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On Prime-Detecting Sequences From Apéry's Recurrence Formulae for $\zeta(3)$ and $\zeta(2)$

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Abstract

We consider the linear three-term recurrence formula

$$X_n = (34(n-1)^3 + 51(n-1)^2 + 27(n-1) + 5)X_{n-1} - (n-1)^6 X_{n-2} \quad (n \geq 2)$$

corresponding to Apéry's non-regular continued fraction for $\zeta(3)$. It is shown that integer sequences $(X_n)_{n\geq 0}$ with $5X_0 \neq X_1$ satisfying the above relation are prime-detecting, i.e., $X_n \not\equiv 0 \pmod{n}$ if and only if n is a prime not dividing $|5X_0 - X_1|$. Similar results are given for integer sequences satisfying the recurrence formula

$$X_n = (11(x-1)^2 + 11(x-1) + 3)X_{n-1} + (n-1)^4 X_{n-2} \quad (n \geq 2)$$

corresponding to Apéry's non-regular continued fraction for $\zeta(2)$ and for sequences related to $\log 2$.

1 Introduction

In 1979, R. Apéry [1] proved the irrationality of $\zeta(3) = \sum_{n=1}^{\infty} 1/n^3$. The jumping-off point of his proof is a recurrence formula,

$$(n+1)^3 X_{n+1} - (34n^3 + 51n^2 + 27n + 5)X_n + n^3 X_{n-1} = 0, \tag{1}$$

which is satisfied by $X_n = a_n$ and $X_n = b_n$ with

$$a_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2, \qquad b_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 c_{n,k}, \tag{2}$$

where

$$c_{n,k} = \sum_{m=1}^{n} \frac{1}{m^3} + \sum_{m=1}^{k} \frac{(-1)^{m-1}}{2m^3 \binom{n}{m}\binom{n+m}{m}} \qquad (1 \le k \le n). \tag{3}$$

One basic fact for the irrationality proof of $\zeta(3)$ is the following inequality:

$$0 \neq \zeta(3) - \frac{b_n}{a_n} = O\big((1+\sqrt{2})^{-8n} \big).$$

When n increases, b_n/a_n converges rapidly to $\zeta(3)$ so that one can conclude the irrationality of $\zeta(3)$. From (1), Apéry's continued fraction expansion of $\zeta(3)$ can be derived, namely

$$\zeta(3) = \cfrac{6}{5 - \cfrac{1^6}{117 - \cfrac{2^6}{535 - \ddots - \cfrac{n^6}{34n^3 + 51n^2 + 27n + 5 - \cdots}}}} \tag{4}$$

(see [4]). F. Beukers [2] proved the congruence

$$a_{((p-1)/2)} \equiv \gamma_p \pmod{p}$$

for all odd primes p, where the integers γ_n are given by the following series expansion of an infinite product:

$$\sum_{n=1}^{\infty} \gamma_n q^n = q \prod_{n=1}^{\infty} (1-q^{2n})^4 (1-q^{4n})^4.$$

Note that $(b_n)_{n\ge 0}$ is a sequence of rationals. For the concept of prime-detecting sequences introduced below we shall need integer sequences. Therefore, we define

$$q_0 = 1, \quad q_n = (n!)^3 a_n \quad (n \ge 1), \quad p_0 = 0, \quad p_n = (n!)^3 b_n \quad (n \ge 1), \tag{5}$$

so that the p_n are integers. It can be shown that both sequences, $(q_n)_{n\ge 0}$ and $(p_n)_{n\ge 0}$, satisfy the recurrence formula

$$X_n = T(n) X_{n-1} - U(n) X_{n-2} \qquad (n \ge 2), \tag{6}$$

where $T(n) = 34(n-1)^3 + 51(n-1)^2 + 27(n-1) + 5$ and $U(n) = (n-1)^6$. This requires some technical computations. Alternatively, for $X_n = q_n$ one can verify (6) by application of the *Zeilberger algorithm* [6, Chapter 7, Algorithm 7.1] using a computer. The same algorithm works for $X_n = a_n$ and (1) ([6, p. 101-102]), but not for p_n and b_n, respectively. We also have

$$\frac{p_n}{q_n} = \frac{b_n}{a_n} \longrightarrow \zeta(3)$$

as n tends to infinity. Finally, computing $p_1 = 6$ and $q_1 = 5$ from (2) and (3), the continued fraction (4) follows from the formula (1) in [7, §§1, 2].

We let $\mathbb{P}$ denote the set of prime numbers. There are several possibilities for suitable functions and sequences to detect primes. We give a short summary of various prime-detecting methods in the concluding section 4 of this paper. Of course, Wilson's theorem plays a significant role.

