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# Transformations Preserving the Hankel Transform

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**Abstract**

We classify all polynomial transformations of integer sequences which preserve the Hankel transform, thus generalizing examples due to Layman and Spivey & Steil. We also show that such transformations form a group under composition.

## 1 Introduction

Given a sequence of integers $\{a_i\} = a_0, a_1, a_2, \ldots$, the Hankel matrix of $A$ is the infinite matrix whose $(i, j)$ entry is $a_{i+j}$ for $i \geq 0, j \geq 0$. The Hankel matrix of order $n$ of $A$ is the $n \times n$ matrix consisting of the first $n$ rows and $n$ columns of the Hankel matrix of $A$, and the Hankel sequence determined by $A$ is the sequence of determinants of the Hankel matrices of order $n$. For example, the Hankel matrix of order 3 is given by

$$\begin{pmatrix} a_0 & a_1 & a_2 \\ a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \end{pmatrix}.$$

Let $R = \mathbb{Z}[a_0, a_1, a_2, \ldots]$. We will say that a sequence $\{b_i\} = \{b_0, b_1, b_2, \ldots\}$ of elements in $R$ is an $HTP$ sequence (Hankel Transform Preserving) if the Hankel transform of $\{b_i\}$ is formally equal to that of $\{a_i\}$; that is, if the following identity holds in $R$ for all $n$:

$$\begin{vmatrix} b_0 & b_1 & \cdots & b_n \\ b_1 & b_2 & \cdots & b_{n+1} \\ \vdots & & \ddots & \vdots \\ b_n & b_{n+1} & \cdots & b_{2n} \end{vmatrix} = \begin{vmatrix} a_0 & a_1 & \cdots & a_n \\ a_1 & a_2 & \cdots & a_{n+1} \\ \vdots & & \ddots & \vdots \\ a_n & a_{n+1} & \cdots & a_{2n} \end{vmatrix}.$$

An $HTP$ sequence $\{b_i\}$ determines a ring homomorphism $b : R \to R$ by the equation $b(a_i) = b_i$. We will sometimes refer to the sequence $\{b_i\}$ as just $b$. It is easy to see that the composition of two ring homomorphisms associated to $HTP$ sequences is the ring homomorphism associated to an $HTP$ sequence, so that the set of all $HTP$ sequences has a semigroup structure (with $b_i = a_i$ representing the identity). We will see (Theorem 3.6) that this is actually a group structure.

*Example* 1.1. Let $\sigma : R \to R$ be given by $\sigma(a_i) = (-1)^i a_i$. Then $\sigma$ determines an $HTP$ sequence. Indeed, the order $n$ Hankel matrix of the sequence $\{(-1)^i a_i\}$ is given by conjugating the order $n$ Hankel matrix $A_n$ of $\{a_i\}$ by the diagonal matrix $D_n$ with $(i,i)$-entry given by $(-1)^i$. The determinant of $D_n^{-1} A_n D_n$ is the same as that of $A_n$.

*Example* 1.2. For an integer $k$, the sequence defined by $b_n = \sum_{i=0}^{n} \binom{n}{i} k^{n-i} a_i$ is an $HTP$ sequence. (Here, if $k = 0$, then we interpret $0^0$ to be 1.) Spivey and Steil [5] called this the falling $k$-binomial transform, and they proved that this preserves the Hankel Transform. When $k = 1$, this gives the binomial transform, which Layman [3] originally proved preserves the Hankel transform.

*Remark* 1.3. In fact, Spivey and Steil allow $k$ to be any real number. If $k$ were not an integer, then $b_n$ is not in $R = \mathbb{Z}[a_0, a_1, a_2, \ldots]$, so in our language, $\{b_n\}$ would not be an HTP sequence. Since we are interested in integer sequences, we have restricted to $\mathbb{Z}[a_0, a_1, a_2, \ldots]$.

**Definition 1.4.** *An $HTP$ sequence $b$ preserves $a_0$ through $a_n$ if $b_i = a_i$ for $0 \leq i \leq n$.*

Suppose that $b(m)$ is a sequence of $HTP$ sequences such that for each $n$, there is a number $M(n)$ such that $b(m)$ preserves $a_0$ through $a_n$ for all $m > M(n)$. Then the *infinite composition*

$$\cdots \circ b(m) \circ b(m-1) \circ \cdots \circ b(0)$$

is itself a well-defined sequence. Indeed, the $n$th term in the sequence of this infinite composite is determined by $b(M(n)) \circ b(M(n) - 1) \circ \cdots \circ b(0)$. It is easy to see that this infinite composition is itself an $HTP$ sequence.

In order to find all $HTP$ sequences, we first describe a special set of sequences, parametrized by $R$ and the positive integers. In fact, for each positive integer $n$, and each $c \in R$, we will define an $HTP$ sequence $b(n, c)$ which preserves $a_0$ through $a_{2n}$. Example 1.2 above arises when $n = 0$ and $c \in \mathbb{Z}$. Our main theorem is

**Theorem 1.5.** *If $b$ is a given $HTP$ sequence, then there is a sequence $c_0, c_1, c_2, \ldots$ in $R$ and an $\epsilon \in \{0, 1\}$ such that*

$$b = \cdots \circ b(n, c_n) \circ \cdots \circ b(1, c_1) \circ b(0, c_0) \circ \sigma^{\epsilon}.$$

Our goal in Section 2 is to define the sequences $b(n, c_n)$, which we do in Definition 2.5. In Section 3 we prove that the set of all $HTP$ sequences forms a group (Theorem 3.6), and we prove our classification Theorem 1.5.

