a cup and the scale is balanced. The basic strategy is the same as for case 4, but the need to keep the scale balanced after our substitutions makes this case hairier than all the others combined. To begin, note again that all the coins besides k must be on some cup.

Lemma 3. *The two coins $k-1$ and $k-2$ must be on the same cup, if they exist (that is, if $k > 2$). Likewise $k-2$ and $k-4$; $k+1$ and $k+2$; and $k+2$ and $k+4$.*

Proof. Suppose the two coins $k-1$ and $k-2$ are not on the same cup. Then we can rotate k, $k-1$, and $k-2$, that is, put k on the cup with $k-1$, put $k-1$ on the cup with $k-2$, and take $k-2$ off the scale. This makes both cups heavier by one gram, producing a weighing with the same outward characteristics as the one we started with, but a different coin off the scale. The same argument applies to the other three pairs of coins we are interested in, mutatis mutandis. □

Lemma 4. *The four coins $k-1$, $k-2$, $k-3$ and $k-4$ must be on the same cup if they exist (that is, if $k \geq 5$).*

Proof. By Lemma 3, the three coins $k-1$, $k-2$, and $k-4$ must be on the same cup. Suppose coin $k-3$ is on the other cup. Then we can swap $k-1$ with $k-3$ and k with $k-4$. Each cup becomes lighter by 2 grams without changing the number of coins present, the balance is maintained, and the Baron's audience is not convinced. □

Lemma 5. *If $k \geq 5$, all coins lighter than k must be on the same cup.*

Proof. By Lemma 4, the four coins $k-1$, $k-2$, $k-3$ and $k-4$ must be on the same cup. Suppose some lighter coin is on the other cup. Call the heaviest such coin c. Then, by choice of c, the coin with weight $c+1$ is on the same cup as the cluster $k-1 \cdots k-4$, and is distinct from coin $k-2$. We can therefore swap c with $c+1$ and swap k with $k-2$. This increases the weight on both cups by 1 gram without changing how many coins are on each, but moves k onto the scale. The Baron's audience is again unconvinced. □

Lemma 6. *Theorem 2 is true for $k \geq 5$.*

Proof. By Lemma 5, all coins lighter than k must be on the same cup. Further, if a coin with weight $k+4$ exists, then by the symmetric version of Lemma 5, all the coins heavier than k must be on the other cup together, and the theorem is true.

If no coin with weight $k+4$ exists, that is, if $n \leq k+3$, how can the theorem be false? All the coins lighter than k must be on one cup, and their total weight is $k(k-1)/2$. Further, in order to falsify the theorem, at least one of the coins heavier than k must also be on that same cup. So the minimum weight of that cup is now $k(k-1)/2 + k + 1$. But we only have at most two coins for the other cup, whose total weight is at most $k+2+k+3 = 2k+5$. For the scale to even have a chance of balancing, we must have

$$k(k-1)/2 + k + 1 \leq 2k + 5 \Leftrightarrow k^2 - 3k - 8 \leq 0.$$

Finding the largest root of that quadratic we see that $k < 5$.

So for $k \geq 5$, the collection of all coins lighter than k is heavy enough that either one needs all the coins heavier than k to balance them, or there are enough coins heavier than k that the theorem is true by symmetric application of Lemma 5. □

Completion of Case 5. It remains to check the case for $k < 5$. If $n > k+3$, then coin $k+4$ exists. If so, all the coins heaver than k must be on the same cup. Furthermore, since k is so small, they will together weigh more than half the available weight, so the scale will be unbalanceable. So $k < 5$ and $n \leq k+3 \leq 7$.

For lack of any better creativity, we will tackle the remaining portion of the problem by complete enumeration of the possible cases, except for the one observation that, to balance the scale with just the coin k off it, the total weight of the remaining coins, $n(n+1)/2 - k$, must be even. This observation cuts our remaining work in half.

Case 5; Seven Coins: $n = 7$**.** Then $5 > k \geq n-3 = 4$, so $k = 4$. Then the weight on each cup must be 12. One of the cups must contain the 7 coin, and no cup can contain the 4 coin, so the only two weighings the Baron could try are $7+5 = 1+2+3+6$, and $7+3+2 = 1+5+6$. But the first of those is unconvincing because $k+1 = 5$ is not on the same cup as $k+2 = 6$, and the second because it has the same shape as $7+3+1 = 2+4+5$ (leaving out the 6-gram coin instead of the asserted 4-gram coin).

