Journal of Integer Sequences, Vol. 10 (2007),
Article 07.5.1

Interspersions and Fractal Sequences Associated with Fractions c^j/d^k

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Abstract

Suppose $c \geq 2$ and $d \geq 2$ are integers, and let S be the set of integers $\lfloor c^j/d^k \rfloor$, where j and k range over the nonnegative integers. Assume that c and d are multiplicatively independent; that is, if p and q are integers for which $c^p = d^q$, then $p = q = 0$. The numbers in S form interspersions in various ways. Related fractal sequences and permutations of the set of nonnegative integers are also discussed.

1 Introduction

Throughout this article, the letters $c, d, j, k, p, q, h, m, n$ represent nonnegative integers such that $c \geq 2$ and $d \geq 2$, and c and d are multiplicatively independent; that is, if $c^p = d^q$, then $p = q = 0$.

Definitions, examples, and references for the terms *interspersion* and *fractal sequence* are easily accessible ([9, 10, 11, 7]), so that only a brief summary is given in this introduction. This introduction also presents certain new arrays defined from the manner in which the fractions c^j/d^k are distributed. The main purpose of the article is to prove that each such array is an interspersion.

Definition. An array $A = (a_{mh})$, $m \geq 1, h \geq 1$, of positive integers is an *interspersion* if
(I1) the rows of A partition the positive integers;
(I2) every row of A is an increasing sequence;

(I3) every column of A is an increasing (possibly finite) sequence;

(I4) if (u_h) and (v_h) are distinct rows of A, and p and q are indices for which $u_p < v_q < u_{p+1}$, then

$$u_{p+1} < v_{q+1} < u_{p+2}.$$

Example 1 below illustrates the manner in which property (I4) matches the name "interspersion"; viz., the terms of each row individually separate and are separated by the terms of all other rows (after initial terms).

Definition of the array $T_{(c,d,k_0)} = \{t(m,h)\}$. Row 1 is defined by $t(1,h) = c^{h-1}$, for $h = 1, 2, \ldots$. For $m \geq 2$, the first term $t(m,1)$ of row m is the least positive integer

$$\lfloor c^j/d^k \rfloor, \quad \text{where } k \geq k_0,$$

that is not in rows $1, 2, ..., m-1$. In order to define the rest of row m, we shall choose a precise k for the representation $t(m,1) = \lfloor c^j/d^k \rfloor$. According to Lemma 2 below, every n has infinitely many representations $\lfloor c^j/d^k \rfloor$, and we choose the one for which k is minimal (with $k \geq k_0$), noting that j is uniquely determined by k. The rest of row m is then defined by

$$t(m,h) = \lfloor c^{j+h-1}/d^k \rfloor, \text{ for } h = 1, 2, \ldots.$$

Example 1. The array $T_{(3,2,0)}$ consists of numbers $\left\lfloor \frac{3^j}{2^k} \cdot 3^{h-1} \right\rfloor$, $h = 1, 2, 3, \ldots$

Table 1. $T_{(3,2,0)}$

1	3	9	27	81	243	729	2187	$\cdots$
2	6	20	60	182	546	1640	4900	
4	13	40	121	364	1093	3280	9841	
5	15	45	136	410	1230	3690	11071	
7	22	68	205	615	1845	5535	16607	
8	25	76	230	691	2075	6227	18683	
10	30	91	273	820	2460	7831	22143	
$\vdots$								

The rows of $T_{(3,2,0)}$, indexed by $m = 1, 2, 3, \ldots$, are given by $(j,k) = (0,0)$, then $(j,k) = (2,2)$, then $(j,k) = (2,1), \ldots$, as indicated here:

Table 2. The pairs $(j,k) = (j_m, k_m)$ for $T_{(3,2,0)}$

m	1	2	3	4	5	6	7	8	9	10	11	12	13	14
j	0	2	2	4	5	7	4	6	8	10	12	7	14	9
k	0	2	1	4	5	8	3	6	9	12	15	7	18	10

Table 1 shows how an interspersion begets a fractal sequence: for each n, we write the number of the row containing n:

$$(1, 2, 1, 3, 4, 2, 5, 6, 1, 7, 8, 9, 3, 10, 4, 11, 12, 13, 14, 2, 15, 5, \ldots),$$

a sequence which contains itself as a proper subsequence (infinitely many times).

To conclude this introduction, we note that the arrays $T_{(c,d,k_0)}$ represent a class of interspersions new to the literature. A few historical notes will help to place the topics of interspersions, dispersions, and fractal sequences within a wider context. Possibly the earliest published array which is an interspersion was published by Kenneth Stolarsky [8] with a revealing title, "A set of generalized Fibonacci sequences such that each natural number belongs to exactly one". In 1980, David Morrison introduced another interspersion, the Wythoff array. Both the Stolarksy and Wythoff arrays are presented in Neil Sloane's *Classic Sequences* [7], which also gives additional twentieth century references, including [2], where the terms "interspersion" and "dispersion" are introduced and proved equivalent, and [3] in which fractal sequences are defined. Twenty-first century references include [1, 4].