## 2 A collection of HTP sequences

**Definition 2.1.** *Let $T : R[x] \to R$ be the $R$-linear homomorphism defined by $T(x^k) = a_k$. For integers $i \geq j \geq 0$, define $T_{ij} : R[x] \to R$ to be the $R$-linear homomorphism given by*

$$T_{ij}(x^k) = (-1)^{i+j} \begin{vmatrix} a_0 & a_1 & a_2 & \cdots & a_i \\ a_1 & a_2 & a_3 & \cdots & a_{i+1} \\ \vdots & & & \ddots & \vdots \\ a_{j-1} & a_j & a_{j+1} & \cdots & a_{j-1+i} \\ a_{j+1} & a_{j+2} & a_{j+3} & \cdots & a_{j+1+i} \\ \vdots & & & \ddots & \vdots \\ a_i & a_{i+1} & a_{i+2} & \cdots & a_{2i} \\ a_k & a_{k+1} & a_{k+2} & \cdots & a_{k+i} \end{vmatrix}.$$

*Remark* 2.2. If $0 \leq k \leq i$ and $k \neq j$, then $T_{ij}(x^k) = 0$, since two rows in the matrix defining $T_{ij}(x^k)$ are equal. Also, keeping track of signs, one sees that $T_{ij}(x^j) = T_{ii}(x^i)$. Finally, we note that $T_{00} = T$.

**Definition 2.3.** *For each $c \in R$ and each integer $i \geq 1$, we define a sequence of polynomials $f_{m,i,c}(x) \in R[x]$ recursively by*

$$f_{0,i,c} = 1, \qquad f_{m+1,i,c} = f_{m,i,c} \cdot (x + cT_{i-1,i-1}(x^{i-1})) - c\left(\sum_{j=0}^{i-1} T_{i-1,j}(f_{m,i,c}) \cdot x^j\right).$$

*Also, we define $f_{m,0,c} \in R[x]$ by $f_{m,0,c} = (x+c)^m$, or (equivalently) recursively by*

$$f_{0,0,c} = 1, \qquad f_{m+1,0,c} = f_{m,0,c} \cdot (x+c).$$

We will show in Lemma 2.9 that for each $i \geq 0$, $m \geq 0$, and $c \in R$, $f_{m,i,c}(x)$ is a degree $m$ polynomial in $x$ with leading coefficient 1.

**Definition 2.4.** *Fixing a choice of $i$ and $c$, define $U_{k,m}$ for each $k, m$ by*

$$f_{m,i,c}(x) = \sum U_{k,m} x^k.$$

*Let $U$ be the infinite matrix whose $(k,m)$ entry is given by $U_{k,m}$.*

Thus $U$ is upper triangular, with diagonal entries all 1. Let $A$ be the infinite Hankel matrix

$$A = \begin{pmatrix} a_0 & a_1 & a_2 & \cdots \\ a_1 & a_2 & a_3 & \cdots \\ a_2 & a_3 & a_4 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

We will show in Lemma 2.14 that the product $U^t A U$ is a Hankel matrix.

**Definition 2.5.** *Let $b(i,c)$ denote the sequence whose Hankel matrix is the matrix $U^tAU$, where $U$ is defined in terms of $i$ and $c$ as above.*

We show in Corollary 2.15 that $b(i,c)$ is an HTP sequence preserving $a_0$ through $a_{2i}$.

*Example* 2.6. If $i = 1$ and $c = 1$, then the recurrence relation of Definition 2.3 becomes

$$f_{m+1,1,1} = f_{m,1,1} \cdot (x + a_0) - T(f_{m,1,1}),$$

$$f_{0,1,1} = 1.$$

Thus,

$$f_{1,1,1} = (x + a_0) - T(1) = x + a_0 - a_0 = x,$$
$$f_{2,1,1} = x(x + a_0) - T(x) = x^2 + a_0x - a_1,$$
$$f_{3,1,1} = (x^2 + a_0x - a_1)(x + a_0) - T(x^2 + a_0x - a_1) = x^3 + 2a_0x^2 + (a_0^2 - a_1)x + -a_0a_1 - a_2.$$

Therefore, the upper left $4 \times 4$ submatrix of $U$ is

$$\begin{pmatrix} 1 & 0 & -a_1 & -a_0a_1 - a_2 \\ 0 & 1 & a_0 & a_0^2 - a_1 \\ 0 & 0 & 1 & 2a_0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Using Mathematica, we compute the upper left $4 \times 4$ submatrix of $U^tAU$ to be

$$\begin{pmatrix} a_0 & a_1 & a_2 & a_3 - a_1^2 + a_0a_2 \\ a_1 & a_2 & a_3 - a_1^2 + a_0a_2 & x \\ a_2 & a_3 - a_1^2 + a_0a_2 & x & y \\ a_3 - a_1^2 + a_0a_2 & x & y & z \end{pmatrix}$$

where

$$x = a_4 - a_0a_1^2 + 2a_0a_3 - 2a_1a_2 + a_0^2a_2$$
$$y = a_5 - 4a_0a_1a_2 + 3a_0a_4 - 2a_1a_3 - a_0^2a_1^2 + 3a_0a_3 - a_2^2 + a_0^3a_2 + a_1^3$$
$$z = a_6 + a_0^4a_2 + 3a_1^2a_2 - a_0^3a_1^2 + 4a_0^3a_3 - 2a_2a_3 - 2a_1a_4 - 6a_0^2a_1a_2$$
$$+6a_0^2a_4 + 2a_0a_1^3 - 3a_0a_2^2 - 6a_0a_1a_3 + 4a_0a_5$$

Thus, the first seven terms of $b(1,1)$ are $a_0, a_1, a_2, a_3 - a_1^2 + a_0a_2, x, y, z$.

