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On Some Properties of Two Simultaneous Polygonal Sequences

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Abstract

This paper focuses on the properties of integers that are simultaneously representable as both m-gonal and n-gonal numbers. Employing hyperbolic geometry, we ascertain both lower and upper bounds on these numbers when there is a finite set of these integers. We then consider the fundamental class of solutions to a generalized Pell-Diophantine equation that relates to these integers, and derive a fast algorithm that can be used to generate them.

1 Introduction

Polygonal numbers can be represented by a group of dots in arithmetic progression. The r-th m-gonal number can be expressed as

$$f_r^m = \frac{(m-2)r^2 - (m-4)r}{2}. \tag{1}$$

For example, the triangular numbers are $1, 3, 6, 10, \ldots$, A000217 and the square numbers are $1, 4, 9, 16, \ldots$, A000290. Geometrical language for these numbers is justified in Fig. 1.

Let $f^{(m,n)}$ be the set of numbers that are simultaneously representable as both m-gonal and n-gonal numbers, where $m, n \in \mathbb{N}$, and let $f_k^{(m,n)}$, be the k-th such number. In other words, there exists integers r and s such that $f_k^{(m,n)} = f_r^m \cap f_s^n$. Note that m and n are symmetric and henceforth we assume, without loss of generality, that $n > m$.

Much of the historical background of polygonal numbers can be found in books [4, 8] and the relationship of these numbers was studied in various articles [1, 5, 10]. In particular, the equal values of three simultaneous polygonal numbers and some Pell equations arising from these numbers were studied [9]. In addition, the proof for the infinitude of $f^{(m,n)}$ can be

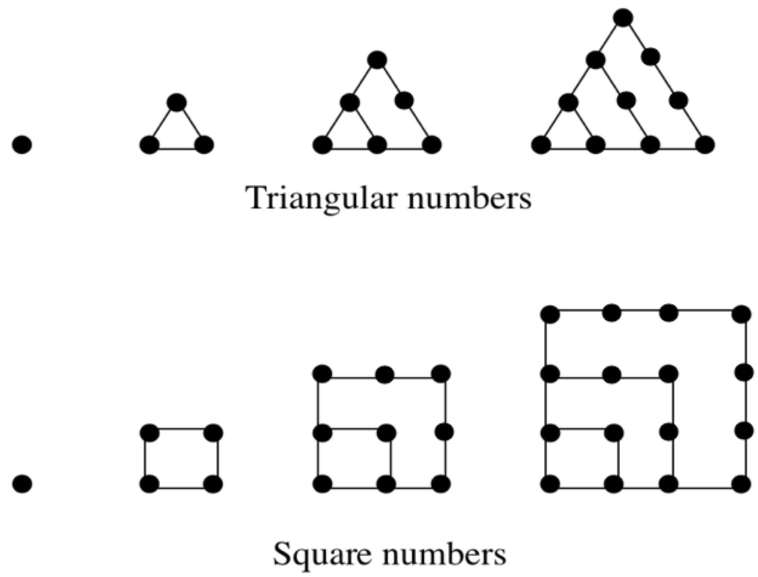

Figure 1: Geometrical Representation of Polygonal Numbers

found in the literature [2, 3, 10], i.e., for $(m-2)(n-2)$ not a perfect square, except when $(m,n) = (3,6)$, $\left|f^{(m,n)}\right| = \infty$, where $|\quad|$ represents cardinality.

The first part of this paper focuses on the case where $\left|f^{(m,n)}\right|$ is finite and estimates the cardinality of such integers, employing the techniques from hyperbolic geometry and the theories from Pell-Diophantine equations, i.e.,

$$x^2 - Dy^2 = \Omega \tag{2}$$

with $D = (m-2)(n-2)$ and $\Omega = (m-2)^2(n-4)^2 - D(m-4)^2$, arriving at

$$1 \leq \left|f^{(m,n)}\right| \leq \lfloor \frac{d(\Omega)}{2} \rfloor - 1$$

where $\Omega = (m-2)^2(n-4)^2 - (m-2)(n-2)(m-4)^2$, $d(\Omega)$ signifies the number of divisors of Ω, and $\lfloor\quad\rfloor$ is the greatest integer function.

The second part of this paper begins with a preliminary discussion on the theory of second-degree Diophantine equations in $\mathbb{Q}(\sqrt{D})$. We define our arsenal in the theory and examine the fundamental class of solutions of (2) as Chu recently did [3]. More importantly, we improve the lower bound in the solution bounds given by Nagell [11] regarding the solutions of a generalized Pell's equation. The improved bounds lead to the systematic determination of solutions under two special circumstances, i.e., (a) $f^{(m,m+2)}$ and (b) $f^{(m,m+1)}$, where $m \geq 5$. The result is a formula for directly ascertaining $f_k^{(m,n)}$, where $k = 1, 2, 3, \cdots$ and $n = m+1$ or $n = m+2$. We establish a few corollaries to illustrate the application of this algorithm. This systematic approach can be readily extended to other algebraic cases.