Proposition 1. *For all integers $n \in \mathbb{N} \setminus \{4\}$ we have*

$$(n-1)! \not\equiv 0 \pmod{n} \quad \Longleftrightarrow \quad n \in \mathbb{P}.$$

Proof. For any prime n we know by Wilson's criterion that $(n-1)! \equiv -1 \pmod{n}$. So it remains to prove $(n-1)! \equiv 0 \pmod{n}$ for any $n = ab \neq 1, 4$ with integers $1 < a, b < n$.

Case 1: $a = b \geq 3$.
Since $n = a^2$ and $\operatorname{lcm}(2, a, 2a, a^2) = \operatorname{lcm}(2, a^2)$, we have

$$\operatorname{lcm}(1, \ldots, a-1, a+1, \ldots, 2a-1, 2a+1, \ldots, n) = \operatorname{lcm}(1, \ldots, n),$$

and so

$$\operatorname{lcm}(1, \ldots, n) \,\Big|\, \Big(1 \cdots (a-1)(a+1) \cdots (2a-1)(2a+1) \cdots n\Big) = \frac{n!}{2a^2} = \frac{(n-1)!}{2}.$$

Case 2: $1 < a < b$.
Since $n = ab = \operatorname{lcm}(a, b, ab)$, we have

$$\operatorname{lcm}(1, \ldots, a-1, a+1, \ldots, b-1, b+1, \ldots, n) = \operatorname{lcm}(1, \ldots, n).$$

Hence

$$\operatorname{lcm}(1, \ldots, n) \,\Big|\, \Big(1 \cdots (a-1)(a+1) \cdots (b-1)(b+1) \cdots n\Big) = \frac{n!}{ab} = (n-1)!.$$

In any case, we get $\operatorname{lcm}(1, \ldots, n) \mid (n-1)!$, in particular $(n-1)! \equiv 0 \pmod{n}$, which completes the proof of Proposition 1. □

In the sequel we consider sequences of integers and contrive a prime-detecting concept. For that purpose we define: A sequence $(x_n)_{n \geq 0}$ of integers is said to be *prime-detecting* if the equivalence $x_n \not\equiv 0 (\bmod n) \Longleftrightarrow n \in \mathbb{P}$ holds for all but finitely many positive integers n. Proposition 1 can be applied to detect primes in a form parallel to the results below based on Apéry's recurrences. Thus we get a very simple primality criterion using a first order recurrence with polynomial coefficients.

Proposition 2. *Let a be a positive integer. We let $p_1, \ldots, p_s$ denote the prime divisors of a. Let*

$$T(x) = x - 1\,,\ \ r_n = \frac{1}{n}\,,$$

$$d_1 = a\,,\ \ d_n = T(n)d_{n-1} \quad (n \geq 2)\,.$$

Then for all integers $n \in \mathbb{N} \setminus \{4\}$ we have

$$d_n \not\equiv 0 \,(\mathrm{mod}\, n) \quad \Longleftrightarrow \quad n \in \mathbb{P} \setminus \{p_1, \ldots, p_s\},$$

and for $p \in \mathbb{P}$

$$d_p \equiv -a \,(\mathrm{mod}\, p).$$

Moreover,

$$d_n \;=\; n! r_n a\,,$$

and, consequently,

$$\lim_{n\to\infty} \frac{d_n}{n! r_n} \;=\; a\,.$$

It is also possible to detect primes by integer solutions of Apéry-type recurrences. In the case of Apéry's recurrence relation (1), we have explicit formulae for X_n involving combinatorial sums. From the arithmetical properties of binomial coefficients (see Eqs. (22)-(25) in Section 2) we can deduce the prime-detecting property of the sequences $(X_n)_{n\geq 0}$. The same can be done for sequences satisfying linear recurrence relations connected with $\zeta(2)$ and $\log 2$. For our results we do not need continued fraction expansions of $\zeta(3)$, $\zeta(2)$, and $\log 2$. However, we state them because they are closely related to the linear recurrence formulae.

Theorem 3. *Let a, b be positive integers such that $5a \neq b$. Let $p_1, \ldots, p_s$ denote the prime divisors of $|5a - b|$. Let*

$$T(x) = 34(x-1)^3 + 51(x-1)^2 + 27(x-1) + 5, \qquad U(x) = (x-1)^6,$$

$$d_0 = a, \quad d_1 = b, \quad d_n = T(n)d_{n-1} - U(n)d_{n-2} \quad (n \geq 2).$$