*Example* 2.7. If $i = 2$ and $c = 1$, then the recurrence relation of Definition 2.3 becomes

$$f_{m+1,2,1} = f_{m,2,1} \cdot (x + T_{1,1}(x)) - T_{1,1}(f_{m,2,1})x - T_{1,0}(f_{m,2,1}),$$

$$f_{0,1,1} = 1.$$

Thus,

$$f_{1,2,1} = (x + a_0a_2 - a_1^2) - 0x - (a_0a_2 - a_1^2) = x,$$
$$f_{2,2,1} = x(x + a_0a_2 - a_1^2) - (a_0a_2 - a_1^2)x + 0 = x^2,$$

$$\begin{aligned} f_{3,2,1} &= x^2(x + a_0a_2 - a_1^2) - (a_0a_3 - a_1a_2)x + a_1a_3 - a_2^2 \\ &= x^3 + (a_0a_2 - a_1^2)x^2 - (a_0a_3 - a_1a_2)x + a_1a_3 - a_2^2. \end{aligned}$$

Therefore, the upper left $4 \times 4$ submatrix of $U$ is

$$\begin{pmatrix} 1 & 0 & 0 & a_1a_3 - a_2^2 \\ 0 & 1 & 0 & a_1a_2 - a_0a_3 \\ 0 & 0 & 1 & a_0a_2 - a_1^2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Using Mathematica, we compute the upper left $4 \times 4$ submatrix of $U^tAU$ to be

$$\begin{pmatrix} a_0 & a_1 & a_2 & a_3 \\ a_1 & a_2 & a_3 & a_4 \\ a_2 & a_3 & a_4 & x \\ a_3 & a_4 & x & y \end{pmatrix}$$

where

$$\begin{aligned} x &= a_5 - a_2^3 - a_0a_3^2 - a_1^2a_4 + 2a_1a_2a_3 + a_0a_2a_4, \\ y &= a_6 - 2a_1^3a_2a_3 - 2a_2^2a_3 + a_1^4a_4 - a_0^2a_2a_3^2 + a_0^2a_2^2a_4 + 2a_0a_1a_2^2a_3 + 2a_1a_3^2 + 2a_1a_2a_4 \\ &\quad + a_1^2a_2^3 + a_0a_1^2a_3^2 - 2a_0a_1^2a_2a_4 - 2a_1^2a_5 - a_0a_2^4 - 2a_0a_3a_4 + 2a_0a_2a_5. \end{aligned}$$

Thus the first seven terms of $b(2,1)$ are $a_0, a_1, a_2, a_3, a_4, x, y$.

**Lemma 2.8.** *Suppose $f$ and $g$ are two polynomials in $R[x]$, and $i \geq 0$ is any integer. Then*

$$\sum_{j=0}^{i-1} T(f \cdot x^j) \cdot T_{i-1,j}(g) = \sum_{j=0}^{i-1} T(g \cdot x^j) \cdot T_{i-1,j}(f).$$

*Proof.* Since both sides of the equation are $R$-linear in both $f$ and $g$, it suffices to consider the case when $f = x^m$ and $g = x^n$, so we need to show that

$$\sum_{j=0}^{i-1} T(x^{m+j}) \cdot T_{i-1,j}(x^n) = \sum_{j=0}^{i-1} T(x^{n+j}) \cdot T_{i-1,j}(x^m).$$

Consider the matrix below:

$$\begin{pmatrix} a_0 & a_1 & a_2 & \cdots & a_{i-1} & a_m \\ a_1 & a_2 & a_3 & \cdots & a_i & a_{m+1} \\ a_2 & a_3 & a_4 & \cdots & a_{i+1} & a_{m+2} \\ \vdots & & & \ddots & & \vdots \\ a_{i-1} & a_i & a_{i+1} & \cdots & a_{2i-2} & a_{m+i-1} \\ a_n & a_{n+1} & a_{n+2} & \cdots & a_{n+i-1} & 0 \end{pmatrix}$$

If we compute the determinant by expanding along the rightmost column, we get

$$\sum_{j=0}^{i-1} (-1)^{i+j} a_{m+j} \cdot T_{i-1,j}(x^n) = -\sum_{j=0}^{i-1} T(x^{m+j}) \cdot T_{i-1,j}(x^n).$$

If we compute the determinant by expanding along the bottom row, we get

$$\sum_{j=0}^{i-1}(-1)^{i+j}a_{n+j}\cdot T_{i-1,j}(x^m) = -\sum_{j=0}^{i-1}T(x^{n+j})\cdot T_{i-1,j}(x^m).$$

□

**Lemma 2.9.** *For each $i \geq 0$, $m \geq 0$, and $c \in R$, $f_{m,i,c}(x)$ is a degree $m$ polynomial in $x$ with leading coefficient* 1.