Case 5; Six Coins: $n = 6$**.** Then $5 > k \geq n - 3 = 3$, and $n(n+1)/2 = 21$ is odd, so k must also be odd. Therefore $k = 3$, and the weight on each cup must be 9. The 6-gram coin has to be on a cup and the 3-gram coin is by presumption out, so the Baron's only chance is the weighing $6 + 2 + 1 = 4 + 5$, but that does not convince his skeptical audience because it looks too much like the weighing $1 + 3 + 4 = 6 + 2$.

Case 5; Five Coins: $n = 5$**.** Then $5 > k \geq n - 3 = 2$, and $n(n+1)/2 = 15$ is odd, so k must also be odd. Therefore $k = 3$, and the weight on each cup must be 6. The only way to do that is the weighing $5 + 1 = 2 + 4$, which does not convince the Baron's audience because it looks too much like $1 + 4 = 2 + 3$.

The remaining cases, $n < 5$, are easy. This concludes Case 5.

Consequently, by an argument similar to the one in case 4 we can show that any number n of coins to which case 5 applies must be the index of a square triangular number.

This concludes the proof of Theorem 2. □

4.1 The Indices of Ones

While proving the theorem we accumulated descriptions of all possible numbers of coins that allow the Baron to confirm a coin in one weighing. We collect that list here to finish the description of the indices of ones in Baron Münchhausen's sequence $a(n)$. The following list corresponds to the five cases in the proof of Theorem 2:

1. n is a triangular number: $n = T_i$. Then the weighing $1 + 2 + 3 + \cdots + i = n$ proves weight of the n-gram coin.

2. $n = 2$. The weighing $1 < 2$ proves the weight of both coins.

3. n is a triangular number plus one: $n = T_i + 1$. Then the weighing $1 + 2 + 3 + \cdots + i < n$ proves the weight of the n-gram coin.

4. n is the index of a triangular number that is a square plus one: $T_n = k^2 + 1$. Then the weighing $1 + 2 + 3 + \cdots + (k-1) < (k+1) + \cdots + n$ proves the weight of the k-gram coin. For example, $T_{25} = 325 = 18^2 + 1$, so the weighing $1 + 2 + \cdots + 17 < 19 + 20 + \cdots + 25$ proves the weight of the 18-gram coin for $n = 25$.

5. n is the index of a square triangular number: $T_n = k^2$. Then the weighing $1 + 2 + 3 + \cdots + (k-1) = (k+1) + \cdots + n$ proves the weight of the k-gram coin. For example, $T_8 = 36 = 6^2$; this is the solution to our original problem.

The sequence of indices of ones in the sequence $a(n)$ starts as: 1, 2, 3, 4, 6, 7, 8, 10, 11, 15, 16, 21, 22, 25, 28, 29, 36, 37, 45, 46, 49, 55, 56, 66, 67.

5 Two Weighings are Always Enough

Our main theorem states that Baron Münchhausen never needs three weighings, for two are always enough.

Theorem 7. $a(n) \leq 2$.

6 Notation

We have already introduced the notation T_x to denote the xth triangular number $x(x+1)/2$. We will also continue to use round brackets to distinguish arithmetic from colocation: $(a+b)$ means the $(a+b)$-gram coin, whereas $a+b$ means the a-gram coin and the b-gram coin.

In addition, we introduce the notation $[x \cdots y]$ for the set of all consecutive coins between x and y, inclusive. Inside expressions in square brackets we will not parenthesize computations: $[3+4 \cdots 11-1]$ is the set of coins weighing from 7 to 10, inclusive, and does not include the coins 3, 4, 11, or 1.

If A denotes a set of coins, then $|A|$ denotes the total weight of those coins (not the cardinality of the set).

When representing a weighing as an equality/inequality we will refer to the left and right sides of the equality/inequality as the left and right cups of the weighing, respectively.

7 Preliminaries

Before we proceed with the main section of the proof, we will prove two lemmas that we are going to need, and that will demonstrate the machinery we will use to prove the main Theorem 7.

Lemma 8. *If n, $n-1$, or $n-2$ is a sum of two triangular numbers, then the Baron can demonstrate the weight of the n-gram coin in two weighings.*

Proof. This is a direct corollary of the argument used to prove Theorem 1. If $n = T_a + T_b$, that argument applies exactly. In the other two cases, the Baron can make judicious use of unbalanced weighings.