Then for all integers $n \in \mathbb{N}$ we have

$$d_n \not\equiv 0 \,(\mathrm{mod}\, n) \quad \Longleftrightarrow \quad n \in \mathbb{P} \setminus \{p_1, \ldots, p_s\}, \tag{7}$$

and for $p \in \mathbb{P}$

$$d_p \;\equiv\; 5a - b \;(\mathrm{mod}\, p).$$

Moreover,

$$d_n = n!^3 \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 \left(a + \frac{b-5a}{6}\, c_{n,k} \right) \quad (n \geq 0), \tag{8}$$

where $c_{n,k}$ is defined in Eq. (3), and

$$\lim_{n\to\infty} \frac{d_n}{n!^3 a_n} = a + \frac{b-5a}{6}\, \zeta(3).$$

R. Apéry [1] also proved the irrationality of $\zeta(2)$ using

$$a'_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}, \qquad b'_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k} c'_{n,k} \tag{9}$$

where

$$c'_{n,k} = 2\sum_{m=1}^{n} \frac{(-1)^{m-1}}{m^2} + \sum_{m=1}^{k} \frac{(-1)^{n+m-1}}{m^2 \binom{n}{m}\binom{n+m}{m}} \qquad (1 \le k \le n). \tag{10}$$

Both $X_n = a'_n$ and $X_n = b'_n$ satisfy the recurrence formula

$$(n+1)^2 X_{n+1} - (11n^2 + 11n + 3)X_n - n^2 X_{n-1} = 0. \tag{11}$$

Here, we have

$$\zeta(2) = \cfrac{5}{3 + \cfrac{1^4}{25 + \cfrac{2^4}{69 + \ddots \quad + \cfrac{n^4}{11n^2 + 11n + 3 \ + \cdots}}}}.$$

Using the coefficient polynomials of the recurrence formula (11), we get

Theorem 4. *Let a, b be positive integers such that $3a \neq b$. Let $p_1, \ldots, p_s$ denote the prime divisors of $|3a - b|$. Let*

$$T(x) = 11(x-1)^2 + 11(x-1) + 3, \qquad U(x) = (x-1)^4,$$

$$d_0 = a, \quad d_1 = b, \quad d_n = T(n)d_{n-1} + U(n)d_{n-2} \quad (n \ge 2).$$

Then for all integers $n \in \mathbb{N}$ we have

$$d_n \not\equiv 0 \pmod{n} \quad \Longleftrightarrow \quad n \in \mathbb{P} \setminus \{p_1, \ldots, p_s\},$$

and for $p \in \mathbb{P}$

$$d_p \equiv b - 3a \pmod{p}.$$

Moreover,

$$d_n = n!^2 \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k} \left(a + \frac{b-3a}{5} c'_{n,k} \right) \quad (n \ge 0),$$

where $c'_{n,k}$ is defined in (10), and

$$\lim_{n\to\infty} \frac{d_n}{n!^2 a'_n} = a + \frac{b-3a}{5} \zeta(2).$$

Theorems 3 and 4 are based on recurrence relations given by Apéry in [1]. Now we consider the recurrence formula

$$(n+1)X_{n+1} - 3(2n+1)X_n + nX_{n-1} = 0, \tag{12}$$

which is satisfied by $X_n = a''_n$ and $X_n = b''_n$ with

$$a''_n = \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k}, \quad b''_n = \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k} c_k, \tag{13}$$

where

$$c_k = \sum_{m=1}^{k} \frac{(-1)^{m-1}}{m} \quad (1 \le k \le n). \tag{14}$$

We prove this result in Section 2 below. From (12) we have the continued fraction expansion

$$\log 2 = \cfrac{2}{3 - \cfrac{1^2}{9 - \cfrac{2^2}{15 - \ddots - \cfrac{n^2}{3(2n+1) - \cdots}}}}.$$

Theorem 5. *Let a, b be positive integers such that $3a \neq b$. Let $p_1, \ldots, p_s$ denote the prime divisors of $|3a - b|$. Let*

$$T(x) = 3(2x-1), \qquad U(x) = (x-1)^2,$$

$$d_0 = a, \qquad d_1 = b, \qquad d_n = T(n)d_{n-1} - U(n)d_{n-2} \quad (n \ge 2).$$

Then for all integers $n \in \mathbb{N} \setminus \{4\}$ we have

$$d_n \not\equiv 0 \pmod{n} \quad \Longleftrightarrow \quad n \in \mathbb{P} \setminus \{p_1, \ldots, p_s\},$$

and for $p \in \mathbb{P}$

$$d_p \equiv 3a - b \pmod{p}.$$

Moreover,

$$d_n = n! \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k}\left(a + \frac{b-3a}{2} c_k\right),$$

where c_k is defined in Eq. (14), and

$$\lim_{n\to\infty} \frac{d_n}{n! a''_n} = a + \frac{b-3a}{2} \log 2.$$

Remark: For $n = 4$ one has $d_4 = 2670b - 306a$, which is not divisible by 4 if and only if $a \not\equiv b \pmod{2}$.