2 When $\left|f^{(m,n)}\right|$ is Finite

Chu [2] and Lucas [10] considered the case when $\left|f^{(m,n)}\right| = \infty$. That is, for $\left|f^{(m,n)}\right|$ to be otherwise finite, we must have $(m-2)(n-2)$ be a perfect square, and $(m,n) \neq (3,6)$.

Using (1) and the fact for every k, $\exists r, s \in \mathbb{N}$, such that $f_k^{(m,n)} = f_r^m \cap f_s^n$, we obtain the equation

$$f_k^{(m,n)} = f_r^m \cap f_s^n = \frac{(m-2)r^2 - (m-4)r}{2} = \frac{(n-2)s^2 - (n-4)s}{2} \tag{3}$$

or, with a little bit of rearrangement,

$$s = \frac{(n-4) \pm \sqrt{(n-4)^2 + 4r(n-2)[(m-2)r - (m-4)]}}{2(n-2)} \tag{4}$$

Since $s \in \mathbb{N}$, the discriminant of the last equation is a square integer. In other words, $\exists q \in \mathbb{N}$, such that

$$(n-4)^2 + 4r(n-2)[(m-2)r - (m-4)] = q^2$$

To rearrange (4) into a Pell equation of the form in (2), we multiply the above equation pairwise by $(m-2)^2$, arriving at

$$\begin{cases} x = (m-2)q \\ y = 2r(m-2) - (m-4) \\ D = (m-2)(n-2) \\ \Omega = (m-2)^2(n-4)^2 - (m-4)^2(m-2)(n-2) \end{cases} \tag{5}$$

Since $s \in \mathbb{N}$, we can rewrite (4) into $s = \frac{(n-4) \pm q}{2(n-2)}$, or $q = 2s(n-2) \mp (n-4)$, whereas from (5), $r = \frac{(m-4)+y}{2(m-2)}$. Note that

$$s = \begin{cases} \frac{(n-4)-q}{2(n-2)}, & \text{when} \quad s = 0 \\ \frac{(n-4)+q}{2(n-2)}, & \text{when} \quad s \geq 1 \end{cases} \tag{6}$$

Since $x = (m-2)q = 2s(m-2)(n-2) \mp (m-2)(n-4)$, we can transform s in terms of x into

$$s = \frac{x \pm (m-2)(n-4)}{2(m-2)(n-2)}. \tag{7}$$

Thus, for every $r, s \in \mathbb{N}$, we must have

$$r = \frac{(m-4)+y}{2(m-2)}, \quad \text{when} \quad r = 0, 1, 2, \cdots \tag{8}$$

and

$$s = \begin{cases} \frac{(n-4)-q}{2(n-2)} = \frac{-x+(m-2)(n-4)}{2(m-2)(n-2)}, & \text{when} \quad s = 0 \\ \frac{(n-4)+q}{2(n-2)} = \frac{x+(m-2)(n-4)}{2(m-2)(n-2)}, & \text{when} \quad s \geq 1 \end{cases} \tag{9}$$

In other words,

$$x \equiv \begin{cases} +(m-2)(n-4)[\text{mod} \quad 2(m-2)(n-2)], & \text{if} \quad s = 0. \\ -(m-2)(n-4)[\text{mod} \quad 2(m-2)(n-2)], & \text{if} \quad s \geq 1. \end{cases} \tag{10}$$

and

$$y \equiv -(m-4)[\text{mod} \quad 2(m-2)], \quad r \geq 0. \tag{11}$$

We note that (3) has solutions when either $(r_0, s_0) = (0,0)$ or $(r_1, s_1) = (1,1)$, or, in other words

$$(r,s) = \begin{cases} (0,0), & \begin{array}{l} x_0 = +(m-2)(n-4) \\ y_0 = -(m-4) \end{array} \\ (1,1), & \begin{array}{l} x_1 = n(m-2) \\ y_1 = m \end{array} \end{cases} \tag{12}$$

We now proceed to estimate bounds on the cardinality of $f^{(m,n)}$.

Theorem 2.1. *There are only finitely many numbers* $f_k^{(m,n)}$ *for* $(m-2)(n-2)$ *a perfect square except when* $(m,n) = (3,6)$. *Lower and upper bounds for* $\left|f^{(m,n)}\right|$ *are given by* $1 \leq \left|f^{(m,n)}\right| \leq \lfloor \frac{d(\Omega)}{2} \rfloor - 1$, *where* $d(\Omega)$ *signifies the number of divisors of* Ω.

Proof. We rewrite (2) into $x^2 - g^2y^2 = \Omega$, where $D = (m-2)(n-2) = g^2$ and $g \in \mathbb{Z}$ with the requirements set forth in (5):

$$x^2 - \gamma^2 = \Omega \tag{13}$$

where $\gamma = gy = \pm\sqrt{x^2 - \Omega}$. Eq. (13) can also be rewritten as $\frac{x^2}{\Omega} - \frac{\gamma^2}{\Omega} = 1$, the equation of a hyperbola with asymptotes

$$\gamma = \pm\frac{\sqrt{\Omega}}{\sqrt{\Omega}}x = \pm x$$

and vertices

$$(\pm\sqrt{\Omega}, 0).$$

The geometric language for the above is justified in Fig. 2. It shows that all the solutions are symmetric over the real axis. This permits us to be concerned only with positive x and γ in the first quadrant.