*Proof.* When $i = 0$ or $m = 0$, the claim is clear. We assume $i \geq 1$. Assume by induction on $m$ that $f_{m,i,c}(x)$ has degree $m$ and leading coefficient 1. It then suffices to show by the recursive definition of $f_{m+1,i,c}(x)$ that $T_{i-1,j}(f_{m,i,c}) = 0$ whenever $i > j > m$. This follows at once from Remark 2.2 and the inductive hypothesis. □

**Lemma 2.10.** *$f_{m,i,c} = x^m$ whenever $m \leq i$.*

*Proof.* Using the recursive definition, we see that it is enough to show that

$$x^m T_{i-1,i-1}(x^{i-1}) = \sum_{j=0}^{i-1} T_{i-1,j}(x^m)x^j$$

whenever $m \leq i-1$. This follows from Remark 2.2, since all terms in the right hand sum are 0, except for the term when $j = m$, and this is $T_{i-1,m}(x^m)x^m = T_{i-1,i-1}(x^{i-1})x^m$. □

**Lemma 2.11.** *For $m, n \geq 0$,*

$$T(f_{m+1,i,c}\cdot f_{n,i,c}) = T(f_{m,i,c}\cdot f_{n+1,i,c}).$$

*Proof.* First,

$$f_{m+1,0,c}\cdot f_{n,0,c} = (x+c)^{m+1}(x+c)^n = (x+c)^m(x+c)^{n+1} = f_{m,0,c}\cdot f_{n+1,0,c}.$$

We now assume $i \geq 1$. We then have

$$f_{m+1,i,c}\cdot f_{n,i,c} - f_{m,i,c}\cdot f_{n+1,i,c} =$$

$$\left(f_{m,i,c}(x + cT_{i-1,i-1}(x^{i-1})) - c\left(\sum_{j=0}^{i-1}T_{i-1,j}(f_{m,i,c})\cdot x^j\right)\right)\cdot f_{n,i,c} -$$

$$f_{m,i,c}\cdot\left(f_{n,i,c}(x + cT_{i-1,i-1}(x^{i-1})) - c\left(\sum_{j=0}^{i-1}T_{i-1,j}(f_{n,i,c})\cdot x^j\right)\right)$$

$$= c\left(f_{m,i,c}\left(\sum_{j=0}^{i-1}T_{i-1,j}(f_{n,i,c})\cdot x^j\right) - \left(\sum_{j=0}^{i-1}T_{i-1,j}(f_{m,i,c})\cdot x^j\right)f_{n,i,c}\right)$$

$$= c\sum_{j=0}^{i-1}\left(f_{m,i,c}\cdot x^j\cdot T_{i-1,j}(f_{n,i,c}) - f_{n,i,c}\cdot x^j\cdot T_{i-1,j}(f_{m,i,c})\right).$$

By Lemma 2.8, this term is in the kernel of $T$. □

**Lemma 2.12.** *$T(f_{m,i,c}(x)) = a_m$ whenever $0 \le m \le 2i$.*

*Proof.* Suppose $m = 2i$. Since $f_{0,i,c} = 1$, and by applying Lemma 2.11 $i$ times,

$$T(f_{2i,i,c}) = T(f_{2i,i,c} \cdot f_{0,i,c}) = T(f_{i,i,c} \cdot f_{i,i,c}).$$

By Lemma 2.10, $f_{i,i,c} = x^i$, so $(f_{i,i,c})^2 = x^{2i}$, and $T(x^{2i}) = a_{2i}$. The other cases follow similarly. $\square$

**Lemma 2.13.**

$$T(f_{2i+1,i,c}(x)) = a_{2i+1} + c \begin{vmatrix} a_0 & a_1 & \cdots & a_i \\ a_1 & a_2 & \cdots & a_{i+1} \\ \vdots & & \ddots & \vdots \\ a_i & a_{i+1} & \cdots & a_{2i} \end{vmatrix}.$$

*Proof.* If we expand the determinant on the right hand side of the equation along the last column, we get

$$\begin{aligned} a_{2i+1} + ca_{2i}T_{i-1,i-1}(x^{i-1}) - c\sum_{j=0}^{i-1} T_{i-1,j}(x^i)a_{i+j} \\ = T\left(x^{2i+1} + cx^{2i}T_{i-1,i-1}(x^{i-1}) - c\sum_{j=0}^{i-1} T_{i-1,j}(x^i)x^{i+j}\right) \\ = T\left(x^i\left(x^i\left(x + cT_{i-1,i-1}(x^{i-1})\right) - c\sum_{j=0}^{i-1} T_{i-1,j}(f_{i,i,c})x^j\right)\right) \\ = T\left(x^i\left(f_{i,i,c} \cdot \left(x + cT_{i-1,i-1}(x^{i-1})\right) - c\sum_{j=0}^{i-1} T_{i-1,j}(f_{i,i,c}) \cdot x^j\right)\right) \\ = T(x^i \cdot f_{i+1,i,c}) = T(f_{i,i,c} \cdot f_{i+1,i,c}) = T(f_{2i+1,i,c}). \end{aligned}$$

Here, we have used Lemma 2.10 to identify $x^i$ with $f_{i,i,c}$ and Lemma 2.11 for the last equality. $\square$

Now recall that $A$ is the infinite Hankel matrix

$$A = \begin{pmatrix} a_0 & a_1 & a_2 & \cdots \\ a_1 & a_2 & a_3 & \cdots \\ a_2 & a_3 & a_4 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

**Lemma 2.14.** *The product $U^tAU$ is a Hankel matrix.*

*Proof.* We have $f_{r,i,c} \cdot f_{u,i,c} = \sum_{s,t} U_{s,r}U_{t,u}x^{s+t}$, so that

$$T(f_{r,i,c} \cdot f_{u,i,c}) = \sum_{s,t} U_{s,r}U_{t,u}a_{s+t} = \sum_{s,t} U_{s,r}a_{s+t}U_{t,u}.$$

This is precisely the $(r, u)$ entry of $U^tAU$. Now the result follows from Lemma 2.11. $\square$

Recall from Definition 2.5 that $b(i,c)$ denotes the sequence whose Hankel matrix is $U^tAU$.