2 Proof of Theorems 3 and 4.

Proof of Theorem 3: First we prove the explicit expression (8) of d_n. From p_n and q_n defined in (5) and their common recurrence formula (6), we see that both $Y_n = q_n$ and $Y_n = p_n$ satisfy the recurrence relation

$$Y_n = T(n)Y_{n-1} - U(n)Y_{n-2} \quad (n \geq 2). \tag{15}$$

Obviously, for any real α and β, $Y_n = \alpha q_n + \beta p_n$ satisfy the relation (15) too. Now we compute α and β according to initial conditions of the sequence $(d_n)_{n\geq 0}$:

$$\begin{array}{rrcl} d_0 = & a = & \alpha q_0 + \beta p_0 & = \alpha, \\ d_1 = & b = & \alpha q_1 + \beta p_1 & = 5\alpha + 6\beta. \end{array}$$

Then $d_n = \alpha q_n + \beta p_n$ are solutions of (15). The system of equations has a unique solution: $\alpha = a$ and $\beta = (b-5a)/6$. Thus, expressing p_n, q_n by Eq. (5) and a_n, b_n by Eq. (2), we have the explicit formula (8) for d_n. Dividing this identity by $n!^3 a_n$ and using $b_n/a_n \to \zeta(3)$, we find the limit $a + (b-5a)\zeta(3)/6$ of the sequence $(d_n/n!^3 a_n)_{n\geq 0}$.

We use the formula (8) for d_n. Observing that $n^3 \big| n!^3$ and

$$\frac{n!^3}{m^3} \equiv 0 \,(\mathrm{mod}\, n) \qquad (1 \leq m \leq n-1),$$

we get

$$\begin{aligned} 12d_n &\equiv 2(b-5a)n!^3 \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 c_{n,k} \quad (\mathrm{mod}\, n) \\ &= 2(b-5a)n!^3 \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 \sum_{m=1}^{n} \frac{1}{m^3} \\ &\quad + 2(b-5a)n!^3 \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 \sum_{m=1}^{k} \frac{(-1)^{m-1}}{2m^3 \binom{n}{m}\binom{n+m}{m}} \\ &\equiv 2(b-5a)n!^3 \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 \frac{1}{n^3} \end{aligned}$$

$$+ \frac{(b-5a)n!^3}{\mathrm{lcm}^3(1,\ldots,n)} \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k} \sum_{m=1}^{k} \frac{(-1)^{m-1}\mathrm{lcm}^3(1,\ldots,n)\binom{n+k}{k}}{m^3 \binom{n}{m}\binom{n+m}{m}} \quad (\mathrm{mod}\, n). \tag{16}$$

It follows from the proof of Proposition 3 in [4] that

$$\frac{\mathrm{lcm}^3(1,\ldots,n)\binom{n+k}{k}}{m^3 \binom{n}{m}\binom{n+m}{m}} \in \mathbb{N} \quad (1 \leq m \leq k \leq n). \tag{17}$$

Case 1: $n \notin \mathbb{P}$.
There is nothing to show for $n = 1$. Moreover, Theorem 3 is also true for $n = 4$ and $n = 6$, since

$$\begin{aligned} d_4 &= 91397560b - 781976a &&\equiv 0 \pmod 4, \\ d_6 &= 1604788039632960b - 13730188564800a &&\equiv 0 \pmod 6. \end{aligned}$$

Now let $n \notin \mathbb{P} \cup \{1, 4, 6\}$. In particular, we have $n \geq 8$. Then, using

$$\begin{aligned} \frac{n!^3}{n^3} &= (n-1)! \cdot (n-1)!^2, \\ (n-1)! &\equiv 0 \,(\mathrm{mod}\, n) \quad (\text{for } n \neq 4 \text{ by Proposition 1}), \\ (n-1)!^2 &\equiv 0 \,(\mathrm{mod}\, 12) \quad (n \geq 4), \end{aligned}$$

we can simplify (16) as follows:

$$12d_n \equiv \frac{(b-5a)n!^3}{\mathrm{lcm}^3(1,\ldots,n)} \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k} \sum_{m=1}^{k} \frac{(-1)^{m-1}\mathrm{lcm}^3(1,\ldots,n)\binom{n+k}{k}}{m^3\binom{n}{m}\binom{n+m}{m}} \,(\mathrm{mod}\, n).$$

Thus $d_n \equiv 0 \,(\mathrm{mod}\, n)$ follows from (17) and

$$\frac{n!^3}{\mathrm{lcm}^3(1,\ldots,n)} \equiv 0 \ (\mathrm{mod}\ 12n) \quad (n \notin \mathbb{P}, \ n \geq 6). \tag{18}$$

