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Some Properties of Hyperfibonacci and Hyperlucas Numbers

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Abstract

In this paper, we discuss the properties of hyperfibonacci numbers and hyperlucas numbers. We derive some identities for hyperfibonacci and hyperlucas numbers by the method of coefficients. Furthermore, we give asymptotic expansions of certain sums involving hyperfibonacci and hyperlucas numbers by Darboux's method.

1 Introduction

Dil and Mező [1] introduced the definitions of "hyperfibonacci" numbers $F_n^{(r)}$ and "hyperlucas" numbers $L_n^{(r)}$

$$F_n^{(r)} = \sum_{k=0}^{n} F_k^{(r-1)}, \quad \text{with} \quad F_n^{(0)} = F_n, \quad F_0^{(r)} = 0, \quad F_1^{(r)} = 1,$$

$$L_n^{(r)} = \sum_{k=0}^{n} L_k^{(r-1)}, \quad \text{with} \quad L_n^{(0)} = L_n, \quad L_0^{(r)} = 2, \quad L_1^{(r)} = 2r+1,$$

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where r is a positive integer, and F_n and L_n are Fibonacci and Lucas numbers, respectively. The generating functions of $F_n^{(r)}$ and $L_n^{(r)}$ are as follows:

$$\begin{aligned}\sum_{n=0}^{\infty} F_n^{(r)} t^n &= \frac{t}{(1-t-t^2)(1-t)^r},\\ \sum_{n=0}^{\infty} L_n^{(r)} t^n &= \frac{2-t}{(1-t-t^2)(1-t)^r}.\end{aligned}$$

The first few values of $F_n^{(r)}$ and $L_n^{(r)}$ are as follows:

$$F_n^{(1)} : 0, 1, 2, 4, 7, 12, 20, 33, 54, 88, 143, 232, 376, 609, 986, 1596, 2583, \ldots;$$
$$F_n^{(2)} : 0, 1, 3, 7, 14, 26, 46, 79, 133, 221, 364, 596, 972, 1581, 2567, 4163, 6746, \ldots;$$
$$L_n^{(1)} : 2, 3, 6, 10, 17, 28, 46, 75, 122, 198, 321, 520, 842, 1363, 2206, 3570, 5777, \ldots;$$
$$L_n^{(2)} : 2, 5, 11, 21, 38, 66, 112, 187, 309, 507, 828, 1348, 2190, 3553, 5759, 9329, 15106, \ldots\ .$$

There are some elementary identities for $F_n^{(r)}$ and $L_n^{(r)}$ when $r = 1, 2$. For example,

$$\begin{aligned}F_n^{(1)} &= \sum_{k=0}^{n-2} k F_{n-k-3},\\ F_n^{(2)} &= F_{n+4} - n - 3\\ &= \sum_{k=0}^{n} (n-k) F_k,\\ L_n^{(1)} &= L_{n+1} - 1\\ &= F_n + F_{n+2} - 1,\\ L_n^{(2)} &= 4(F_{n+1} - 1) + 3F_n - n\\ &= L_{n+3} - (n+4).\end{aligned}$$

For the above values and elementary identities of $F_n^{(r)}$ and $L_n^{(r)}$, see [4] (A000071, A001924, A001610, A023548).

Hyperfibonacci numbers and hyperlucas numbers are useful, and Dil and Mező [1] derived some equalities for Fibonacci and Lucas numbers by applying them. Hence, hyperfibonacci numbers and hyperlucas numbers deserve to be investigated. In this paper, we discuss properties of $F_n^{(r)}$ and $L_n^{(r)}$. We establish some identities for $F_n^{(r)}$ and $L_n^{(r)}$. Furthermore, we give asymptotic expansions of certain sums related to $F_n^{(r)}$ and $L_n^{(r)}$.