We denote d as the vertical distance from the asymptote $\gamma = x$ to the curve $\gamma = \sqrt{x^2 - \Omega}$. Thus

$$d = x - \sqrt{x^2 - \Omega}. \tag{14}$$

The trivial solution when $x = y = 0$ is eliminated. As seen from Fig. 2, $\sqrt{\Omega}$ cannot be an integer otherwise $\gamma = y = 0$. Hence

$$\max(d) = \lfloor\sqrt{\Omega}\rfloor + 1 - \sqrt{(\lfloor\sqrt{\Omega}\rfloor + 1)^2 - \Omega}, \qquad \min(d) = 1.$$

where $\sqrt{\Omega} \in \mathbb{R}$. Therefore

$$1 \leq d \leq \lfloor\sqrt{\Omega}\rfloor + 1 - \sqrt{(\lfloor\sqrt{\Omega}\rfloor + 1)^2 - \Omega} = \Phi. \tag{15}$$

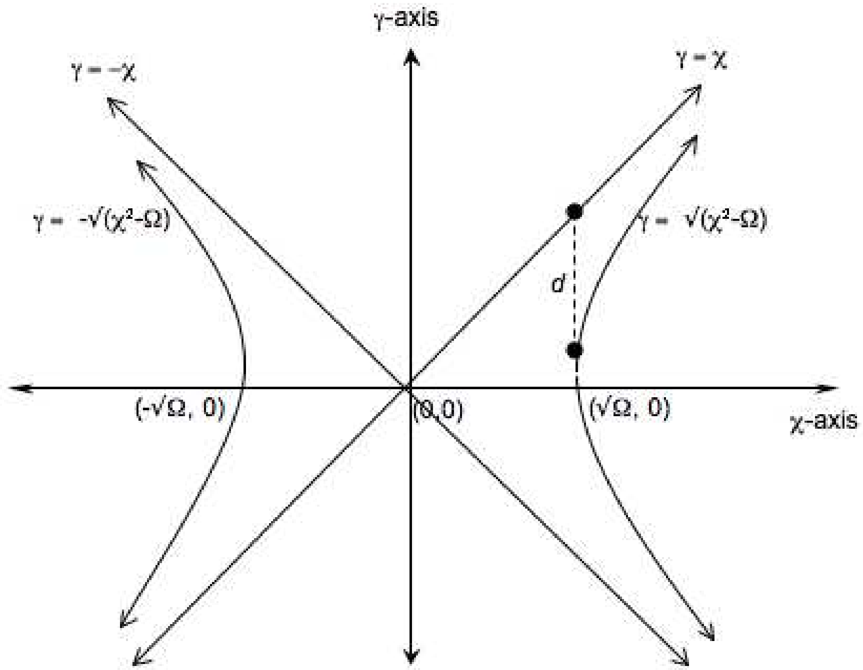

Figure 2: An Equation of Hyperbola: $x^2 - \gamma^2 = \Omega$

From (14) we see that $x \in \mathbb{N}$ hence $\sqrt{x^2 - \Omega} \in \mathbb{N}$, leading to integral solutions (x, γ). Solving for x in (14), we get

$$x = \frac{\Omega + d^2}{2d}. \tag{16}$$

Similarly

$$y = \sqrt{\frac{x^2 - \Omega}{g^2}} = \frac{\Omega - d^2}{2dg}.$$

It immediately follows from (16) that $x \in \mathbb{N}$, $2d\,|\Omega + d^2,$ and $\Omega \equiv -d^2 \pmod{2d}$, hence $d\,|\Omega$. On the other hand, it is apparent that $\Omega = (m-2)^2(n-4)^2 - (m-4)^2(m-2)(n-2)$ is an even number, therefore (16) implies that d is an even number as well. Hence, the cardinality of d becomes

$$1 \le |d| \le \sum_{d|\Omega\,,2 \le d \le \Phi} 1. \tag{17}$$

Since $(x, y) = (\frac{\Omega+d^2}{2d}, \frac{\Omega-d^2}{2dg})$ for each d satisfying (17),

$$1 \le |(x, y)| \le \sum_{d|\Omega\,,2 \le d \le \Phi} 1,$$

which implies

$$1 \le \left|f^{(m,n)}\right| \le \sum_{d|\Omega\,,2 \le d \le \Phi} 1. \tag{18}$$

To get sharper bounds, we need two lemmas — each of whose proofs is self-evident.

Lemma 2.1. $\lim_{\Omega\to\infty} \Phi = \sqrt{\Omega}$.

Lemma 2.2. *There are exactly* $\lfloor \frac{d(\Omega)}{2} \rfloor - 1$ d*'s such that* $2 \le d \le \sqrt{\Omega}$ *with* $d \,|\Omega$.