For (18) it suffices to prove the two congruences

$$\frac{n!}{\mathrm{lcm}\,(1,\ldots,n)} \equiv 0 \ (\mathrm{mod}\ 12) \quad (n \geq 6), \tag{19}$$

$$\frac{n!}{\mathrm{lcm}\,(1,\ldots,n)} \equiv 0 \ (\mathrm{mod}\ n) \quad (n \notin \mathbb{P}, \ n \geq 6). \tag{20}$$

Both congruences (19) and (20) hold for $n = 6, 7$ and $n = 6$, respectively. In the sequel we assume that $n \geq 8$ and $n \notin \mathbb{P}$. First, we observe for $1 \leq m \leq n$ that

$$\mathrm{lcm}\,(1,\ldots,n) \,\Big|\, \mathrm{lcm}\,(1,\ldots,m)\mathrm{lcm}\,(m+1,\ldots,n) \,\Big|\, m!\,(m+1)(m+2)\ldots n = n!. \tag{21}$$

Therefore, it follows from $n \geq 8$ that

$$\begin{aligned} \frac{n!}{\mathrm{lcm}\,(1,\ldots,6)\mathrm{lcm}\,(7,8,\ldots,n)} &= \frac{6! \cdot (7 \cdot 8 \cdots n)}{\mathrm{lcm}\,(1,\ldots,6)\mathrm{lcm}\,(7,8,\ldots,n)} \\ &= 12 \cdot \frac{7 \cdot 8 \cdots n}{\mathrm{lcm}\,(7,8,\ldots,n)} \equiv 0 \quad (\mathrm{mod}\ 12), \end{aligned}$$

so that (21) implies (19). The congruence (20) is already shown in the proof of Proposition 1, and therefore one conlusion in (7) of Theorem 3 holds.

Case 2: $n \in \mathbb{P}$.
For $n = p \in \{2, 3\}$ we have

$$\begin{aligned} d_2 &= 117b - a &\equiv 5a - b \pmod 2, \\ d_3 &= 62531b - 535a &\equiv 5a - b \pmod 3. \end{aligned}$$

In the sequel we assume $p \geq 5$ is a prime. We need some arithmetic properties of binomial coefficients:

$$\binom{p}{k} \not\equiv 0 \,(\mathrm{mod}\, p) \iff k \in \{0, p\}, \tag{22}$$

$$\begin{aligned} \binom{p+k}{k} &= \frac{(p+1)(p+2)\cdot(p+k)}{1\cdot 2\ldots k} \equiv \frac{1\cdot 2\ldots k}{1\cdot 2\ldots k} \pmod p \\ &\equiv 1 \,(\mathrm{mod}\, p) \iff k \in \{0, 1, 2, \ldots, p-1\}, \end{aligned} \tag{23}$$

$$\begin{aligned} \binom{2p}{p} &= \frac{(p+1)(p+2)\ldots(2p)}{1\cdot 2\ldots p} = 2\,\frac{(p+1)(p+2)\ldots(2p-1)}{1\cdot 2\ldots(p-1)} \\ &\equiv 2\,\frac{1\cdot 2\ldots(p-1)}{1\cdot 2\ldots(p-1)} \equiv 2 \pmod p, \end{aligned} \tag{24}$$

$$e_p\left(\binom{p}{k}\right) \in \{0, 1\} \quad (k \in \{0, 1, \ldots, p\}), \tag{25}$$

where $e_p(m)$ is the exponent of p in the decomposition of m. We denote the first term on the right side of (16) by S_1 and compute its residue class modulo p using (22) and (24):

$$\begin{aligned} S_1 &= 2(b-5a)p!^3 \sum_{k=0}^{p} \binom{p}{k}^2 \binom{p+k}{k}^2 \frac{1}{p^3} \\ &= 2(b-5a)(p-1)!^3 \sum_{k=0}^{p} \binom{p}{k}^2 \binom{p+k}{k}^2 \\ &\equiv -2(b-5a) \sum_{k\in\{0\,,\,p\}} \binom{p}{k}^2 \binom{p+k}{k}^2 \pmod p \\ &\equiv -2(b-5a)(1+4) \equiv 10(5a-b) \pmod p. \end{aligned} \tag{26}$$

Next, we treat the second term S_2 on the right side of (16):

$$S_2 = (b-5a)p!^3 \sum_{k=0}^{p} \binom{p}{k}^2 \binom{p+k}{k}^2 \sum_{m=1}^{k} \frac{(-1)^{m-1}}{m^3 \binom{p}{m}\binom{p+m}{m}}. \tag{27}$$

It is convenient to compute the sum of the terms with $1 \le k \le p-1$ separately. By (22), (23), and (25) we have

$$\begin{aligned} e_p\left(\binom{p}{k}^2 \binom{p+k}{k}^2 \right) &= 2, \\ e_p\left(m^3 \binom{p}{m} \binom{p+m}{m} \right) &= 1 \qquad (1 \le m \le k). \end{aligned}$$