If $n = T_a + T_b + 1$, for $a \le b$, then one of the weighings needs to be unbalanced, for example

$$\begin{aligned} [1 \cdots b] &= T_b \\ [1 \cdots a] + T_b &< n. \end{aligned}$$

If $n = T_a + T_b + 2$, then both weighings should be unbalanced:

$$\begin{aligned} [1 \cdots b] &< (T_b + 1) \\ [1 \cdots a] + (T_b + 1) &< n. \end{aligned}$$

□

Since triangular numbers are pretty dense among the small integers, Theorem 2 and Lemma 8 account for many small n. This is good, because the main proof in Section 8 does not go through for small n. In particular, the reader is invited to verify that the smallest n that does not fall under the purview of Lemma 8 is $n = 54$; for example by consulting sequence A020756 in OEIS [3].

The following covers a special case we will encounter in the main proof, and coincidentally demonstrates the argument we will use in the main proof that the complicated weighings we will present will, in fact, convince the Baron's audience.

Lemma 9. *If there exists an a such that $2n = T_a + T_{a+1}$, the Baron can prove the weight of the n-gram coin in two weighings.*

Proof. We know that $T_a < n < T_{a+1}$ and, in fact, $n = T_a + \frac{a+1}{2}$. Suppose we can find coins x and y with $a + 1 < x < y = x + \frac{a+1}{2} < n$. Then the Baron can present the following two weighings:

$$\begin{aligned} [1 \cdots a] + y &= x + n \\ [1 \cdots a + 1] + x &= y + n. \end{aligned}$$

They will balance by the choice of x and y. Why will they convince the Baron's audience?

Let the audience consider the sum of the two weighings. The coins x and y appear on both sides of the sum, so they do not affect the balance of the total. Besides them, a coins appeared twice on the left, and one additional coin appeared once on the left; and this huge pile of stuff was balanced by just two appearances of a single coin on the right. How is this possible? The least possible total weight of the left-hand sides (except x and y) occurs if the coins that appeared twice have weights $[1 \cdots a]$, and the coin that appeared once has weight $a + 1$, for a total weight of $T_a + T_{a+1}$. The greatest possible total weight of the right-hand sides (again excluding x and y) occurs if the solitary coin on the right weighs n grams. But the known fact that $2n = T_a + T_{a+1}$ guarantees that, even in this extreme case the scale will just barely balance; so any other set of weights would cause the left cup to overbalance the right in at least one of the weighings Baron Münchhausen conducts. Therefore, the Baron's audience is forced to conclude that the solitary coin on the right must weigh n grams, as was the Baron's intention.

Now, when can we find such coins x and y? We can safely take the $a+2$ coin for x. Then the desired y coin will exist if $n > a+2+\frac{a+1}{2}$, which is equivalent to $T_a > a+2$, which holds for $a \geq 3$; to wit $n \geq 8$. Smaller n are covered by Lemma 8. □

8 Proof of the Main Theorem

There are two magical steps. First, let $a \leq b \leq c$ be such that

$$T_a + T_b + T_c = n + T_n. \tag{1}$$

By the triangular number theorem, proved by Gauss in his diary [4, 6], such a decomposition of $T_n + n$ into three triangular numbers is always possible. We should remark at this point that $c > n$ would imply $T_c \geq T_{n+1} > T_n + n$ so is impossible; and that $c = n$ would imply $T_c = T_n$ so $T_a + T_b = n$, allowing the Baron to proceed by the method in Lemma 8. So we can assume $c < n$.

Second, let us try to represent $T_c - n$ as the sum of some subset S of weights from the range $[a + 1 \cdots n - 1]$. Now there are three non-magical steps. We will prove that if such a representation exists, then the Baron can convince his audience of the weight of the n-gram coin in two weighings, by a particular method to be described forthwith; then we will note some properties of sums of subsets of ranges of integers; and then we will systematically examine possible choices of a, b, and c, and prove that the above-mentioned subset S really does exist, except in one case, for which Lemma 9 supplies an alternate method of solution.

8.1 Step 1: What to do with S

Lemma 10. *Let a, b, and c satisfy*

$$T_a + T_b + T_c = n + T_n.$$

Let S be a subset of $[a+1 \cdots n-1]$ for which

$$|S| = T_c - n.$$

Then there exist two weighings that uniquely identify the n-gram coin.