By both Lemmas 2.1 and 2.2, it is apparent that

$$\sum_{d|\Omega\,,2\le d\le \Phi} 1 = \lfloor \frac{d(\Omega)}{2} \rfloor - 1.$$

Therefore an analogy to (18) is

$$1 \le \left| f^{(m,n)} \right| \le \lfloor \frac{d(\Omega)}{2} \rfloor - 1. \tag{19}$$

We thereby complete our proof. □

Example 2.2. *In the case of* $f^{(3,11)}$, $\Omega = 40$ *and* $1 \le \left| f^{(3,11)} \right| \le 3$.

Example 2.3. *In the case of* $f^{(27,102)}$, $\Omega = 4680000$ *and* $1 \le \left| f^{(27,102)} \right| \le 209$.

3 Preliminaries - Diophantine Equations of the Second Degree

We define a common language used in describing the properties of the equation (2) expressed in the quadratic field $\mathbb{Q}(\sqrt{D})$. The following definitions are well-known and have appeared in various literature [2, 3, 4, 11]. Both Definitions 3.2 and 3.4 can be proven by finite induction.

Definition 3.1. *Let* D *and* Ω *be two integers. If* $x = u$ *and* $y = v$ *are integers which satisfy (2), we say, for simplicity, that the ordered pair*

$$(u, v) = u + v\sqrt{D}$$

is a solution to (2). In general, we consider all of the solutions (x, y) *of*

$$x^2 - Dy^2 = 1, \qquad \Omega = 1 \tag{20}$$

with $x, y > 0$*. Among these there is a least integer ordered pair* (x', y') *, such that* x' *and* y' *have their least positive values. The ordered pair* (x', y')*, or* $x' + y'\sqrt{D}$*, is called the fundamental solution of (20).*

Definition 3.2. *If* (x', y') *is the fundamental solution of (20), then we may obtain all of the solutions,* (x^*, y^*)*, of (20) through*

$$(x^*, y^*) = (x', y')^n,$$

where $n = 1, 2, 3, \ldots, etc.$

Example 3.3. *The fundamental solution of* $x^2 - 5y^2 = 1$ *is* $(9, 4)$*, so all of the solutions of this equation are given by* $(9, 4)^n$*, where* $n = 1, 2, 3, \ldots,$ *etc.*

Definition 3.4. *There can be multiple classes of solutions to (2). If* (u_1, v_1) *is the fundamental solution that occurs in the class* K *of (2), then we may obtain all of the solutions* (u', v') *belonging to the class* K *through*

$$(u', v') = (u_1, v_1)(x^*, y^*) = (u_1, v_1)(x', y')^n,$$

where (x^*, y^*) *and* $(x', y')^n$ *are as defined in Definition 3.2.*

Example 3.5. *There are only two classes of solutions to the equation* $u^2 - 5v^2 = 4$*.* $(3, \pm 1)$ *are the least integer ordered pairs satisfying the equation. Therefore each of* $(3, \pm 1)$ *is the fundamental solution of the said equation in its own respective class. If we take* $(3, 1)$*, and the fundamental solution (9,4) of* $x^2 - 5y^2 = 1$*, then all of the solutions belonging to that class of (2) are given by (3,1)(9,4)*^{n}*, where* $n = 1, 2, 3, \ldots,$ *etc, whereas all of the solutions belonging to the other class of (2) are given by (3,-1)(9,4)*^{n}*, where* $n = 1, 2, 3, \ldots,$ *etc.*

Definition 3.6. K'=$(u, v)(x, y)^n$ *and* $\overline{K}''$=$(u, -v)(x, y)^n$ *are the solutions of (2) in two distinct classes,* K *and* $\overline{K}$*, respectively, where* $n = 1, 2, 3, \ldots,$ *etc, and* (x, y) *is the fundamental solution of (20) and* (u, v) *of (2). In general, if the solutions* K' *and* $\overline{K}''$ *coincide with each other, or* $K' = \overline{K}''$ *for* $n = 1, 2, 3, \ldots,$ *etc, we call either* K *or* $\overline{K}$ *an ambiguous class. In the case when either* u *or* $v = 0$*, the class that contains this* (u, v) *is an ambiguous class.*

Next we need to introduce an important lemma, which is the generalization of the bounds presented in Nagell's book [11].

Lemma 3.1. *The fundamental solutions,* (u, v)*, of all of the classes of (2) satisfy the bounds*

$$0 \leq v \leq y' \sqrt{\tfrac{\Omega}{2(x'+1)}},$$
$$\sqrt{\Omega} \leq |u| \leq \sqrt{\tfrac{(x'+1)\Omega}{2}}$$

where (x', y') *is the fundamental solution of (20) and* $u = x$ *and* $v = y$ *are integers which satisfy (2).*

Proof. The lower bound on u, given as 0 by Nagell, can be improved to $\sqrt{\Omega}$ since $v \geq 0$. □

4 Class Number Arguments

We begin this section by introducing the idea of mapping and defining the *proper* solutions that satisfy a set of rules involving the number of the classes of solutions to the general Pell equation, (2).

Definition 4.1. *Let A be a set of the integral solutions (u, v) of $u^2 - Dv^2 = \Omega$ and B be a set of the integer solutions (r, s), where $r, s \in \mathbb{N}$, such that we can map A into B. In other words, we denote θ as which assigns to each (u, v) of A an (r, s) of B, i.e., $\theta : A \to B$.*

Note that both r and s are as defined in the aforesaid relationship, *i.e*, $f_k^{(m,n)} = f_r^m \cap f_s^n$.