Hence we get

$$p!^3 \sum_{k=1}^{p-1} \binom{p}{k}^2 \binom{p+k}{k}^2 \sum_{m=1}^{k} \frac{(-1)^{m-1}}{m^3 \binom{p}{m}\binom{p+m}{m}} \equiv 0 \pmod{p}.$$

Then the sum in (27) simplifies to

$$\begin{aligned} S_2 &\equiv (b-5a)p!^3 \sum_{k\in\{0\,,p\}} \binom{p}{k}^2 \binom{p+k}{k}^2 \sum_{m=1}^{k} \frac{(-1)^{m-1}}{m^3 \binom{p}{m}\binom{p+m}{m}} \\ &\equiv 4(b-5a)p!^3 \sum_{m=1}^{p} \frac{(-1)^{m-1}}{m^3 \binom{p}{m}\binom{p+m}{m}} \pmod{p}. \end{aligned}$$

It follows from Eqs. (22), (23) and (24) that

$$p!^3 \sum_{m=1}^{p} \frac{(-1)^{m-1}}{m^3 \binom{p}{m}\binom{p+m}{m}} = \frac{p!^3}{p^3\binom{2p}{p}} + p!^3 \sum_{m=1}^{p-1} \frac{(-1)^{m-1}}{m^3 \binom{p}{m}\binom{p+m}{m}} \equiv \frac{p!^3}{2p^3} \pmod{p},$$

which yields

$$S_2 \equiv (b-5a)\,\frac{4p!^3}{2p^3} = 2(b-5a)(p-1)!^3 \equiv 2(5a-b) \pmod{p}. \tag{28}$$

The congruences (26) and (28) for S_1 and S_2 give

$$12d_p \equiv S_1 + S_2 \equiv 10(5a-b) + 2(5a-b) = 12(5a-b) \pmod{p} \quad (p \ge 5).$$

Since $p \ge 5$ we have $d_p \equiv 5a - b \pmod{p}$. This completes the proof. □

Proof of Theorem 4: Putting

$$q'_0 = 1, \quad q'_n = (n!)^2 a'_n \quad (n \ge 1), \quad p'_0 = 0, \quad p'_n = (n!)^2 b'_n \quad (n \ge 1)$$

with a'_n and b'_n defined in (9), both sequences, $(q'_n)_{n\geq 0}$ and $(p'_n)_{n\geq 0}$, satisfy the recurrence formula

$$X_n = T(n)X_{n-1} + U(n)X_{n-2} \quad (n \geq 2), \tag{29}$$

where $T(n) = 11(n-1)^2 + 11(n-1) + 3$ and $U(n) = (n-1)^4$. For any real α and β, $Y_n = \alpha q'_n + \beta p'_n$ satisfy the relation (29) too. Again we compute α and β according to the initial conditions of the sequence $(d_n)_{n\geq 0}$:

$$\begin{array}{rrrl} d_0 = & a = & \alpha q'_0 + \beta p'_0 & = \alpha, \\ d_1 = & b = & \alpha q'_1 + \beta p'_1 & = 3\alpha + 5\beta. \end{array}$$

Then $d_n = \alpha q'_n + \beta p'_n$ are solutions of (29). The system of equations has a unique solution: $\alpha = a$ and $\beta = (b-3a)/5$. Thus, expressing a'_n and b'_n by (9), we get the formula for d_n. The limit of the sequence $(d_n/n!^2 a'_n)_{n\geq 0}$ can be computed using the explicit formula of d_n and $b'_n/a'_n \to \zeta(2)$.

In what follows we use exactly the same arguments as in the proof of Theorem 3. Using the explicit formula for d_n we get

$$\begin{aligned} 5d_n &\equiv (b-3a)n!^2 \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k} c'_{n,k} \pmod{n} \\ &= 2(b-3a)n!^2 \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k} \sum_{m=1}^{n} \frac{(-1)^{m-1}}{m^2} \\ &\quad + (b-3a)n!^2 \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k} \sum_{m=1}^{k} \frac{(-1)^{n+m-1}}{m^2 \binom{n}{m}\binom{n+m}{m}} \\ &\equiv 2(b-3a)n!^2 \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k} \frac{(-1)^{n-1}}{n^2} \\ &\quad + \frac{(b-3a)n!^2}{\operatorname{lcm}^2(1,\ldots,n)} \sum_{k=0}^{n} \binom{n}{k}^2 \sum_{m=1}^{k} \frac{(-1)^{n+m-1}\operatorname{lcm}^2(1,\ldots,n)\binom{n+k}{k}}{m^2\binom{n}{m}\binom{n+m}{m}} \pmod{n} \\ &=: S_1 + S_2. \end{aligned}$$