**Corollary 2.15.** *We have $b(i,c)_n = T(f_{n,i,c})$, and $b(i,c)$ is an $HTP$ sequence preserving $a_0$ through $a_{2i}$. Moreover*

$$b(i,c)_{2i+1} = a_{2i+1} + c\begin{vmatrix} a_0 & a_1 & \cdots & a_i \\ a_1 & a_2 & \cdots & a_{i+1} \\ \vdots & & \ddots & \vdots \\ a_i & a_{i+1} & \cdots & a_{2i} \end{vmatrix}.$$

*Proof.* Since $U^t$ and $U$ are each triangular with 1s on the diagonal, it follows that the Hankel matrices of finite order associated to $U^tAU$ have the same determinants as those of $A$. Thus the sequence of entries on the top row of $U^tAU$ represents a transformation of $A$ which preserves the Hankel transform. This is the same as the sequence of entries in the top row of $AU$ since $U^t$ preserves the top row. But it is easy to see that this sequence is given by $T(f_{n,i,c})$. The first $2i$ terms of the sequence are given by Lemma 2.12, while the $2i+1$ term is given by Lemma 2.13. $\square$

# 3 Classifying all $HTP$ sequences

**Lemma 3.1.** *For each integer $n \geq 0$, the determinant*

$$\begin{vmatrix} a_0 & a_1 & \cdots & a_n \\ a_1 & a_2 & \cdots & a_{n+1} \\ \vdots & & \ddots & \vdots \\ a_n & a_{n+1} & \cdots & a_{2n} \end{vmatrix}$$

*is irreducible in $R$.*

*Proof.* The statement is obvious if $n = 0$, so suppose $n \geq 1$. We make $R$ into a graded ring by assigning $\deg(a_i) = 2n+1-i$. Then the determinant is a homogeneous polynomial of degree $(n+1)^2$. As a polynomial in $a_{2n}$ with coefficients in $\mathbb{Z}[a_0, a_1, \ldots, a_{2n-1}]$, the determinant can be written

$$\begin{vmatrix} a_0 & a_1 & \cdots & a_{n-1} \\ a_1 & a_2 & \cdots & a_n \\ \vdots & & \ddots & \vdots \\ a_{n-1} & a_n & \cdots & a_{2n-2} \end{vmatrix} \cdot a_{2n} + \begin{vmatrix} a_0 & a_1 & \cdots & a_{n-1} & a_n \\ a_1 & a_2 & \cdots & a_n & a_{n+1} \\ \vdots & & \ddots & & \vdots \\ a_{n-1} & a_n & \cdots & a_{2n-2} & a_{2n-1} \\ a_n & a_{n+1} & \cdots & a_{2n-1} & 0 \end{vmatrix}.$$

The coefficient of $a_{2n}$ has degree $(n+1)^2 - 1$, while the constant coefficient has degree $(n+1)^2$. But no element in $\mathbb{Z}[a_0, a_1, \ldots, a_{2n-1}]$ has degree 1. Therefore, the coefficient of $a_{2n}$ does not divide the constant coefficient. By induction, the coefficient of $a_{2n}$ is irreducible. The result follows from this. $\square$

**Lemma 3.2.** *Suppose that $b$ is an HTP sequence which preserves $a_0$ through $a_{2n-1}$. Then $b_{2n} = a_{2n}$.*

*Proof.* In order to have

$$\begin{vmatrix} a_0 & a_1 & \cdots & a_n \\ a_1 & a_2 & \cdots & a_{n+1} \\ \vdots & & \ddots & \vdots \\ a_n & a_{n+1} & \cdots & b_{2n} \end{vmatrix} = \begin{vmatrix} a_0 & a_1 & \cdots & a_n \\ a_1 & a_2 & \cdots & a_{n+1} \\ \vdots & & \ddots & \vdots \\ a_n & a_{n+1} & \cdots & a_{2n} \end{vmatrix},$$

we must have

$$(b_{2n} - a_{2n}) \begin{vmatrix} a_0 & a_1 & \cdots & a_{n-1} \\ a_1 & a_2 & \cdots & a_n \\ \vdots & & \ddots & \vdots \\ a_{n-1} & a_n & \cdots & a_{2n-2} \end{vmatrix} = 0.$$

Thus, $b_{2n} - a_{2n}$ must be 0. □

**Lemma 3.3.** *Suppose $b$ is an HTP sequence. Then either $b_1 - a_1$ or $b_1 + a_1$ is divisible by $a_0$ in $R$.*

*Proof.* Clearly, $b_0 = a_0$, since the 0th terms in the Hankel transforms must coincide. Now, in order to have

$$\begin{vmatrix} a_0 & b_1 \\ b_1 & b_2 \end{vmatrix} = \begin{vmatrix} a_0 & a_1 \\ a_1 & a_2 \end{vmatrix}$$

we must have $b_2 a_0 - b_1^2 = a_2 a_0 - a_1^2$, so $a_0(b_2 - a_2) = b_1^2 - a_1^2 = (b_1 - a_1)(b_1 + a_1)$. Therefore, $a_0$ divides either $b_1 - a_1$ or $b_1 + a_1$. □