2 Verification of interspersion properties

Lemma 1. *Suppose s/r is a positive irrational number and $0 < \delta < \epsilon$. Then there exist arbitrarily large integers j and k such that*

$$\delta < jr - ks < \epsilon. \tag{1}$$

Proof. First, suppose $\delta = 0$. Let j_i/k_i be the ith convergent to s/r, so by [5], for all sufficiently large i, we have

$$|s/r - j_i/k_i| < 1/k_i^2.$$

Let i be large enough that $k_i > r/\epsilon$ and $j_i/k_i > s/r$. Then

$$|s/r - j_i/k_i| < \epsilon/rk_i,$$

whence $0 < j_i r - k_i s < \epsilon$, as desired.

Now suppose there exists $\delta > 0$ such that for some J and K, the inequality (1) fails for all (j, k) satisfying $j \geq J$ and $k \geq K$. Let j' and k' satisfy $j' \geq J$, $k' \geq K$, and

$$0 < j'r - k's < \epsilon - \delta,$$

and let $\delta_1 = j'r - k's$. Then

$$\epsilon/\delta_1 - \delta/\delta_1 > 1,$$

so that

$$\delta/\delta_1 < q < \epsilon/\delta_1$$

for some $q \geq 1$. Thus, taking $j = qj'$ and $k = qk'$, we have $\delta < jr - ks < \epsilon$, a contradiction. □

Lemma 2. *Every n can be represented as $\lfloor c^j/d^k \rfloor$ using arbitrarily large j and k.*

Proof. In Lemma 1, put $s = \ln c$ and $t = \ln d$; put $\delta = \ln n$ and $\epsilon = \ln(n + 1)$, and let j and k be arbitrarily large integers satisfying (1):

$$\ln n < j \ln c - k \ln d < \ln(n + 1).$$

Equivalently, $n < c^j/d^k < n + 1$, so that $n = \lfloor c^j/d^k \rfloor$. □

Lemma 3. *Suppose n is a term in $T = T_{(c,d,x_0)}$, so that $n = t(m,h)$ for some (m,h). Then the row-successor of n is given by*

$$t(m,h+1) = cn + q \text{ for some } q \text{ satisfying } 0 \le q \le c-1.$$

Proof. We have $n = \lfloor c^j/d^k \rfloor = c^j/d^k - \delta$, where $0 < \delta < 1$, so that $cn = c^{j+1}/d^k - c\delta$. Also, $t(m,h+1) = c^{j+1}/d^k - \epsilon$, where $0 < \epsilon < 1$, so that

$$t(m,h+1) - cn = c\delta - \epsilon.$$

Now $0 < c\delta < c$, so that $-1 < c\delta - \epsilon < c$. Because $c\delta - \epsilon$ is an integer, we conclude that it is in $\{0,1,\ldots,c-1\}$. □

Lemma 4. *No two terms of the array $T = T_{(c,d,k_0)}$ are equal.*

Proof. Suppose, to the contrary, that there are distinct terms $n = \lfloor c^j/d^k \rfloor$ and $n_1 = \lfloor c^{j_1}/d^{k_1} \rfloor$ such that $n = n_1$. Assume, without loss of generality, that j is the least exponent for which $\lfloor c^{j_1}/d^{k_1} \rfloor = \lfloor c^j/d^k \rfloor$ for some j_1 and k_1.

Case 1: neither n nor n_1 lies in column 1 of T. By Lemma 3,

$$n = c\left\lfloor c^{j-1}/d^k \right\rfloor + q \quad \text{and} \quad n_1 = c\left\lfloor c^{j_1-1}/d^{k_1} \right\rfloor + q_1,$$

where $0 \le q \le c-1$ and $0 \le q_1 \le c-1$. Thus,

$$c\left\lfloor c^{j-1}/d^k \right\rfloor + q = c\left\lfloor c^{j_1-1}/d^{k_1} \right\rfloor + q_1,$$

so that, assuming without loss that $\lfloor c^{j-1}/d^k \rfloor \ge \lfloor c^{j_1-1}/d^{k_1} \rfloor$, we have

$$\left\lfloor c^{j-1}/d^k \right\rfloor - \left\lfloor c^{j_1-1}/d^{k_1} \right\rfloor = (q_1 - q)/c.$$

But $0 \le (q_1 - q)/c < 1$, so that, as $(q_1 - q)/c$ is an integer, we have $q_1 = q$ and $\lfloor c^{j-1}/d^k \rfloor = \lfloor c^{j_1-1}/d^{k_1} \rfloor$, contrary to the minimality of j.