For convenience, we first recall some notation. Let $\alpha = (1+\sqrt{5})/2$. It is well known that

$$F_n = \frac{\alpha^n - (-1)^n \alpha^{-n}}{\sqrt{5}}, \quad L_n = \alpha^n + (-1)^n \alpha^{-n},$$

and F_n and L_n satisfy the following recurrence relation

$$W_{n+1} = W_n + W_{n-1}, \quad n \geq 1. \tag{1}$$

As usual, the binomial coefficient $\binom{n}{m}$ is defined by

$$\binom{n}{m} = \begin{cases} \dfrac{n!}{m!(n-m)!}, & \text{if} \quad n \geq m; \\ 0, & \text{if } n < m; \end{cases}$$

where n and m are nonnegative integers. Throughout, $[z^n]f(z)$ denotes the coefficient of z^n in $f(z)$, where

$$f(z) = \sum_{n=0}^{\infty} f_n z^n.$$

If $f(t)$ and $g(t)$ are formal power series, the following relations hold [2]

$$[t^n](af(t) + bg(t)) = a[t^n]f(t) + b[t^n]g(t), \tag{2}$$

$$[t^n]tf(t) = [t^{n-1}]f(t), \tag{3}$$

$$[t^n]f(t)g(t) = \sum_{k=0}^{n} [y^k]f(y)[t^{n-k}]g(t). \tag{4}$$

The above relations (2)–(4) will be used later on.

Now we recall the notation of the binomial transform of a sequence, the inverse binomial transform of a sequence and Euler-Seidel infinite matrix [1, 5]. Let $\{a_k\}_{k=0}^{\infty}$ be a sequence. The binomial transform of $\{a_k\}$ is given by $\sum_{k=0}^{n} \binom{n}{k} a_k$, the inverse binomial transform of $\{a_k\}$ is given by $\sum_{k=0}^{n} \binom{n}{k} (-1)^k a_k$, and the Euler-Seidel infinite matrix corresponding to the sequence $\{a_k\}$ is determined by the following formulas

$$a_n^{[0]} = a_n \quad (n \geq 0),$$

$$a_n^{[k]} = a_n^{[k-1]} + a_{n+1}^{[k-1]} \quad (n \geq 0, \quad k \geq 0),$$

where $a_n^{[k]}$ is the element at the $(k+1)th$ row and the $(n+1)th$ column. The sequences $\{a_n^{[0]}\}$ and $\{a_0^{[n]}\}$ satisfy [1]

$$a_0^{[n]} = \sum_{k=0}^{n} \binom{n}{k} a_k^{[0]}, \tag{5}$$

$$a_n^{[0]} = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} a_0^{[k]}. \tag{6}$$

Let $a(t)$ and $\bar{a}(t)$ be the generating function of $\{a_n^{[0]}\}$ and $\{a_0^{[n]}\}$, respectively,

$$a(t) = \sum_{n=0}^{\infty} a_n^{[0]} t^n, \quad \bar{a}(t) = \sum_{n=0}^{\infty} a_0^{[n]} t^n.$$

The functions $a(t)$ and $\bar{a}(t)$ satisfy [1]

$$\bar{a}(t) = \frac{1}{1-t} a\left(\frac{t}{1-t}\right), \tag{7}$$

$$a(t) = \frac{1}{t+1} \bar{a}\left(\frac{t}{t+1}\right). \tag{8}$$

2 Main Results

In this section, we derive some identities for $F_n^{(r)}$ and $L_n^{(r)}$. Later, we give asymptotic expansions of certain sums involving $F_n^{(r)}$ and $L_n^{(r)}$.

Various identities involving Fibonacci and Lucas numbers were established. The following sums were investigated [3, 7, 8]

$$\sum_{j_1+j_2+\cdots+j_k=n} F_{j_1}F_{j_2}\cdots F_{j_k}, \qquad \sum_{j_1+j_2+\cdots+j_k=n} L_{j_1}L_{j_2}\cdots L_{j_k}.$$

For example,

$$\begin{aligned} \sum_{j_1+j_2=n} F_{j_1}F_{j_2} &= \frac{(n-1)L_n+2F_{n-1}}{5}, \\ \sum_{j_1+j_2=n} L_{j_1}L_{j_2} &= (n+1)L_n+2F_{n+1}. \end{aligned}$$