Proof. Let $\bar{S}$ denote the complement of S in $[a+1 \cdots n-1]$. We want to make a useful weighing out of S. Rewrite

$$T_c = |S| + n$$

with explicit ranges of coin weights:

$$[1 \cdots a] + [a+1 \cdots c] = S \cap [a+1 \cdots c] + S \cap [c+1 \cdots n-1] + n.$$

Now cancel coins appearing on both sides to get

$$[1 \cdots a] + \bar{S} \cap [a+1 \cdots c] = S \cap [c+1 \cdots n-1] + n. \tag{2}$$

Observe that (2) forms a legal weighing; let us take it as the first one.

Now we want to make another weighing that will, together with (2), demonstrate the weight of the n-gram coin. Let us begin by massaging (1):

$$\begin{aligned} T_a + T_b + T_c &= n + T_n \\ T_a + T_b &= T_{n-1} - T_c + 2n \\ T_a + T_b + |\bar{S} \cap [b+1 \cdots c]| &= |\bar{S} \cap [b+1 \cdots c]| + |[c+1 \cdots n-1]| + 2n. \end{aligned}$$

Now, converting the last equality into coins and subtracting (2), we get

$$[1 \cdots a] + S \cap [a+1 \cdots b] = \bar{S} \cap [b+1 \cdots c] + \bar{S} \cap [c+1 \cdots n-1] + n. \tag{3}$$

Again, each coin occurs at most once, so the Baron can legitimately take (3) as his second weighing.

Now, why do the balanced weighings (2) and (3) uniquely identify the n-gram coin? Consider which coins appear on which sides of those two equations. Let L_1 and R_1 be the left- and right-hand sides of the first weighing (2), respectively, and likewise L_2 and R_2 for the second weighing (3). Also, let O_1 and O_2 be the sets of coins that do not participate in (2) and (3), respectively. Then

$$\begin{aligned} [1 \cdots a] &= L_1 \cap L_2, \\ \bar{S} \cap [a+1 \cdots b] &= L_1 \cap O_2, \\ S \cap [a+1 \cdots b] &= O_1 \cap L_2, \\ S \cap [b+1 \cdots c] &= O_1 \cap O_2, \\ \bar{S} \cap [b+1 \cdots c] &= L_1 \cap R_2, \\ \bar{S} \cap [c+1 \cdots n-1] &= O_1 \cap R_2, \\ S \cap [c+1 \cdots n-1] &= R_1 \cap O_2, \\ n &= R_1 \cap R_2. \end{aligned}$$

Seeing the two weighings (2) and (3), Baron Münchhausen's audience reasons analogously to how they did in the proof of Lemma 9. They consider the sum of the two weighings, which tells them

$$|L_1| + |L_2| = |R_1| + |R_2|.$$

They see that some coins, namely $L_1 \cap R_2$, (which the Baron knows to be $\bar{S} \cap [b+1 \cdots c]$) appeared first on the left and then on the right, so those coins do not affect the balance of the sum. The audience also sees that

1. a coins appeared on the left both times ($L_1 \cap L_2$);
2. $b - a$ coins appeared on the left once and never on the right ($(L_1 \cap O_2) \cup (O_1 \cap L_2)$);
3. $n-1-c$ coins appeared on the right once and never on the left ($(R_1 \cap O_2) \cup (O_1 \cap R_2)$); and
4. just one coin appeared on the right both times ($R_1 \cap R_2$).

Now, a and $b - a$ are going to be much bigger than $n - c - 1$ and 1, so the audience will be surprised that so many coins can be balanced by so few. And they will wonder how to minimize the total weight

$$2|L_1 \cap L_2| + |(L_1 \cap O_2) \cup (O_1 \cap L_2)|$$

of the many, and how to maximize the total weight

$$|(R_1 \cap O_2) \cup (O_1 \cap R_2)| + 2|R_1 \cap R_2|$$

of the few. And they will see that to do this, they must

1. let the coins in $L_1 \cap L_2$ have the weights $[1 \cdots a]$;
2. let the coins in $(L_1 \cap O_2) \cup (O_1 \cap L_2)$ have the weights $[a + 1 \cdots b]$;
3. let the coins in $(R_1 \cap O_2) \cup (O_1 \cap R_2)$ have the weights $[c + 1 \cdots n - 1]$; and
4. let the sole coin in $R_1 \cap R_2$ have weight n.