Definition 4.2. *(u, v) of (2) is a proper solution if:*

i. *(u, v) satisfies the bounds in Lemma 3.1 or, in other words, (u, v) is a fundamental solution pair of a solution class of (2), and*

ii. *(u, v) produces a valid mapping, i.e., $\theta : A \to B$.*

For example, we know from Definition 3.4, $(3, 1)$ is a fundamental solution which constitutes one class of solutions to the equation $x^2 - 5y^2 = 4$, hence satisfying the bounds in Lemma 3.1. We term $(3, 1)$ a *proper solution* because $(3, 1)(9, 4)^n$ also produces a valid mapping $\theta : A \to B$ for some $n \in \mathbb{N}$ and (r, s) of B.

Theorem 4.1. *In most cases, there are 4, and perhaps more proper solutions associated with $f^{(m,n)}$, a set of numbers that are simultaneously representable as both m-gonal and n-gonal numbers, for $m, n \in \mathbb{N}$.*

Proof. From (12) we know that $(x_o, \pm y_o) = [(n - 4)(m - 2), \pm(m - 4)]$ and $(x_1, \pm y_1) = [n(m - 2), \pm m]$ are the solutions to (2). In general, if (a, b) and $(a\prime, b\prime)$ belong to the same class, then it is easy to check that $\Omega \,|a'', b'',$ where $(a'', b'') = (aa' \pm bb'D, ab' \pm ba')$. It is evident that if K is the class consisting of the solutions $(x_i, y_i), i = 1, 2, 3, \ldots,$ and $\overline{K}$, or the conjugate of $K, (x_i, -y_i)$, $i = 1, 2, 3, \ldots,$ will constitute another class. Since the case $(-x, -y)$ collapses onto class K, the same for the case $(-x, y)$, which collapses onto $\overline{K}$. Therefore under necessary and sufficient conditions, $[(n - 4)(m - 2), \pm(m - 4)]$ and $[n(m - 2), \pm m]$ represent four different classes of solutions, though they are not necessarily *fundamental*. We can thus have at least 4 *proper* solutions, i.e., $[(n-4)(m-2), \pm(m-4)]$ or $[n(m-2), \pm m]$. $\square$

Next, we speak of an ambiguous class. We check certain conditions that will put both of the aforesaid (x_o, y_o) and (x_1, y_1) in the same solution class. As Ω in Lemma 3.1 becomes large, we can have a considerable but finite number of fundamental solutions belonging to different classes that satisfy the bounds. We will see that the number of *proper* solutions limits to, in some cases, no more than four.

Theorem 4.2. *There are at least two proper solutions when*

i. *$n = m + 1$ and $n = m + 2$ for $m \geq 5$;*

ii. *when $n - m > 2$, $(3, 7), (3, 8)$ and $(3, 10)$ are the only cases.*

Each of i *and* ii *represents the necessary and sufficient conditions such that (x_o, y_o) and (x_1, y_1) both belong to the same solutions class, i.e., both are ambiguous.*

Proof. From Theorem 4.1, $(x_o, \pm y_o) = [(m-2)(n-4), \pm(m-4)]$ and $(x_1, \pm y_1) = [n(m-2), \pm m]$ can represent at most 4 classes of solutions. We can write $(x_o, \pm y_o)(x', y') = (x_1, \pm y_1)$ as

$$\begin{cases} x_1 & = (m-2)(n-4)x' \pm (m-4)(m-2)(n-2)y' = n(m-2) \\ y_1 & = (m-2)(n-4)y' \pm (m-4)x' = \pm m \end{cases}$$

Multiplying x_1 by $\frac{(m-4)}{(m-2)}$ and y_1 by $(n-4)$ we get

$$\begin{cases} \frac{(m-4)}{(m-2)}x_1 = & (m-4)(n-4)x' \pm (m-4)^2(n-2)y' = n(m-4) \\ (n-4)y_1 = & (m-2)(n-4)^2y' \pm (m-4)(n-4)x' = \pm m(n-4) \end{cases}$$

Note that the signs above are member-wise linked with each other, so by subtracting $\frac{(m-4)}{(m-2)}x_1$ from $(n-4)y_1$, we can only have

$$y'[(m-2)(n-4)^2 - (m-4)^2(n-2)] = \begin{cases} 4(n-m) \\ -2(mn-2m-2n) \end{cases},$$

whereas

$$y'[(m-2)(n-4)^2 + (m-4)^2(n-2)] = \begin{cases} 4(n-m) \\ -2(mn-2m-2n) \end{cases}$$

is impossible. Further simplified,

$$y'(n-m)(mn-2m-2n) = \begin{cases} 4(n-m) \\ -2(mn-2m-2n) \end{cases}$$

and

$$y' = \begin{cases} \frac{4}{mn-2m-2n} \\ \frac{-2}{n-m} \end{cases}.$$

Knowing $n > m$ and $y' \in \mathbb{Z}$, we easily deduce that when $y' = \frac{-2}{n-m}$, $n = m+1$ and $n = m+2$ are the two satisfying cases. In the case when $y' = \frac{4}{mn-2m-2n}$,

$$mn - 2m - 2n = \begin{cases} \pm 1 \\ \pm 2 \\ \pm 4 \end{cases}$$

giving us

$$n = \begin{cases} \frac{2m \pm 1}{m-2} \\ \frac{2m \pm 2}{m-2} \\ \frac{2m \pm 4}{m-2} \end{cases}.$$

A simple calculation yields the only 3 satisfying cases when $n - m > 2$: $(m, n) = (3, 7)$, $(3, 8)$, and $(3, 10)$.