Now we derive some identities for $F_n^{(r)}$ and $L_n^{(r)}$. Denote

$$A_{n,k,r} = \sum_{j_1+j_2+\cdots+j_k=n} F_{j_1}^{(r)}F_{j_2}^{(r)}\cdots F_{j_k}^{(r)}, \quad B_{n,k,r} = \sum_{j_1+j_2+\cdots+j_k=n} L_{j_1}^{(r)}L_{j_2}^{(r)}\cdots L_{j_k}^{(r)}.$$

These sums are interesting because they can help us to find some new convolution properties. For $A_{n,k,r}$ and $B_{n,k,r}$, we have

Theorem 1. *Let k, $n \geq 1$ and $r \geq 1$ be positive integers. For $A_{n,k,r}$ and $B_{n,k,r}$, we have*

$$A_{n,2,1} = n+5-2F_{n+3}+\frac{(n-1)L_{n+4}+2F_{n-1}}{5}, \tag{9}$$

$$B_{n,2,1} = n+9-10F_{n+3}+\frac{(5n-1)L_{n+4}+4L_{n+6}+10F_{n-1}}{5}, \tag{10}$$

$$A_{n,k+1,r} = \sum_{j=0}^{n} A_{n,k,r}F_j^{(r)}, \tag{11}$$

$$B_{n,k+1,r} = \sum_{j=0}^{n} B_{n,k,r}L_j^{(r)}. \tag{12}$$

Proof. Let

$$F_r(t) = \frac{t}{(1-t-t^2)(1-t)^r}, \quad L_r(t) = \frac{2-t}{(1-t-t^2)(1-t)^r}.$$

Clearly,

$$\begin{aligned}
F_1(t) &= \left(\frac{\alpha^2-\alpha^{-2}}{t-1} - \frac{\alpha^2}{t-\alpha^{-1}} + \frac{\alpha^{-2}}{t+\alpha}\right)\frac{t}{\sqrt{5}} \\
&= \left(\sum_{n=0}^{\infty} F_{n+3}t^n - \sum_{n=0}^{\infty} t^n\right)t, \\
A_{n,2,1} &= [t^n]F_1^2(t), \\
B_{n,2,1} &= [t^n]L_1^2(t) \\
&= [t^n]F_1^2(t) - 4[t^{n+1}]F_1^2(t) + 4[t^{n+2}]F_1^2(t) \\
&= A_{n,2,1} - 4A_{n+1,2,1} + 4A_{n+2,2,1}.
\end{aligned}$$

Then we get

$$\begin{aligned}
\sum_{j_1+j_2=n} F_{j_1}^{(1)}F_{j_2}^{(1)} &= [t^n]F_1^2(t) \\
&= [t^n]\frac{t^2}{5}\Bigg[\frac{(\alpha^2-\alpha^{-2})^2}{(t-1)^2} + \frac{\alpha^4}{(t-\alpha^{-1})^2} + \frac{\alpha^{-4}}{(t+\alpha)^2} - \frac{2\alpha^2(\alpha^2-\alpha^{-2})}{(t-1)(t-\alpha^{-1})} \\
&\quad + \frac{2\alpha^{-2}(\alpha^2-\alpha^{-2})}{(t-1)(t+\alpha)} - \frac{2}{(t-\alpha^{-1})(t+\alpha)}\Bigg] \\
&= [t^n]\frac{t^2}{5}\Bigg\{\frac{(\alpha^2-\alpha^{-2})^2}{(t-1)^2} + \frac{\alpha^6}{(1-\alpha t)^2} + \frac{\alpha^{-6}}{(1+\alpha^{-1}t)^2} \\
&\quad - \frac{2(\alpha^4-\alpha^{-4})(\alpha^2-\alpha^{-2})}{t-1} + \left[-2\alpha^5(\alpha^2-\alpha^{-2}) + \frac{2\alpha}{\alpha+\alpha^{-1}}\right]\frac{1}{1-\alpha t} \\
&\quad + \left[-2\alpha^{-5}(\alpha^2-\alpha^{-2}) + \frac{2\alpha^{-1}}{\alpha+\alpha^{-1}}\right]\frac{1}{1+\alpha^{-1}t}\Bigg\} \\
&\quad = [t^n]\frac{t^2}{5}\Bigg\{(\alpha^2-\alpha^{-2})^2\sum_{n=0}^{\infty}(n+1)t^n + \sum_{n=0}^{\infty}(n+1)\alpha^{n+6}t^n \\
&\quad + \sum_{n=0}^{\infty}(n+1)(-1)^n\alpha^{-n-6}t^n + 2(\alpha^4-\alpha^{-4})(\alpha^2-\alpha^{-2})\sum_{n=0}^{\infty}t^n \\
&\quad + \left[-2\alpha^5(\alpha^2-\alpha^{-2}) + \frac{2\alpha}{\alpha+\alpha^{-1}}\right]\sum_{n=0}^{\infty}\alpha^n t^n \\
&\quad + \left[-2\alpha^{-5}(\alpha^2-\alpha^{-2}) + \frac{2\alpha^{-1}}{\alpha+\alpha^{-1}}\right]\sum_{n=0}^{\infty}(-1)^n\alpha^{-n}t^n\Bigg\} \\
&= [t^n]t^2\Bigg\{\sum_{n=0}^{\infty}(n+1)t^n + \frac{1}{5}\sum_{n=0}^{\infty}(n+1)[\alpha^{n+6}+(-1)^n\alpha^{-n-6}]t^n + 2F_4\sum_{n=0}^{\infty}t^n \\
&\quad -2\sum_{n=0}^{\infty}\frac{\alpha^{n+5}+(-1)^n\alpha^{-n-5}}{\sqrt{5}}t^n + \frac{2}{5}\sum_{n=0}^{\infty}\frac{\alpha^{n+1}+(-1)^n\alpha^{-n-1}}{\sqrt{5}}t^n\Bigg\}
\end{aligned}$$