And then they will see from (1), which can be rewritten as

$$\begin{aligned} T_a + T_b + T_c &= T_n + n \\ T_a + T_b &= T_{n-1} - T_c + 2n \\ 2|[1 \cdots a]| + |[a + 1 \cdots b]| &= |[c + 1 \cdots n - 1]| + 2n \end{aligned}$$

that even if they minimize the left and maximize the right, the scale will just barely balance. And then they will know that any other weights than those would have made the left heavier than the right, and since the scale did balance, those are the weights that must have been, and they will wonder in awe at the Baron's skill in convincing them of the weight of his chosen coin out of n in only two weighings. □

We have established that the existence of a subset S of $[a + 1 \cdots n - 1]$ that adds up to $|S| = T_c - n$ suffices to let the Baron convince his audience of the weight of the coin labeled n in two weighings. Now, when does such a subset reliably exist?

8.2 Step 2: Sums of subsets of ranges

What are the possible sums of subsets of a given range of positive integers? Straightforward calculations provide the needed answer; we state the relevant facts without consuming space with their proofs.

Lemma 11. *The set of possible sums of subsets of size k of a range $[s \cdots t]$ is exactly the range $[ks + T_{k-1} \cdots kt - T_{k-1}]$.*

Lemma 12. *The set of possible sums of subsets of size k or $k+1$ of the range $[s \cdots t]$ is a single contiguous range if and only if*

$$kt - T_{k-1} + 1 \geq (k+1)s + T_k. \tag{4}$$

In this case, the sums are the range $[ks + T_{k-1} \cdots (k+1)t - T_k]$.

If (4) holds for $k < (t-s)/2$, then (4) also holds for $k' = k+1$. The range contiguity behavior is also preserved under the symmetry $k \to t - s - k$.

Lemma 13. *If $k < (t-s)/2$ is such that*

$$kt - T_{k-1} + 1 \geq (k+1)s + T_k,$$

then the set of possible sums of subsets of sizes $[k \cdots t-s-k]$ is the contiguous range $[ks + T_{k-1} \cdots (t-s-k)t - T_{t-s-k-1}]$.

Considering the cases when $k = 1$ or $k = 2$ gives

Corollary 14. *If $t + 1 \geq 2s + 1$, or, equivalently, $s \leq t/2$, the subsets of the range $[s \cdots t]$ can achieve any sum in*

$$[s \cdots T_t - T_s].$$

Corollary 15. *If $2t \geq 3s + 3$, or, equivalently, $s \leq \frac{2}{3}t - 1$, the subsets of the range $[s \cdots t]$ can achieve any sum in*

$$[2s + 1 \cdots T_t - T_{s+1}].$$

These two facts will help characterize when $T_c - n$, from above, can be achieved as the sum of some set of coins from a given range.

8.3 Step 3: Systematic study of possible a, b, and c

We are now ready to finish this proof. Recall the setup: The Baron has n coins; we have made a decomposition into three triangular numbers

$$n + T_n = T_a + T_b + T_c, \quad a \leq b \leq c;$$

and we know that if we can find a subset S of $[a+1 \cdots n-1]$ for which

$$T_c - n = |S|,$$

the Baron can convince his audience of the weight of the n-gram coin in two weighings. We also know, from the corollaries above, that

1. If $2a+3 \le n$, sums of subsets of $[a+1 \cdots n-1]$ include the range $[a+1 \cdots T_{n-1}-T_{a+1}]$, and

2. If $3a+6 \le 2n-2$, sums of subsets of $[a+1 \cdots n-1]$ include the range $[2a+3 \cdots T_{n-1} - T_{a+2}]$.

Here is the main remaining idea: As T_a is the smallest number in our decomposition, we know that $T_a \le \frac{T_n+n}{3} < \frac{T_{n+1}}{3}$. We can conclude from this that $a < \frac{n+1}{\sqrt{3}}$. Since $\frac{n+1}{\sqrt{3}}$ grows slower than $\frac{2n}{3}$, for large n we expect the condition $3a+6 \le 2n-2$ in Corollary 15 to hold. Then it will suffice to prove that $T_c - n$ falls into the rather large range $[2a+3 \cdots T_{n-1} - T_{a+2}]$. The lower bound will be easy; and for the upper bound we will find that if T_c is large, then T_a will be small. This analysis will cover $c \le n-4$; and $c > n-4$ will be extreme enough to handle via Corollary 14.