**Lemma 3.4.** *Fix an integer $n \geq 1$. Suppose $b$ is an HTP sequence preserving $a_0$ through $a_{2n}$. Then $b_{2n+1} - a_{2n+1}$ is divisible by*

$$\begin{vmatrix} a_0 & a_1 & \cdots & a_n \\ a_1 & a_2 & \cdots & a_{n+1} \\ \vdots & & \ddots & \vdots \\ a_n & a_{n+1} & \cdots & a_{2n} \end{vmatrix}.$$

*Proof.* We have

$$\begin{vmatrix} a_0 & a_1 & \cdots & a_n & a_{n+1} \\ a_1 & a_2 & \cdots & a_{n+1} & a_{n+2} \\ \vdots & & \ddots & & \vdots \\ a_n & a_{n+1} & \cdots & a_{2n} & b_{2n+1} \\ a_{n+1} & a_{n+2} & \cdots & b_{2n+1} & b_{2n+2} \end{vmatrix}$$

$$= b_{2n+2} \begin{vmatrix} a_0 & a_1 & \cdots & a_{n-1} & a_n \\ a_1 & a_2 & \cdots & a_n & a_{n+1} \\ \vdots & & \ddots & & \vdots \\ a_{n-1} & a_n & \cdots & a_{2n-2} & a_{2n-1} \\ a_n & a_{n+1} & \cdots & a_{2n-1} & a_{2n} \end{vmatrix} + \begin{vmatrix} a_0 & a_1 & \cdots & a_n & a_{n+1} \\ a_1 & a_2 & \cdots & a_{n+1} & a_{n+2} \\ \vdots & & \ddots & & \vdots \\ a_n & a_{n+1} & \cdots & a_{2n} & b_{2n+1} \\ a_{n+1} & a_{n+2} & \cdots & b_{2n+1} & 0 \end{vmatrix}$$

The determinant on the right above can be written

$$\begin{vmatrix} a_0 & a_1 & \cdots & a_n & a_{n+1} \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \cdots & 0 & b_{2n+1} \\ 0 & 0 & \cdots & b_{2n+1} & 0 \end{vmatrix} + \begin{vmatrix} a_0 & a_1 & \cdots & a_n & a_{n+1} \\ \vdots & & \ddots & & \vdots \\ a_n & a_{n+1} & \cdots & a_{2n} & 0 \\ 0 & 0 & \cdots & b_{2n+1} & 0 \end{vmatrix}$$

$$+ \begin{vmatrix} a_0 & a_1 & \cdots & a_n & a_{n+1} \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \cdots & 0 & b_{2n+1} \\ a_{n+1} & a_{n+2} & \cdots & 0 & 0 \end{vmatrix} + \begin{vmatrix} a_0 & a_1 & \cdots & a_n & a_{n+1} \\ \vdots & & \ddots & & \vdots \\ a_n & a_{n+1} & \cdots & a_{2n} & 0 \\ a_{n+1} & a_{n+2} & \cdots & 0 & 0 \end{vmatrix}.$$

(In each of the four above matrices, all rows except the last two are the same.) Now, we can write this as

$$= -b_{2n+1}^2 \begin{vmatrix} a_0 & a_1 & \cdots & a_{n-1} \\ \vdots & & \ddots & \vdots \\ a_{n-1} & a_n & \cdots & a_{2n-2} \end{vmatrix} - b_{2n+1} \begin{vmatrix} a_0 & a_1 & \cdots & a_{n-1} & a_{n+1} \\ \vdots & & \ddots & & \vdots \\ a_n & a_{n+1} & \cdots & a_{2n-1} & 0 \end{vmatrix}$$

$$-b_{2n+1} \begin{vmatrix} a_0 & a_1 & \cdots & a_{n-1} & a_n \\ \vdots & & \ddots & & \vdots \\ a_{n-1} & a_n & \cdots & a_{2n-2} & a_{2n-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{2n} & 0 \end{vmatrix} + \begin{vmatrix} a_0 & a_1 & \cdots & a_n & a_{n+1} \\ \vdots & & \ddots & & \vdots \\ a_n & a_{n+1} & \cdots & a_{2n} & 0 \\ a_{n+1} & a_{n+2} & \cdots & 0 & 0 \end{vmatrix}.$$

The second and third terms in the above sum are equal, since the determinant of a matrix is equal to the determinant of its transpose.

Now suppose

$$\begin{vmatrix} a_0 & a_1 & \cdots & a_n & a_{n+1} \\ a_1 & a_2 & \cdots & a_{n+1} & a_{n+2} \\ \vdots & & \ddots & & \vdots \\ a_n & a_{n+1} & \cdots & a_{2n} & b_{2n+1} \\ a_{n+1} & a_{n+2} & \cdots & b_{2n+1} & b_{2n+2} \end{vmatrix} = \begin{vmatrix} a_0 & a_1 & \cdots & a_n & a_{n+1} \\ a_1 & a_2 & \cdots & a_{n+1} & a_{n+2} \\ \vdots & & \ddots & & \vdots \\ a_n & a_{n+1} & \cdots & a_{2n} & a_{2n+1} \\ a_{n+1} & a_{n+2} & \cdots & a_{2n+1} & a_{2n+2} \end{vmatrix}.$$