$$
\begin{aligned}
= \quad & [t^n]t^2\Bigg[\sum_{n=0}^{\infty}(n+7)t^n + \frac{1}{5}\sum_{n=0}^{\infty}(n+1)L_{n+6}t^n - 2\sum_{n=0}^{\infty}F_{n+5}t^n \\
& + \frac{2}{5}\sum_{n=0}^{\infty}F_{n+1}t^n\Bigg].
\end{aligned}
$$

By applying (3) and the definitions of F_n and L_n, we have (9). Naturally, we deduce that

$$
\begin{aligned}
B_{n,2,1} \quad = \quad & n+5-2F_{n+3} + \frac{(n-1)L_{n+4}+2F_{n-1}}{5} + 4\Bigg[n+7-2F_{n+5} \\
& + \frac{(n+1)L_{n+6}+2F_{n+1}}{5}\Bigg] - 4\Bigg[n+6-2F_{n+4}+\frac{nL_{n+5}+2F_n}{5}\Bigg].
\end{aligned}
$$

By using (1), we prove that (10) holds. By means of (4), we can show that (11) and (12) hold. □

It is interesting that we can get congruence relations from (9) and (10)

$$
A_{n,2,1} \equiv \left(n - 2F_{n+3} + \frac{(n-1)L_{n+4}+2F_{n-1}}{5}\right) \pmod 5,
$$

$$
B_{n,2,1} \equiv \left(n - 10F_{n+3} + \frac{(5n-1)L_{n+4}+4L_{n+6}+10F_{n-1}}{5}\right) \pmod 9.
$$

When k or r gets large, it is difficult to compute the closed forms of $A_{n,k,r}$ and $B_{n,k,r}$. However, we can give their asymptotic values. Now we recall a lemma [6].

Lemma 2. *Assume that $f(t) = \sum_{n\geq 0} a_n t^n$ is an analytic function for $|t| < r$ and with a finite number of algebraic singularities on the circle $|t| = r$. $\alpha_1, \alpha_2, \cdots, \alpha_l$ are singularities of order ω, where ω is the highest order of all singularities. Then*

$$
a_n = (n^{\omega-1}/\Gamma(\omega)) \times \left(\sum_{k=1}^{l} g_k(\alpha_k)\alpha_k^{-n} + o(r^{-n})\right), \tag{13}
$$

where $\Gamma(\omega)$ is the gamma function, and

$$
g_k(\alpha_k) = \lim_{t\to\alpha_k}(1-(t/\alpha_k))^{\omega} f(t).
$$

Theorem 3. *Suppose that k and r are fixed positive integers. For $A_{n,k,r}$ and $B_{n,k,r}$, when $n \to \infty$,*