Case 2: one of the terms, n or n_1, lies in column 1. By definition of column 1, n and n_1 cannot both lie in column 1. Assume that n but not n_1 lies in column 1. Write $n = t(m,1)$ and $n_1 = t(m_1,h)$, where $h \ge 2$. Then by definition of $t(m,1)$, we have $m_1 \ge m$, so that

$$n \le T(m_1,1) < n_1,$$

contrary to the assumption that $n = n_1$. □

Theorem 5. *The array $T_{(c,d,k_0)}$ is an interspersion.*

Proof. By Lemma 4, property (I1) in the introduction holds, and clearly (I2) and (I3) hold. To see that (I4) holds, suppose

$$t(m,h) < t(m',h') < t(m,h+1).$$

We must prove

$$t(m,h+1) < t(m',h'+1) < t(m,h+2).$$

Since $t(m,h) < t(m',h')$, we have $t(m',h') - t(m,h) \geq 1$, so that

$$ct(m',h') - ct(m,h) \geq c.$$

Consequently, if $0 \leq q_1 \leq c-1$ and $0 \leq q_2 \leq c-1$, then $ct(m',h') - ct(m,h) \geq q_1 - q_2$, so that

$$ct(m,h) + q_1 \leq ct(m',h') + q_2,$$

which by Lemma 3 implies $t(m,h+1) \leq t(m',h'+1)$, so that by Lemma 4,

$$t(m,h+1) < t(m',h'+1)$$

Likewise, the inequality

$$ct(m,h+1) - ct(m',h') \geq c$$

implies $t(m',h'+1) < t(m,h+2)$. □

3 Permutations of $\mathbb{N}$

Suppose c, d, k_0 are as already stipulated, and abbreviate $T_{(c,d,k_0)}$ as T. In this section, we shall show that the exponents k in the representation $\lfloor c^j/d^k \rfloor$ for the numbers in T form a permutation of the sequence $\mathbb{N} = (0, 1, 2, \ldots)$. For example, as indicated in Table 2, for $(c, d, k_0) = (3, 2, 0)$, the sequence of values of k is

$$(0, 2, 1, 4, 5, 8, 3, 6, 9, 12, 15, 7, 18, 10, \ldots).$$

Theorem 6. *Regarding the interspersion $T_{(c,d,k_0)}$, let*

$$\left\lfloor (c^{j_m}/d^{k_m})c^{h-1} \right\rfloor, \text{ for } h = 1, 2, 3, \ldots,$$

be the numbers in row m. Then each $n \geq k_0$ occurs exactly once in the sequence (k_m).

Proof. Suppose, to the contrary, that there is a least $K \geq k_0$ for which, for every j,

$$\left\lfloor c^j/d^K \right\rfloor = \left\lfloor c^{p_j}/d^k \right\rfloor$$

for some k satisfying $k_0 \leq k < K$ and p_j. Then

$$\left| \frac{c^j}{d^K} - \frac{c^{p_j}}{d^k} \right| < 1.$$

Moreover, as $k < K$, we have $p_j < j$ and can write $K = k + e$ where $e > 0$ and $j = p_j + e_j$ where $e_j > 0$, so that

$$\left| \frac{c^{e_j}}{d^e} - 1 \right| < \frac{d^k}{c^{p_j}}.$$

As $j \to \infty$, clearly $p_j \to \infty$, so that $\frac{d^k}{c^{p_j}} \to 0$. Consequently, $c^{e_j} = d^e$ for all sufficiently large j, contrary to the independence of c and d, as defined and hypothesized in the introduction. Thus, there is no such K, which is to say that for every $k \geq k_0$, there exists a row of T such that the numbers in that row are the numbers $\lfloor (c^j/d^k)c^{h-1} \rfloor$ for some j. By definition of $t(m,1)$ as the least $\lfloor c^j/d^k \rfloor = \lfloor c^{j_m}/d^{k_m} \rfloor$ not in a row numbered $1, 2, ..., m-1$, the numbers k_m are distinct. □

Regarding the set $\mathbb{N}$ of natural numbers to be $\{1, 2, 3, ...\}$, Theorem 6 shows that the sequence $(k_m - k_0 + 1)$ is a permutation of $\mathbb{N}$. Do such permutations have notable asymptotics? Can they be efficiently computed? We leave these questions open.

4 Examples

In Theorem 5, the index k_0 can be any nonnegative integer, and in Example 1, $k_0 = 0$. In Table 3, we keep $(c, d) = (3, 2)$ as in Table 1 but change k_0 to 1. In infinitely many cases, a row of $T_{(3,2,0)}$ is identical to a row of $T_{(3,2,1)}$, and in infinitely many cases a row of $T_{(3,2,0)}$ is not identical to a row of $T_{(3,2,1)}$. These easily proved observations remain true for $k_0 = 2, 3, 4, \ldots$.