For $n \notin \mathbb{P}$ we proceed as in the proof of Theorem 3. Next, we treat the case $n \in \mathbb{P}$. For $n = p \in \{2,3,5\}$ we have

$$\begin{array}{rcrcl} d_2 &=& 25b + a &\equiv& b - 3a \pmod{2}, \\ d_3 &=& 1741b + 69a &\equiv& b - 3a \pmod{3}, \\ d_5 &=& 53310076b + 2112972a &\equiv& b - 3a \pmod{5}. \end{array}$$

Now we compute the residue classes of S_1 and S_2 modulo p for $n = p \in \mathbb{P} \setminus \{2,3,5\}$. For S_1

we get by (22) and (24):

$$
\begin{aligned}
S_1 &= 2(b-3a)p!^2\sum_{k=0}^{p}\binom{p}{k}^2\binom{p+k}{k}\frac{(-1)^{p-1}}{p^2}\\
&= 2(b-3a)(p-1)!^2\sum_{k=0}^{p}\binom{p}{k}^2\binom{p+k}{k}\\
&\equiv 2(b-3a)\sum_{k\in\{0\,,\,p\}}\binom{p}{k}^2\binom{p+k}{k}\\
&\equiv 2(b-3a)(1+2)\equiv 6(b-3a)\pmod p.
\end{aligned}\tag{30}
$$

Before treating S_2 we observe for $1\le k\le p-1$ that

$$
\begin{aligned}
e_p\left(\binom{p}{k}^2\binom{p+k}{k}\right) &= 2,\\
e_p\left(m^2\binom{p}{m}\binom{p+m}{m}\right) &= 1\quad(1\le m\le k).
\end{aligned}
$$

Then we get

$$
\begin{aligned}
S_2 &\equiv (b-3a)p!^2\sum_{k\in\{0\,,\,p\}}\binom{p}{k}^2\binom{p+k}{k}\sum_{m=1}^{k}\frac{(-1)^{p+m-1}}{m^2\binom{p}{m}\binom{p+m}{m}}\\
&\equiv 2(b-3a)p!^2\sum_{m=1}^{p}\frac{(-1)^m}{m^2\binom{p}{m}\binom{p+m}{m}}\\
&\equiv (b-3a)\,\frac{-2p!^2}{2p^2}=-(b-3a)(p-1)!^2\equiv-(b-3a)\pmod p.
\end{aligned}
$$

This together with Eq. (30) yields

$$
5d_p\equiv S_1+S_2\equiv 6(b-3a)-(b-3a)=5(b-3a)\pmod p.
$$

By $p\ge 7$ we have $d_p\equiv b-3a\pmod p$. This completes the proof. □

3 On a linear recurrence sequence for $\log 2$.

In this section we first prove that the sequences $(a''_n)_{n\ge 0}$ and $(b''_n)_{n\ge 0}$ satisfy the relation (12). First, we consider $(a''_n)_{n\ge 0}$. Let

$$
\lambda_{n,k}=\binom{n}{k}\binom{n+k}{k},\ \ A_{n,k}=-(4n+2)\lambda_{n,k}\quad(0\le k\le n),
$$

and $A_{n,n+1} = A_{n,-1} = 0$ for $n \geq 0$. Note that $\binom{n}{k} = 0$ for $k < 0$ or $k > n$. Using

$$\frac{\lambda_{n,k-1}}{\lambda_{n,k}} = \frac{k^2}{(n+k)(n-k+1)}\,, \quad \frac{\lambda_{n+1,k}}{\lambda_{n,k}} = \frac{n+k-1}{n-k+1}\,, \quad \frac{\lambda_{n-1,k}}{\lambda_{n,k}} = \frac{n-k}{n+k} \quad (1 \leq k \leq n),$$

we have

$$A_{n,k} - A_{n,k-1} = (n+1)\lambda_{n+1,k} - 3(2n+1)\lambda_{n,k} + n\lambda_{n-1,k} \quad (1 \leq k \leq n). \tag{31}$$

Therefore, we get

$$0 = A_{n,n+1} - A_{n,-1} = \sum_{k=0}^{n+1}(A_{n,k} - A_{n,k-1}) = (n+1)a''_{n+1} - 3(2n+1)a''_n + na''_{n-1},$$

which proves that $(a''_n)_{n\geq 0}$ satisfies (12).