$$
A_{n,k,r} = \frac{n^{k-1}}{(k-1)!}\left[\left(\frac{\alpha}{\alpha^2+1}\right)^k (1+\alpha)^{kr}\alpha^n + o(\alpha^n))\right], \tag{14}
$$

$$
B_{n,k,r} = \frac{n^{k-1}}{(k-1)!}\left[\left(\frac{2\alpha^2-\alpha}{\alpha^2+1}\right)^k (1+\alpha)^{kr}\alpha^n + o(\alpha^n))\right]. \tag{15}
$$

Proof. Let

$$f_{k,r}(t) = \frac{t^k}{(1-t-t^2)^k(1-t)^{kr}}.$$

We know that $f_{k,r}(t)$ is analytic for $|t| < 1/\alpha$, and with one algebraic singularity on the circle $|t| = 1/\alpha$. The order of $1/\alpha$ is k. One can compute that

$$\lim_{t\to 1/\alpha}(1-\alpha t)^k f_{k,r}(t) = \left(\frac{\alpha}{\alpha^2+1}\right)^k (1+\alpha)^{kr}.$$

By using Lemma 2, we can prove that (14) holds. Using the same method, we can prove that (15) holds. □

In addition, we give asymptotic expansions of other sums for $F_n^{(r)}$ and $L_n^{(r)}$.

Theorem 4. *Let n be a positive integer. When $n \to \infty$,*

$$\sum_{k=0}^{n}\binom{n}{k}F_k^{(r)} = \frac{\alpha(1+\alpha)^r}{(\alpha^2+1)(2-\alpha)^n} + o((2-\alpha)^{-n}), \tag{16}$$

$$\sum_{k=0}^{n}\binom{n}{k}L_k^{(r)} = \frac{(1+\alpha)^r}{(2-\alpha)^n} + o((2-\alpha)^{-n}), \tag{17}$$

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^k F_k^{(r)} = \frac{-\alpha^{n+1}(2-\alpha)^r}{\alpha^2+1} + o((-\alpha)^n), \tag{18}$$

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^k L_k^{(r)} = (2-\alpha)^r\alpha^n + o((-\alpha)^n). \tag{19}$$

Proof. We only give the proofs of (16) and (18). The proofs of (17) and (19) are similar to those of (16) and (18), respectively, and they are omitted here. Let $F_k^{(r)}$ be the first row, and we get the Euler-Seidel infinite matrix A. Then let $F_k^{(r)}$ be the first column, and we get Euler-Seidel infinite matrix B. The elements of A and B are denoted by $a_n^{[k]}$ and $b_n^{[k]}$. By using (5) and (6), we obtain

$$a_0^{[n]} = \sum_{k=0}^{n}\binom{n}{k}F_k^{(r)},$$

$$b_n^{[0]} = \sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}F_k^{(r)}.$$

By means of (7) and (8), we get

$$\begin{aligned}\sum_{k=0}^{n}\binom{n}{k}F_k^{(r)} &= a_0^{[n]}\\ &= [t^n]\frac{t(1-t)^r}{(t^2-3t+1)(1-2t)^r},\\ \sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}F_k^{(r)} &= b_n^{[0]}\\ &= [t^n]\frac{-t(1+t)^r}{t^2-t-1}.\end{aligned}$$

We know that the function $\bar{a}(t)=\dfrac{t(1-t)^r}{(t^2-3t+1)(1-2t)^r}$ is analytic for $|t|<2-\alpha$ with one algebraic singularity $\alpha_1=2-\alpha$ on the circle $|t|=2-\alpha$, and $a(t)=\dfrac{-t(1+t)^r}{t^2-t-1}$ is analytic for $|t|<\alpha^{-1}$ with one algebraic singularity $\beta_1=-\alpha^{-1}$ on the circle $|t|=\alpha^{-1}$. The orders of α_1 and β_1 are 1. We can compute that

$$\begin{aligned}\lim_{t\to 2-\alpha}(1-\frac{t}{2-\alpha})\bar{a}(t) &= \frac{\alpha(1+\alpha)^r}{\alpha^2+1},\\ \lim_{t\to -\alpha^{-1}}\left(1+\frac{t}{\alpha^{-1}}\right)a(t) &= -\frac{\alpha(2-\alpha)^r}{\alpha^2+1}.\end{aligned}$$