Table 3. $T_{(3,2,1)}$

1	4	13	40	121	364	1093	3280	⋯
2	6	20	60	182	546	1640	4920	
3	10	30	91	273	820	2460	7381	
5	15	45	136	410	1230	3690	11071	
7	22	68	205	615	1845	5535	16607	
8	25	76	230	691	2075	6227	18683	
9	28	86	259	778	2335	7006	21018	
⋮								

The rows of $T_{(3,2,1)}$, indexed by $m = 1, 2, 3, \ldots$, are given by $(j, k) = (1, 1)$, then $(j, k) = (2, 2)$, then $(j, k) = (3, 3), \ldots$, as indicated here:

Table 4. The pairs $(j, k) = (j_m, k_m)$ for $T_{(3,2,1)}$

m	1	2	3	4	5	6	7	8	9	10	11	12	13	14
j	1	2	3	4	5	7	9	6	8	10	12	7	14	9
k	1	2	3	4	5	8	11	6	9	12	15	7	18	19

The fractal sequence corresponding to $T_{(3,2,1)}$ is

$$(1, 2, 3, 1, 4, 2, 5, 6, 7, 3, 8, 9, 1, 10, 4, 11, 12, 13, 14, 2, 15, 5, 16, 17, 6, \ldots).$$

Next, we change k_0 to 3 :

Table 5. $T_{(3,2,3)}$								
1	3	10	30	91	273	820	2460	$\cdots$
2	7	22	68	205	615	1845	5535	
4	12	38	115	345	1037	3113	9341	
5	15	45	136	410	1230	3690	11071	
6	19	57	172	518	1556	4670	14012	
8	25	76	230	691	2075	6227	18683	
9	28	86	259	778	2335	7006	21018	
$\vdots$								

The rows of $T_{(3,2,3)}$, indexed by $m = 1, 2, 3, \ldots$, are given by $(j, k) = (2, 3)$, then $(j, k) = (4, 5)$, then $(j, k) = (7, 9), \ldots$, as indicated here:

Table 6. The pairs $(j, k) = (j_m, k_m)$ for $T_{(3,2,3)}$														
m	1	2	3	4	5	6	7	8	9	10	11	12	13	14
j	2	4	7	4	8	7	9	6	15	10	12	7	14	16
k	3	5	9	4	10	8	11	6	20	12	15	7	18	21

The fractal sequence corresponding to $T_{(3,2,3)}$ is

$$(1, 2, 1, 3, 4, 5, 2, 6, 7, 1, 8, 3, 9, 10, 4, 11, 12, 13, 5, 14, 15, 2, 16, 17, 6, \ldots).$$

As a final example, consider the interspersion $T_{(2,3,0)}$:

Table 7. $T_{(2,3,0)}$								
1	2	4	8	16	32	64	128	$\cdots$
3	7	14	28	56	113	227	455	
5	10	21	42	85	170	341	682	
6	12	25	50	101	202	404	809	
9	189	37	75	151	303	606	1213	
11	22	44	89	179	359	719	1438	
13	26	53	106	213	426	852	1704	
$\vdots$								

The rows of $T_{(2,3,0)}$, indexed by $m = 1, 2, 3, \ldots$, are given by $(j, k) = (0, 0)$, then $(j, k) = (5, 2)$, then $(j, k) = (4, 1), \ldots$, as indicated here:

Table 8. The pairs $(j, k) = (j_m, k_m)$ for $T_{(2,3,0)}$														
m	1	2	3	4	5	6	7	8	9	10	11	12	13	14
j	0	5	4	9	8	13	18	23	20	17	44	22	30	46
k	0	2	1	4	3	6	9	12	10	8	25	11	16	26

The fractal sequence corresponding to $T_{(2,3,0)}$ is

$$(1, 1, 2, 1, 3, 4, 2, 1, 5, 3, 6, 4, 7, 2, 8, 1, 9, 5, 10, 11, 3, 6, 12, 13, 4, 7, 14, 2, \ldots).$$

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2000 *Mathematics Subject Classification*: Primary 11B99
Keywords: interspersion, fractal sequence.

(Concerned with sequences A007337, A022447, A114537, A114577, A120862, A120863, A124904, A124905, A124906, A124907, A124908, A124909, A124910, A124911, A124912, A124913, A124914, A124915, A124916, A124917, A124918, A124919, A125150, A125151, A125152, A125153, A125154, A125155, A125156, A125157, A125158, A125159, A125160, and A125161.)

Received December 30 2006; revised version received May 4 2007. Published in *Journal of Integer Sequences*, May 6 2007.

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