Expanding these determinants as above and setting them equal, we get

$$b_{2n+2} \begin{vmatrix} a_0 & a_1 & \cdots & a_n \\ a_1 & a_2 & \cdots & a_{n+1} \\ \vdots & & \ddots & \vdots \\ a_n & a_{n+1} & \cdots & a_{2n} \end{vmatrix} - b_{2n+1}^2 \begin{vmatrix} a_0 & \cdots & a_{n-1} \\ a_1 & \cdots & a_n \\ \vdots & \ddots & \vdots \\ a_{n-1} & \cdots & a_{2n-2} \end{vmatrix} - 2b_{2n+1} \begin{vmatrix} a_0 & a_1 & \cdots & a_{n-1} & a_n \\ a_1 & a_2 & \cdots & a_n & a_{n+1} \\ \vdots & & \ddots & & \vdots \\ a_{n-1} & a_n & \cdots & a_{2n-2} & a_{2n-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{2n-1} & 0 \end{vmatrix} =$$

$$a_{2n+2} \begin{vmatrix} a_0 & a_1 & \cdots & a_n \\ a_1 & a_2 & \cdots & a_{n+1} \\ \vdots & & \ddots & \vdots \\ a_n & a_{n+1} & \cdots & a_{2n} \end{vmatrix} - a_{2n+1}^2 \begin{vmatrix} a_0 & \cdots & a_{n-1} \\ a_1 & \cdots & a_n \\ \vdots & \ddots & \vdots \\ a_{n-1} & \cdots & a_{2n-2} \end{vmatrix} - 2a_{2n+1} \begin{vmatrix} a_0 & a_1 & \cdots & a_{n-1} & a_n \\ a_1 & a_2 & \cdots & a_n & a_{n+1} \\ \vdots & & \ddots & & \vdots \\ a_{n-1} & a_n & \cdots & a_{2n-2} & a_{2n-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{2n-1} & 0 \end{vmatrix}.$$

Thus, we must have

$$(b_{2n+2} - a_{2n+2}) \begin{vmatrix} a_0 & a_1 & \cdots & a_n \\ a_1 & a_2 & \cdots & a_{n+1} \\ \vdots & & \ddots & \vdots \\ a_n & a_{n+1} & \cdots & a_{2n} \end{vmatrix}$$

$$= (b_{2n+1}^2 - a_{2n+1}^2) \begin{vmatrix} a_0 & a_1 & \cdots & a_{n-1} \\ a_1 & a_2 & \cdots & a_n \\ \vdots & & \ddots & \vdots \\ a_{n-1} & a_n & \cdots & a_{2n-2} \end{vmatrix} + 2(b_{2n+1} - a_{2n+1}) \begin{vmatrix} a_0 & a_1 & \cdots & a_{n-1} & a_n \\ a_1 & a_2 & \cdots & a_n & a_{n+1} \\ \vdots & & \ddots & & \vdots \\ a_{n-1} & a_n & \cdots & a_{2n-2} & a_{2n-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{2n-1} & 0 \end{vmatrix}.$$

By Lemma 3.1, either the claim is true or

$$\begin{vmatrix} a_0 & a_1 & \cdots & a_n \\ a_1 & a_2 & \cdots & a_{n+1} \\ \vdots & & \ddots & \vdots \\ a_n & a_{n+1} & \cdots & a_{2n} \end{vmatrix}$$

must divide

$$(b_{2n+1} + a_{2n+1}) \begin{vmatrix} a_0 & a_1 & \cdots & a_{n-1} \\ a_1 & a_2 & \cdots & a_n \\ \vdots & & \ddots & \vdots \\ a_{n-1} & a_n & \cdots & a_{2n-2} \end{vmatrix} + 2 \begin{vmatrix} a_0 & a_1 & \cdots & a_{n-1} & a_n \\ a_1 & a_2 & \cdots & a_n & a_{n+1} \\ \vdots & & \ddots & & \vdots \\ a_{n-1} & a_n & \cdots & a_{2n-2} & a_{2n-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{2n-1} & 0 \end{vmatrix}.$$

But in the quotient by the ideal $I$ generated by $a_0, a_1, \ldots, a_{n-1}, a_{n+2}, a_{n+3}, \ldots, a_{2n+1}$, the first determinant becomes $a_n^{n+1}$ (up to a sign), while the sum above becomes $2a_n^n * a_{n+1}$ (again up to a sign). Since $a_n$ does not divide $a_{n+1}$ in $R/I$, the claim must be true.

□

We now turn to showing that the set of $HTP$ sequences forms a group under composition. For this, we first need the following Lemma.