Lemma 16. *For $n \ge 37$ having decompositions of $n + T_n$ into $T_a + T_b + T_c$ with $a \le b \le c$ and $c \le n-4$, Theorem 7 holds.*

Proof. Since $a < \frac{n+1}{\sqrt{3}}$,

$$3a + 6 < (n+1)\sqrt{3} + 6. \tag{5}$$

For $n \ge 37$, the right hand side of (5) is less than $2n-2$, so Corollary 15 applies. Subsets of $[a+1 \cdots n-1]$ cover all sums in $[2a+3 \cdots T_{n-1} - T_{a+2}]$.

Does $T_c - n$ fall into this range? For the upper bound, we have the sequence of equivalent inequalities, starting with the desired one

$$\begin{aligned} T_c - n &\le T_{n-1} - T_{a+2} \\ T_a + T_c + (a+1) + (a+2) + n &\le T_{n-1} + 2n = T_a + T_b + T_c \\ n + 2a + 3 &\le T_b. \end{aligned}$$

The last of these also implies $T_c - n \ge T_b - n \ge 2a+3$, so for $n \ge 37$, as long as $T_b \ge n + 2a + 3$, a subset S of $[a+1 \cdots n-1]$ can be found that sums to $T_c - n$, permitting the Baron to convince his audience of the weight of the coin labeled n.

When can we guarantee that $T_b \ge n + 2a + 3$? We know that $a < \frac{n+1}{\sqrt{3}}$, so it is enough to guarantee that

$$T_b \ge n + \frac{2(n+1)}{\sqrt{3}} + 3.$$

As $T_b \ge T_a$, it is enough to guarantee that

$$T_a + T_b = T_n - T_c + n \ge 2n + \frac{4(n+1)}{\sqrt{3}} + 6.$$

If $c \le n-4$, then $T_n - T_c + n \ge 5n - 6$. For $n \ge 37$, $5n - 6 > 2n + \frac{4(n+1)}{\sqrt{3}} + 6$, so $T_c - n$ does, in fact, fit into the desired range. □

Lemma 17. *For $n \ge 21$ having decompositions of $n + T_n$ into $T_a + T_b + T_c$ with $a \le b \le c$ and $c > n-4$, Theorem 7 holds.*

Proof. As remarked earlier, $c > n$ is impossible, and $c = n$ implies that $n = T_a + T_b$, so two weighings suffice by Lemma 8. So we are left with three cases: $c = n - 1$, $c = n - 2$ and $c = n - 3$. For such c, T_c is at least T_{n-3}, so

$$T_a + T_b \leq 2n + (n-1) + (n-2) = 4n - 3.$$

Therefore $T_a \leq 2n - \frac{3}{2}$. For $n \geq 21$, this implies $2a + 3 \leq n$. This fact allows us to use Corollary 14, meaning that we have full use of the range $[a + 1 \cdots T_{n-1} - T_{a+1}]$.

Does $T_c - n$ fall into this range? For $n \geq 21$, the lower bound follows from

$$T_c - n \geq T_{n-3} - n \gg n > a + 1.$$

We prove the upper bound case by case.

Case 1. $c = n - 3$. Rearrange the upper bound condition:

$$T_{n-3} - n = T_c - n \leq T_{n-1} - T_{a+1} \quad \Leftrightarrow \tag{6}$$
$$T_a + a + 1 \leq n + (n-1) + (n-2) = 3n - 3. \tag{7}$$

For n this large, the known $2a + 3 \leq n$ and $T_a \leq 2n - \frac{3}{2}$ together imply (7), so S exists and the Baron succeeds.

Case 2. $c = n - 2$. Then $T_a + T_b = 3n - 1$; therefore $T_a \leq \frac{3n-1}{2}$. For the upper bound, we want

$$T_{n-2} - n = T_c - n \leq T_{n-1} - T_{a+1} \quad \Leftrightarrow$$
$$T_a + a + 1 \leq n + (n-1).$$

Since we know $T_a \leq \frac{3n-1}{2}$, it suffices that $a + 1 \leq \frac{n-1}{2}$, which is mercifully equivalent to the already established condition $2a + 3 \leq n$. Therefore, the desired subset S exists and the Baron succeeds.