We prove and therefore single out the cases when $(x_o, \pm y_o)$ and $(x_1, \pm y_1)$ belong to the same class, or when $(x_1, \pm y_1)$ is generated from $(x_o, \pm y_o)$. We need to ascertain whether $(x_o, \pm y_o)$ satisfies the bounds in Lemma 3.1. If they do then $(x_o, \pm y_o)$ are the two *proper* solutions, otherwise we must assume there exists at least two other *proper* solutions that will each generate $(x_o, \pm y_o)$. However, we must note that $x_o, y_o \neq 0$, i.e., $[(m-2)(n-4), \pm(m-4)] \neq (0,0)$, otherwise according to Definition 3.6, we will have one less class and the remaining one is ambiguous. The case when this does not occur is $m \geq 5$.

□

We restrict our attention to the two aforesaid special cases, focusing our interest on $m \geq 5$: (a) $f^{(m,m+2)}$ and (b) $f^{(m,m+1)}$. We prove that $(x_o, \pm y_o)$ indeed satisfy the bounds and therefore are the *proper* solutions. The following lemma is useful and readily obtained:

Lemma 4.1. *We designate (x', y') as the fundamental solution of $x^2 - Dy^2 = 1$. Therefore, when $n = m+2$, $(x', y') = (m-1, 1)$, whereas $n = m+1$, $(x', y') = (2m-3, 2)$.*

Theorem 4.3. *In the case of $f^{(m,m+2)}$, when $m \geq 4$, two proper solutions of (2) are*

$$[(\lfloor\sqrt{\frac{\Omega}{2m}}\rfloor + 1)^2, \pm(\lfloor\sqrt{\frac{\Omega}{2m}}\rfloor - 1)]$$

Proof. From Theorem 4.1, $(x_o, \pm y_o) = [(m-2)^2, \pm(m-4)]$ by substitution. We also have

$$\begin{cases} D = m(m-2) \\ \Omega = (m-2)^4 - m(m-2)(m-4)^2 = 2m^3 - 8m^2 + 16 \end{cases} \tag{21}$$

Applying Lemma 3.1, we get

$$\begin{gathered} 0 \leq v \leq \sqrt{\tfrac{\Omega}{2m}} \\ \sqrt{\Omega} \leq |u| \leq \sqrt{\tfrac{m\Omega}{2}} \end{gathered} \tag{22}$$

Considering the upper bound for v, we yield $\sqrt{(m-3)^2} < \sqrt{\frac{\Omega}{2m}} = \sqrt{m^2 - 4m + \frac{8}{m}} < \sqrt{(m-2)^2}$, which means $\lfloor\sqrt{\frac{\Omega}{2m}}\rfloor = m-3$. It is inferred from (22) that

$$0 \leq \lfloor\sqrt{\frac{\Omega}{2m}}\rfloor - 1 = m-4 < \sqrt{\frac{\Omega}{2m}}, \qquad m \geq 4.$$

Because $\lfloor\sqrt{\frac{\Omega}{2m}}\rfloor - 1 = m-4$ satisfies the bounds, we know that it is a fundamental solution of a particular class. Since the mapping $\theta : (x_o, y_o) \to (r_o, s_o)$ is valid, we know that (x_o, y_o) and its conjugate, $(x_o, -y_o)$, are the 2 proper solutions. In other words, $y_o = \lfloor\sqrt{\frac{\Omega}{2m}}\rfloor - 1$. By substitution we get $(x_o, \pm y_o) = [(m-2)^2, \pm(m-4)] = [(\lfloor\sqrt{\frac{\Omega}{2m}}\rfloor + 1)^2, \pm(\lfloor\sqrt{\frac{\Omega}{2m}}\rfloor - 1)]$.