**Lemma 3.5.** *Suppose that $b$ and $b'$ are two $HTP$ sequences which each preserve $a_0$ through $a_{2n}$, $n \geq 1$. Also, suppose that $b_{2n+1} + b'_{2n+1} = 2a_{2n+1}$. Then $b \circ b'$ preserves $a_0$ through $a_{2n+2}$.*

*Proof.* Let $d = b \circ b'$. Clearly $d$ preserves $a_0$ through $a_{2n}$. By Lemma 3.4, there is a $c$ such that

$$b_{2n+1} = a_{2n+1} + c \begin{vmatrix} a_0 & a_1 & \cdots & a_n \\ a_1 & a_2 & \cdots & a_{n+1} \\ \vdots & & \ddots & \vdots \\ a_n & a_{n+1} & \cdots & a_{2n} \end{vmatrix}.$$

Since $b_{2n+1} + b'_{2n+1} = 2a_{2n+1}$, we must have

$$b'_{2n+1} = a_{2n+1} - c \begin{vmatrix} a_0 & a_1 & \cdots & a_n \\ a_1 & a_2 & \cdots & a_{n+1} \\ \vdots & & \ddots & \vdots \\ a_n & a_{n+1} & \cdots & a_{2n} \end{vmatrix}.$$

Since both $b$ and $b'$ preserve $a_0$ through $a_{2n}$, it follows that $d_{2n+1} = a_{2n+1}$. Now $d_{2n+2} = a_{2n+2}$ by Lemma 3.2. □

**Theorem 3.6.** *The set of $HTP$ sequences forms a group under composition.*

*Proof.* It suffices to show that any $HTP$ sequence $b$ has a left inverse. By Lemma 3.3, either $b_1 - a_1$ is divisible by $a_0$ or $b_1 + a_1$ is divisible by $a_0$. We will assume first that $b_1 - a_1$ is divisible by $a_0$. Now let $n$ be the smallest number such that $b_{2n+1} \neq a_{2n+1}$ (so $n \geq 0$, since $b_0 = a_0$.) By Lemma 3.2, we know that $b_{2n} = a_{2n}$. Choose $c_n$ as in Lemma 3.4 (or 3.3) so that

$$b_{2n+1} = a_{2n+1} - c_n \begin{vmatrix} a_0 & a_1 & \cdots & a_n \\ a_1 & a_2 & \cdots & a_{n+1} \\ \vdots & & \ddots & \vdots \\ a_n & a_{n+1} & \cdots & a_{2n} \end{vmatrix}.$$

Then by Lemma 3.5 and Corollary 2.15, the composition $b(n, c_n) \circ b$ preserves $a_0$ through $a_{2n+2}$.

Now, we inductively define a sequence $c_0, c_1, c_2, \ldots$ so that the composition

$$b(n, c_n) \circ b(n-1, c_{n-1}) \circ \cdots \circ b(0, c_0) \circ b$$

preserves $a_0$ through $a_{2n+2}$. Thus

$$(\cdots \circ b(n, c_n) \circ b(n-1, c_{n-1}) \circ \cdots \circ b(0, c_0)) \circ b = \mathrm{id}.$$

We assumed above that $b_1 - a_1$ is divisible by $a_0$. If $b_1 + a_1$ is divisible by $a_0$, then we can reduce to the former case by replacing $b$ by $b \circ \sigma$ (see Example 1.1). Then, as above, we can find a sequence $c_0, c_1, c_2, \ldots$ so that

$$\cdots \circ b(n, c_n) \circ b(n-1, c_{n-1}) \circ \cdots \circ b(0, c_0) \circ b \circ \sigma = \mathrm{id}.$$

Composing and precomposing both sides with $\sigma$ (which satisfies $\sigma^2 = \mathrm{id}$), we get

$$\sigma \circ (\cdots \circ b(n, c_n) \circ b(n-1, c_{n-1}) \circ \cdots \circ b(0, c_0)) \circ b = \mathrm{id}.$$

□

**Lemma 3.7.** *The inverse $b(n,c)^{-1}$ of $b(n,c)$ preserves $a_0$ through $a_{2n}$ and satisfies*

$$b(n,c)^{-1}_{2n+1} = a_{2n+1} - c \begin{vmatrix} a_0 & a_1 & \cdots & a_n \\ a_1 & a_2 & \cdots & a_{n+1} \\ \vdots & & \ddots & \vdots \\ a_n & a_{n+1} & \cdots & a_{2n} \end{vmatrix}.$$

*Proof.* Since $b(n,c)$ preserves $a_0$ through $a_{2n}$ by Corollary 2.15, it is clear that $b(n,c)^{-1}$ also preserves $a_0$ through $a_{2n}$. For the second part, $b(n,-c) = b(n,-c) \circ b(n,c) \circ b(n,c)^{-1}$. Since $b(n,-c) \circ b(n,c)$ preserves $a_0$ through $a_{2n+2}$ by Lemma 3.5 and Corollary 2.15, it follows that $b(n,c)^{-1}$ must have the same $2n+1$ term as $b(n,-c)$. □

*Remark* 3.8. We do not know whether or not $b(n,c)^{-1} = b(n,-c)$ in general.

We now prove our main theorem.

*Proof.* As in Theorem 3.6, we can inductively define $c_0, c_1, \ldots$ such that either

$$b' := b \circ b(0,c_0)^{-1} \circ b(1,c_1)^{-1} \circ \cdots \circ b(n,c_n)^{-1}$$

or

$$b' := b \circ \sigma \circ b(0,c_0)^{-1} \circ b(1,c_1)^{-1} \circ \cdots \circ b(n,c_n)^{-1}$$

preserves $a_0$ through $a_{2n+2}$. Here, we use Lemma 3.7 together with Lemma 3.5 to complete the inductive step. Now either $b$ or $b \circ \sigma$ can be written as

$$b' \circ b(n,c_n) \circ \cdots \circ b(1,c_1) \circ b(0,c_0).$$

It follows that

$$\cdots \circ b(n,c_n) \circ \cdots \circ b(1,c_1) \circ b(0,c_0)$$

agrees on all terms with $b$ or $b \circ \sigma$. In the latter case, we multiply both sides by $\sigma$. □

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