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A NOTE ON INTEGRAL INEQUALITIES AND EMBEDDINGS OF BESOV SPACES

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ABSTRACT. It is shown that certain known integral inequalities imply directly a well-known embedding theorem of Besov spaces.

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In [7] the authors prove a theorem which links estimates on the modulus of continuity of a real-valued function to the finiteness of a certain integral. Their result reads as follows:

Theorem 1. *Let* $\Psi : \mathbb{R} \to \mathbb{R}_+$ *satisfy* $\Psi(\xi) = \Psi(-\xi)$, $\Psi(\infty) = \infty$ *and* Ψ *non-decreasing for* $\xi \geq 0$. *Let* $p : [-1,1] \to \mathbb{R}_+$ *be continuous and satisfy* $p(\xi) = p(-\xi)$, $p(0) = 0$ *and* p *non-decreasing for* $\xi \geq 0$. *Set*

$$\Psi^{-1}(\xi) = \sup\{\eta, \Psi(\eta) \leq \xi\} \quad \textit{for } \xi \geq \Psi(0) \quad \textit{and} \tag{1}$$

$$p^{-1}(\xi) = \max\{\eta, p(\eta) \leq \xi\} \quad \textit{for } 0 \leq \xi \leq p(1)\ . \tag{2}$$

If one has for a function $f \in C([0,1])$ *that*

$$\int_0^1 \int_0^1 \Psi\left(\frac{f(x) - f(y)}{p(x-y)}\right) dx\, dy \leq B < \infty,$$

then one has for all $s, t \in [0,1]$*:*

$$|f(s) - f(t)| \leq 8 \int_0^{|s-t|} \Psi^{-1}\left(\frac{4B}{\xi^2}\right) dp(\xi).$$

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As shown in [6] Theorem 1 can be improved in certain boundary cases. Here we aim to show that Theorem 1, known as the GRR-lemma, can be used to derive directly an embedding theorem from certain Besov spaces into the spaces of Hölder continuous functions. Although this observation is straightforward it is remarkable since the proof of the GRR-lemma is not very complicated. Moreover one gets a better understanding of the arising constant.

Originally, the authors of [7] apply Theorem 1 to study continuity of Gaussian processes. Another application of this theorem is an easy derivation of the Kolmogorov-Prohorov criterion for weak compactness of probability measures or an extension of the Burkholder-Davis-Gundy inequality, see [4]. A generalized version of Theorem 1 has been obtained in [2] and used in [8] where upper bounds for the growth of the diameter of a given set exposed in a diffusive stochastic flow are proved. The $n-$dimensional version of Theorem 1 reads in one of its possible forms as follows [1]:

Theorem 2. *Let (X,d) and (Y,ρ) be metric spaces. Let $f : X \to Y$ be a continuous function and let m be a nonnegative Radon measure. Let further $\Psi : \mathbb{R}_+ \to \mathbb{R}_+$ be a right-continuous function, nondecreasing satisfying $\Psi(0) = 0$ and $\Psi(x) > 0$ for all $x > 0$. Define Ψ^{-1} as in (1). Assume that:*

$$V := \int\int \Psi\left(\frac{\rho(f(x),f(y))}{d(x,y)}\right) m(dx)\, m(dy) < \infty.$$

Then one has for all $x,y \in X$:

$$\rho(f(x),f(y)) \leq 6 \int_0^{d(x,y)} \left\{ \Psi^{-1}\left(\frac{4V}{m(B_r(x))^2}\right) + \Psi^{-1}\left(\frac{4V}{m(B_r(y))^2}\right)\right\} dr.$$

Note that here the assumptions on Ψ vary slightly from the ones made in Theorem 1. Let us define Sobolev-spaces of fractional order. For a given open connected set $\Omega \subset \mathbb{R}^n$, and parameters $s \in (0,1)$, $p \geq 1$ the Banach-space $W^{(s,p)}(\Omega)$ is defined as the set of all functions $f \in L^2(\Omega)$ for which the norm

$$\|f\|^p_{s,p,\Omega} := \int_\Omega |f|^p\, dx + \int_\Omega\int_\Omega \frac{|f(x)-f(y)|^p}{|x-y|^{n+sp}}\, dy\, dx$$

is finite. These spaces are called Sobolev-Slobodecki spaces and form a special case of the so called Besov spaces. They appear naturally as trace spaces of Sobolev spaces of integer order of differentiation and in the study of boundary value problems for partial differential equations. The monographs [1, 10, 3, 11, 12, 9] are a good choice out of the broad literature on Besov spaces and embedding theorems.

The following well-known embedding theorem follows from Theorem 2.

Theorem 3. *Assume that $s \in \left(\frac{1}{2}, 1\right)$ and $\frac{n}{s} < p \leq \frac{n}{1-s}$. Consider a open connected set $\Omega \subset \mathbb{R}^n$. If Ω satisfies the property that $C(\Omega)$ is dense in $W^{(s,p)}(\Omega)$ then bounded sets of $W^{(s,p)}(\Omega)$ are also bounded sets in $C^\alpha(\Omega)$ with $\alpha \leq s - \frac{n}{p}$.*

The assumption that $C(\Omega)$ is dense in $W^{(s,p)}(\Omega)$ is satisfied for nice sets like $\Omega = \mathbb{R}^n$. The assumption stays valid for a wide class of domains, for this subtle matter the reader is referred to [5, 1, 9, 11, 10]. The theorem holds without the restriction $p \leq \frac{n}{1-s}$. This is the only concession to the use of Theorem 2.

Proof of Theorem 3. Choose $k = n + sp > 2n$. Choose $\gamma = \frac{p}{n+sp}$. Note that $\gamma \leq 1$. Choose in Theorem 2 $X = Y = \mathbb{R}^n$, $d(x,y) = |x-y|$, $\rho(x,y) = |x-y|^\gamma$. Let m be the Lebesgue measure supported on Ω. Choose $\Psi(z) = z^k$ and note that Ψ satisfies all assumptions in Theorem 2. Let

[1]The author thanks M. Scheutzow for providing him with his notes on the GRR-lemma.

S be a set in $W^{s,p}(\Omega)$ such that $\|f\|_{s,p,\Omega} \leq K$. Since $k\gamma = p$ the assumptions yield that for $f \in S$:

$$V = \int\int \Psi\left(\frac{\rho(f(x), f(y))}{d(x,y)}\right) m(dx)\, m(dy) = \int_\Omega \int_\Omega \left(\frac{|f(x) - f(y)|^\gamma}{|x-y|}\right)^k dy\, dx < C.$$

Theorem 2 now states that for any $x, y \in \Omega$:

$$\begin{aligned} |f(x) - f(y)|^\gamma &\leq 12 \int_0^{|x-y|} \Psi^{-1}\left(\frac{4V}{C(n)r^{2n}}\right) dr \\ &\leq C(n,k)(4V)^{\frac{1}{k}} \left(\frac{k}{k-2n}\right) |x-y|^{\left(\frac{k-2n}{k}\right)}. \end{aligned}$$

This leads to:

$$|f(x) - f(y)| \leq C(n,k,V)|x-y|^{\left(\frac{k-2n}{k\gamma}\right)} = C(s,p,V)|x-y|^{s-\frac{n}{p}}.$$

The theorem is thus proved. □

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