By using Lemma 2, we prove that (16) and (18) hold. □

Theorem 5. *Suppose that m and r are fixed positive integers. When $n\to\infty$,*

$$\sum_{j=0}^{n}F_{n-j}^{(r)}\binom{j+m-1}{j} = \frac{\alpha^{n+1}(1+\alpha)^{r+m}}{\alpha^2+1}+o(\alpha^n), \tag{20}$$

$$\sum_{j=0}^{n}L_{n-j}^{(r)}\binom{j+m-1}{j} = (1+\alpha)^{r+m}\alpha^n+o(\alpha^n). \tag{21}$$

Proof. We can verify that

$$\begin{aligned}\sum_{j=0}^{n}F_{n-j}^{(r)}\binom{j+m-1}{j} &= \sum_{j=0}^{n}[t^{n-j}]\frac{t}{(1-t-t^2)(1-t)^r}[t^j]\frac{1}{(1-t)^m}\\ &= [t^n]\frac{t}{(1-t-t^2)(1-t)^{m+r}},\\ \sum_{j=0}^{n}L_{n-j}^{(r)}\binom{j+m-1}{j} &= [t^n]\frac{2-t}{(1-t-t^2)(1-t)^{m+r}}.\end{aligned}$$

By using Lemma 2, we have (20) and (21). □

In the final of this section, we compare the accurate values with the asymptotic values. In Theorem 4, let

$$X_n = \sum_{k=0}^{n} \binom{n}{k} F_k^{(r)}, \quad Y_n = \frac{\alpha(1+\alpha)^r}{(\alpha^2+1)(2-\alpha)^n}.$$

n	X_n	Y_n	$\lvert X_n - Y_n\rvert/X_n$
50	$9.2737156629317196 \times 10^{20}$	$9.2737269219308562 \times 10^{20}$	1.2141×10^{-6}
100	$7.345448671565505 \times 10^{41}$	$7.3454486715782857 \times 10^{41}$	1.7399×10^{-12}
150	$5.818115698360039 \times 10^{62}$	$5.8181156983601641 \times 10^{62}$	2.1352×10^{-14}

Table 1: some values of X_n and Y_n

From the above table, we find that the value of $|X_n - Y_n|/X_n$ gets smaller and smaller with the increasing of n.

3 Remarks

Consider the sequences $\{u_n^{(r)}\}$ and $\{v_n^{(r)}\}$ defined by

$$\sum_{n=0}^{\infty} u_n^{(r)} z^n = \frac{z}{(1-pz-z^2)(1-z)^r},$$
$$\sum_{n=0}^{\infty} v_n^{(r)} z^n = \frac{2-pz}{(1-pz-z^2)(1-z)^r},$$

where $p > 0$. It is clear that $u_n^{(r)} = F_n^{(r)}$ and $v_n^{(r)} = L_n^{(r)}$ when $p = 1$. The conclusions of $F_n^{(r)}$ and $L_n^{(r)}$ can be generalized to the cases of $\{u_n^{(r)}\}$ and $\{v_n^{(r)}\}$. For example, put

$$U_{n,k,r} = \sum_{j_1+j_2+\cdots+j_k=n} u_{j_1}^{(r)} u_{j_2}^{(r)} \cdots u_{j_k}^{(r)}, \quad V_{n,k,r} = \sum_{j_1+j_2+\cdots+j_k=n} v_{j_1}^{(r)} v_{j_2}^{(r)} \cdots v_{j_k}^{(r)}.$$