Case 3. $c = n - 1$. Then $T_a + T_b = 2n$; therefore $T_a \leq n$. If $b = a$, then $T_a = n$ and the Baron succeeds in one weighing. If $b = a + 1$, then the Baron succeeds in two weighings by Lemma 9.

Now let us assume that $b \geq a+2$. Therefore, $T_b \geq T_{a+2} > T_a + (a+1) + (a+1)$. Therefore

$$\begin{aligned} 2n = T_a + T_b &> 2(T_a + (a+1)) \\ -n &< -T_{a+1} \\ T_c - n &< T_{n-1} - T_{a+1}, \end{aligned}$$

so $T_c - n$ fits in the desired range and the Baron succeeds. □

The arguments above prove that the Baron can convince his audience of the weight of the n-gram coin among n coins for $n \geq 37$. The theorem is completed by noting that Lemma 8 covers all smaller n.

9 Discussion

Is it surprising that the answer is two? That no matter how many coins there are, Baron Münchhausen can always prove the weight of one of them in just two weighings on the scale? It surprised us and it surprised most people we gave this puzzle to. At first, everyone expects that this sequence should tend to infinity, or at least grow without bound.

So we were most intrigued when we proved Theorem 1 and discovered that the problem always has a simple solution in three weighings. On reflection, however, maybe that discovery should have been less of a surprise. In standard coin-weighing puzzles, the person constructing the weighings is trying to find something out; so they are limited if nothing else by information-theoretic considerations, and as the number of coins involved increases, the problem usually becomes unequivocally more difficult. In this puzzle, however, Baron Münchhausen has complete information. So on the one hand, as the number of coins increases, the audience knows less and the Baron's task becomes more difficult; but on the other hand, the available resources for constructing interesting weighings also grow. In this case, it turns out that these forces balance to produce a bounded sequence.

In fact, demonstrating the weight of one coin among n in two weighings is *easy*. Think about what the Baron actually needs to do to satisfy the conditions outlined in the beginning of the proof, in Section 8. He must find some decomposition of $n + T_n$ into three triangular numbers, and find some subset of a certain collection of his coins that adds up to some number. The proof is long and hairy because we are trying to prove that this subset *always* exists, but the vast majority of the time this is trivial. How many ways are there to pick a subset of integers from fifty to a hundred, so that their sum will be three thousand? Or three thousand one? Gazillions![1] If the range one has to work with is reasonably large, and the target sum is comfortably between zero and the total sum of all the integers in one's range, then of course one can find a subset, or a hundred subsets.

Even more, any number n generally has *many* decompositions into a sum of three triangular numbers—on the order of the square root of n [5]. The proof in Section 8 is hairy also because we were proving that an *arbitrary* decomposition of $n+T_n$ into three triangular numbers leads to a solution, but in practice the Baron has the freedom to pick and choose among a great number of possible decompositions.

The ease of proving one coin in two weighings suggests two future directions. One can explore how many ways there are for Baron Münchhausen to prove himself right. One can also explore harder tasks that can be asked of him.

For the outermost example, we can ask the Baron to prove the weight of *all* the given coins. For $n = 6$ this task will match the following puzzle, authored by Sergey Tokarev [9], that appeared at the last round of the Moscow Math Olympiad in 1991:

> You have 6 coins weighing 1, 2, 3, 4, 5 and 6 grams that look the same, except for their labels. The number (1, 2, 3, 4, 5, 6) on the top of each coin should correspond to its weight. How can you determine whether all the numbers are correct, using a balance scale only twice?

[1] Yes, "gazillions" is a technical term in advanced combinatorics.

This task is clearly harder, and indeed this sequence does tend to infinity: If the total number of coins is n, then the needed number of weighings is always greater than $\log_3 n$ [7]. And again, we give it as homework for the reader to prove this lower bound, as well as an upper bound of $n-1$ weighings.

There is also a huge spectrum of possible intermediate tasks. For example, how many coins can the Baron show at once with at most two weighings? What is the smallest number of weighings the Baron needs to specify two coins? Or, given the total number of coins, how many weighings does the Baron need to show the weight of a particular coin? What if the audience can choose which coin's weight the Baron must prove? Which of these tasks can be done in a fixed maximum number of weighings, and which can not? What asymptotic behaviors of the number of needed weighings occur? What happens if we start using different families of sets of available coins, not just $[1 \cdots n]$? There is plenty to be curious about!

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