Next, we prove that $(b''_n)_{n\geq 0}$ satisfies the relation (12). Let

$$S_{n,k} = (n+1)\lambda_{n+1,k}c_k - 3(2n+1)\lambda_{n,k}c_k + n\lambda_{n-1,k}c_k \quad (1 \leq k \leq n),$$

$B_{n,k} = A_{n,k}c_k$ for $0 \leq k \leq n$, and $B_{n,n+1} = B_{n,-1} = 0$ for $n \geq 0$. By (31) we have

$$\begin{aligned} B_{n,k} - B_{n,k-1} &= (A_{n,k} - A_{n,k-1})c_k + A_{n,k-1}(c_k - c_{k-1}) \\ &= S_{n,k} + A_{n,k-1}\frac{(-1)^{k-1}}{k} \quad (1 \leq k \leq n). \end{aligned}$$

This yields

$$\begin{aligned} 0 &= B_{n,n+1} - B_{n,-1} = \sum_{k=0}^{n+1}(B_{n,k} - B_{n,k-1}) \\ &= \sum_{k=0}^{n+1} S_{n,k} - (4n+2)\sum_{k=1}^{n+1}\binom{n}{k-1}\binom{n+k-1}{k-1}\frac{(-1)^{k-1}}{k} \\ &= (n+1)b''_{n+1} - 3(2n+1)b''_n + nb''_{n-1} - (4n+2)\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}\frac{(-1)^k}{k+1} \\ &= (n+1)b''_{n+1} - 3(2n+1)b''_n + nb''_{n-1} \quad (n \geq 1), \end{aligned}$$

since, by Vandermonde's theorem on the hypergeometric series ${}_2F_1(a,b;c;x)$,

$$\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}\frac{(-1)^k}{k+1} = {}_2F_1\big(n+1,-n;2;1\big) = \frac{(1-n)_n}{(2)_n} = 0\ .$$

It remains to show that $\lim_{n\to\infty} b''_n/a''_n = \log 2$. For this purpose we shall apply a theorem of O. Toeplitz concerning linear series transformations (cf. [8, p. 10, no. 66], [10]). From Toeplitz's result we have

$$\lim_{n\to\infty}\frac{\binom{n}{\nu}\binom{n+\nu}{\nu}}{\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}} = 0 \ (\nu \geq 0) \quad \iff \quad \lim_{n\to\infty}\frac{b''_n}{a''_n} = \log 2.$$

The limit on the left-hand side follows from the inequality

$$\frac{\binom{n}{\nu}\binom{n+\nu}{\nu}}{\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}} \le \frac{\binom{n}{\nu}\binom{n+\nu}{\nu}}{\binom{n}{\nu}\binom{n+\nu}{\nu}+\binom{n}{\nu+1}\binom{n+\nu+1}{\nu+1}} = \frac{1}{1+\frac{(n+\nu+1)(n-\nu)}{(\nu+1)^2}},$$

in which we choose $n > \nu$. Theorem 5 can be proven in the same way as it was done for Theorems 3, 4 in Section 2.

4 Concluding remarks.

We complete the above results by a short summary of known prime-detecting methods. First, by Proposition 1 or Wilson's theorem, it is clear that $(\Gamma(n))_{n\geq 1}$ is a prime-detecting sequence formed by the Gamma function.

1. Detecting primes by polynomials. Legendre showed that there is no rational algebraic function which takes always primes. However, polynomials in many variables with integer coefficients are known whose positive values are exactly the prime numbers obtained as the variables run through all nonnegative integers, [9, p. 158]. The background of this result is given by the fact that the set of primes can be described by diophantine equations.

2. Detecting primes by binomial coefficients. Deutsch [5] has proven the following result for all integers $n \geq 2$:

$$\binom{n-1}{k} \equiv (-1)^k \pmod{n} \quad (0 \le k \le n-1) \quad \Longleftrightarrow \quad n \in \mathbb{P}.$$

3. Detecting primes by Dirichlet series. Prime-detecting sequences can be constructed from Dirichlet series. Let $s \geq 2$ be an integer, let $(a_m)_{m\geq 1}$ be a sequence of integers such that $a_m = O(m^{s-1-\varepsilon})$ for any $\varepsilon > 0$ as $m \to \infty$. Then the Dirichlet series $\sum_{m=1}^{\infty} a_m/m^s$ converges. Assume the weak condition that a_p does not vanish for primes p. Then the sequence $(x_n)_{n\geq 1}$ defined by $x_n = (n!)^s \sum_{m=1}^{n} a_m/m^s$ is prime-detecting since for $n \in \mathbb{N} \setminus \{4\}$ we have by Proposition 1 that

$$\begin{aligned} x_n &= n^s \sum_{m=1}^{n-1} a_n \left(\frac{(n-1)!}{m}\right)^s + a_n\big((n-1)!\big)^s \\ &\equiv a_n\big((n-1)!\big)^s \equiv \begin{cases} (-1)^s a_n (\operatorname{mod} n), & \text{if } n \in \mathbb{P}; \\ 0 \quad (\operatorname{mod} n), & \text{if } n \notin \mathbb{P}. \end{cases} \end{aligned}$$

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