□

In a similar fashion, we also get two *proper* solutions in the case of $f^{(m,m+1)}$. For brevity we omit the proof herein:

Theorem 4.4. *In the case of $f^{(m,m+1)}$, when $m \geq 4$, two proper solutions of (2) are*

$$[(\lfloor\sqrt{\frac{\Omega}{m-1}}\rfloor\lfloor\sqrt{\frac{\Omega}{m-1}}\rfloor+\lfloor\sqrt{\frac{\Omega}{m-1}}\rfloor),\pm(\lfloor\sqrt{\frac{\Omega}{m-1}}\rfloor-1)]$$

We will go on to prove that $(x_o, \pm y_o)$ from Theorem 4.4 are indeed the *only* two proper solutions:

Theorem 4.5. *There are only two proper solutions, i.e., $(x_o, \pm y_o)$, in the case $n = m+1$, as long as $m \geq 5$.*

Proof. By Lemma 4.1, we take $(x', y') = (2m-3, 2)$ as the fundamental solution of $x^2 - Dy^2 = 1$. From Theorem 4.2, we know that for $n = m+1$,

$$(x_o, y_o)(x', y') = (x_1, y_1)$$

or

$$[\underbrace{(m-2)(m-3)}_{x_o},\underbrace{(m-4)}_{y_o}](\underbrace{2m-3}_{x'},\underbrace{2}_{y'}) = [\underbrace{(m+1)(m-2)}_{x_1},\underbrace{m}_{y_1}]$$

For brevity, we say in $\mathbb{Q}[\sqrt{D}]$,

$$(x_1, y_1) \equiv (x_o x' + y_o y' D, x' y_o + x_o y') \bmod \begin{cases} 2(n-2) \\ 2(m-2) \end{cases}. \tag{23}$$

hence by way of both (10) and (11)

$$(x_1, y_1) \equiv [-(m-2)(n-4), -(m-4)] \bmod \begin{cases} 2(n-2) \\ 2(m-2) \end{cases}.$$

Since $y' = 2$ and $D = (m-2)(n-2)$, we deduce (23) to

$$(x_1, y_1) \equiv (x_o x', x' y_o + x_o y') \bmod \begin{cases} 2(n-2) \\ 2(m-2) \end{cases}. \tag{24}$$

We have to prove the non-existence of another proper solution, besides the two from Theorem 4.4, i.e.,

$$(x_o, \pm y_o) = [(\lfloor\sqrt{\frac{\Omega}{m-1}}\rfloor\lfloor\sqrt{\frac{\Omega}{m-1}}\rfloor+\lfloor\sqrt{\frac{\Omega}{m-1}}\rfloor),\pm(\lfloor\sqrt{\frac{\Omega}{m-1}}\rfloor-1)].$$

We will proceed with this by assuming that there exists a proper solution (a, b), such that

$$\theta : (a, b) \to (r, s)$$

is valid, where $0 < a < x_o$, $0 < b < y_o$. We shall also see that

$$\begin{cases} a \not\equiv \pm(m-2)(n-4) \bmod 2(n-2) \\ b \not\equiv -(m-4) \bmod 2(m-2) \end{cases}, \tag{25}$$

otherwise (a,b) will be equal to some $(x,y)\in\mathbb{N}$ by the fact that the map $(x,y)\xrightarrow{\theta}(r,s)$ is injective. So, if $(a_k,b_k)\xrightarrow{\theta}(r,s)$ is valid, (r,s) must be generated by

$$(a_k,b_k)=(a,b)(x',y')^k,\qquad k=1,3,5,\ldots \tag{26}$$

It is clear that k cannot be even, else by putting $k=2j$,

$$(x',y')^{2j}\equiv(1,2jx'y')\bmod\begin{cases}2(n-2)\\2(m-2)\end{cases} \tag{27}$$

and the result of $(a,b)(x',y')^{2j}$ will not satisfy the condition in (25). Putting $k=1$, we have

$$(a,b)(x',y')=(ax'+by'D,bx'+ay')$$

and since $y'=2$ and $D=(m-2)(n-2)$,

$$(a,b)(x',y')=(ax',bx'+ay')\bmod\begin{cases}2(n-2)\\2(m-2)\end{cases}.$$

For $k=1,3,5,\ldots$, we have by way of (27),

$$\begin{aligned}(a_k,b_k)=(a,b)(x',y')(x',y')^{2j}\ &\equiv[ax'+2jxy(bx'+ay')D,bx'+ay'+2ajx'^2y']\\&\equiv[ax',bx'+ay'+2ajx'^2y']\\&\qquad\qquad\bmod\begin{cases}2(n-2)\\2(m-2)\end{cases}.\end{aligned}$$

We have to also assume, for some k, (26) will produce a valid mapping. Thus, from (24) we see that both

$$\begin{cases}a_k\equiv ax'\\x_1\equiv x_ox'\end{cases}\bmod 2(n-2).$$

Because a_k is a solution, so both

$$\begin{cases}a_k\\x_1\end{cases}\equiv-(m-2)(n-4)\bmod 2(n-2).$$

This implies that a and x_o must satisfy the same congruence, but this is in contradiction to what we have assumed, (25). Therefore (a,b) cannot exist, leaving us with the only two *proper* solutions aforementioned, i.e., $(x_o,\pm y_o)$. The same argument, i.e., $m\geq 5$, from Theorem 4.2 applies.