Then we have

$$\begin{aligned} U_{n,2,1} &= \frac{n-1}{p^2(p^2+4)}(v_n^{(0)} + 2v_{n+1}^{(0)} + v_{n+2}^{(0)}) - \frac{2}{p^3}(u_{n+1}^{(0)} + 2u_n^{(0)} + u_{n-1}^{(0)}) \\ &\quad + \frac{2}{p(p^2+4)} u_{n-1}^{(0)} + \frac{np+p+4}{p^3}, \end{aligned} \tag{22}$$

$$U_{n,k+1,r} = \sum_{j=0}^{n} U_{n,k,r} u_j^{(r)}. \tag{23}$$

We can verify that

$$
\begin{aligned}
U_{n,2,1} &= [t^n]\frac{t^2}{(1-pt-t^2)^2(1-t)^2} \\
&= [t^n]\Big\{\frac{1}{\Delta(1+\tau)^2}\frac{t^2}{(t+\tau)^2}+\frac{1}{\Delta}\Big(\frac{1}{1-\tau^{-1}}-\frac{1}{1+\tau}\Big)^2\frac{t^2}{(1-t)^2} \\
&\quad +\frac{1}{\Delta}\frac{1}{(1-\tau^{-1})^2}\frac{t^2}{(t-\tau^{-1})^2}-\frac{2}{\Delta(1+\tau)}\Big(\frac{1}{1-\tau^{-1}}-\frac{1}{1+\tau}\Big)\frac{t^2}{\tau+1}\Big(\frac{1}{t+\tau}+\frac{1}{1-t}\Big) \\
&\quad -\frac{2t^2}{\Delta(1+\tau)(1-\tau^{-1})}\Big(\frac{-1}{t+\tau}+\frac{1}{t-\tau^{-1}}\Big)\frac{1}{\sqrt{\Delta}} \\
&\quad +\frac{2t^2}{\Delta(1-\tau^{-1})}\Big(\frac{1}{1-\tau^{-1}}-\frac{1}{1+\tau}\Big)\Big(\frac{1}{t-\tau^{-1}}+\frac{1}{1-t}\Big)\frac{1}{1-\tau^{-1}}\Big\} \\
&= [t^n]\Big\{\frac{1}{p^2\Delta}\sum_{n=0}^{\infty}(n+1)t^{n+2}(v_{n+2}^{(0)}+2v_{n+3}^{(0)}+v_{n+4}^{(0)})+\frac{1}{p^2}\sum_{n=0}^{\infty}(n+1)t^{n+2} \\
&\quad -\frac{2}{p^3}\sum_{n=0}^{\infty}t^{n+2}(u_{n+3}^{(0)}+2u_{n+2}^{(0)}+u_{n-1}^{(0)})+\frac{2}{p\Delta}\sum_{n=0}^{\infty}t^{n+2}u_{n+1}^{(0)}+\frac{4+2p}{p^3}\sum_{n=0}^{\infty}t^{n+2}\Big\},
\end{aligned}
$$

where $\Delta = p^2+4$, $\tau = \frac{p+\sqrt{\Delta}}{2}$. Hence (22) holds. The proof of (23) is similar to that of (11), and it is omitted here. The identities about $\{v_n^{(0)}\}$ can be found in the same way.

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References

[1] A. Dil and I. Mező, A symmetric algorithm for hyperharmonic and Fibonacci numbers, *Appl. Math. Comput.* **206** (2008), 942–951.

[2] D. Merlini, R. Sprugnoli and M. C. Verri, The method of coefficients, *Amer. Math. Monthly* **114** (2007), 40–57.

[3] N. Robbins, Some convolution-type and combinatorial identities pertaining to binary linear recurrences, *Fibonacci Quart.* **29** (1991), 249–255.

[4] N. J. A. Sloane, *The On–Line Encyclopedia of Integer Sequences*. Published electronically at http://www.research.att.com/~njas/sequences, 2010.

[5] M. Z. Spivey, Combinatorial sums and finite difference, *Discrete Math.* **307** (2007), 3130–3146.

[6] G. Szegő, *Orthogonal Polynomials*, Amer. Math. Soc. Colloq. Publ., Vol. 23, rev. ed., 1959.

[7] Wenpeng Zhang, Some identities involving the Fibonacci numbers, *Fibonacci Quart.* **35** (1997), 225–228.

[8] Fengzhen Zhao and Tianming Wang, Generalizations of some identities involving the Fibonacci numbers, *Fibonacci Quart.* **39** (2001), 165–167.

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