□

Theorem 4.6. *There are only two proper solutions, i.e., $(x_o,\pm y_o)$, in the case $n=m+2$ for m even and ≥ 6.*

Proof. Using the proof method of the last theorem in the case $n = m + 2$, we have

$$[\underbrace{(m-2)^2}_{x_o}, \underbrace{(m-4)}_{y_o}](\underbrace{m-1}_{x'}, \underbrace{1}_{y'}) = [\underbrace{(m+2)(m-2)}_{x_1}, \underbrace{m}_{y_1}].$$

Working in $\mathbb{Q}[\sqrt{D}]$ yields

$$(x_1, y_1) \equiv (x_o x' + y_o y' D, x' y_o + x_o y') \bmod \left\{ \begin{array}{l} 2(n-2) \\ 2(m-2) \end{array} \right. . \tag{28}$$

Therefore if m is an even number, y_o is also even. Providing $y' = 1$ and $D = (m-2)(n-2)$, we can deduce (28) down to

$$(x_1, y_1) \equiv (x_o x', x' y_o + x_o y') \bmod \left\{ \begin{array}{l} 2(n-2) \\ 2(m-2) \end{array} \right. .$$

Thus, we can assume the existence of (a, b) and arrive at a contradiction. Finally, the proof follows just as illustrated above.

□

The following corollaries illustrate the application of the general algorithm provided above.

Corollary 4.1. $\left|f^{(3,5)}\right| = \infty$ *and*

$$f_k^{(3,5)} = \frac{2w + \sqrt{3w^2 + 12w}}{48},$$

where $w = (2+\sqrt{3})^{2k-1} - (2-\sqrt{3})^{2k-1}$.

Example 4.2. $f_1^{(3,5)} = 1, f_2^{(3,5)} = 210,\ f_3^{(3,5)} = 40755, \ldots,$ *A014979*.

Corollary 4.3. $\left|f^{(3,7)}\right| = \infty$ *and*

$$f^{(3,7)} = \frac{S^2 - 9}{40},$$

where $S = 3\sqrt{5}(w + \overline{w}) \pm 5(w - \overline{w})$ *and* $\left\{ \begin{array}{l} w = (9+4\sqrt{5})^{2\lfloor \frac{k+1}{2} \rfloor - 1} \\ \overline{w} = (9-4\sqrt{5})^{2\lfloor \frac{k+1}{2} \rfloor - 1} \end{array} \right.$, $k = 1, 2, 3, \ldots,$ *yielding certain values of* $f^{(3,7)}$.

Example 4.4. $f^{(3,7)} = 1,\ f^{(3,7)} = 55,\ f^{(3,7)} = 121771, \ldots,$ *A046194*.

Corollary 4.5. $\left|f^{(3,6)}\right| = \infty$ *and* $f_k^{(3,6)} = (2k-1)k$.

Example 4.6. $f^{(3,6)} = 1, f^{(3,6)} = 6, f^{(3,6)} = 15, \ldots,$ *A000384*.

By Theorems 4.5 and 4.6 we readily have, without excessive calculations,

Corollary 4.7. $\left|f^{(1993,1994)}\right| = \infty$, *where* $f_1^{(1993,1994)} = 1,\ f_2^{(1993,1994)} = 1004438512832684175 \rightarrow \left\{ \begin{array}{l} r = 31764429 \\ s = 31756455 \end{array} \right. , \ldots$

Corollary 4.8. $\left|f^{(1994,1996)}\right| = \infty$, *where* $f_1^{(1994,1996)} = 1,\ f_2^{(1994,1996)} = 63014073509833425 \rightarrow \left\{ \begin{array}{l} r = 7954065 \\ s = 7950075 \end{array} \right. , \ldots$

5 Conclusion

In this paper we considered the set of integers, $f^{(m,n)}$, that are simultaneously representable as m- and n-gonal numbers, deriving some of the interesting properties about them.

First, we derived bounds on the cardinality of $f^{(m,n)}$ when $(m-2)(n-2)$ is a perfect square except when $(m,n)=(3,6)$. We employed techniques from hyperbolic geometry to bound the number of solutions to (2). It is obvious that in this case we would have no more than $d(\Omega)$ solutions since $x^2 - Dy^2 = (x+D'y)(x-D'y)$, where $D'D' = D$. A sharp upper bound is also obtained: $\lfloor \frac{d(\Omega)}{2} \rfloor - 1$. We may point out that in the case when $\left|f^{(m,n)}\right| = \infty$, or $(m-2)(n-2)$ is not a perfect square except when $(m,n)=(3,6)$, (2) always has infinitely many solutions with $\Omega > 2\sqrt{D}+1$, and therefore D is a quadratic residue of Ω and hence of its divisors.

Second, we examined and singled out the solution classes, or proper solutions, that can be used to generate $f^{(m,n)}$, using the theory of Pell diophantine equations and the improved bounds of [11]. In the cases $n = m+1$ and $n = m+2$, there are only two proper solutions. This fact enabled us to quickly generate $f_k^{(m,m+1)}$ and $f_k^{(m,m+2)}$ where $k = 1, 2, 3, \ldots.$ The approach used in arriving at the two special cases can be readily extended to algebraic cases involving other m and n as well.

Pell-Diophantine equations play an important role in this paper, as well in the field of binary quadratic forms [7]. For example, in the case when we put $(m,n) = (3,7)$, or $\Omega = 4$, (2) of this kind is heavily used in the theory of biquadratic forms [6] and the Lucas test for Mersenne primes [12]. The Pell equation encountered in this paper may have some applications to these fields.

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