## ON SOME INEQUALITIES WITH POWER-EXPONENTIAL FUNCTIONS

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Abstract:	In this paper, we prove the open inequality $a^{ea} + b^{eb} \ge a^{eb} + b^{ea}$ for either	F
	$a \ge b \ge \frac{1}{e}$ or $\frac{1}{e} \ge a \ge b > 0$ . In addition, other related results and conjectures	
	are presented.	



Vasile Cîrtoaje vol. 10, iss. 1, art. 21, 2009

Title Page		
Contents		
••	••	
•	►	
Page 1 of 14		
Go Back		
Full Screen		
Close		

## journal of **inequalities** in pure and applied mathematics

## Contents

1	Introduction	3	
2	Main Results	4	
3	Proofs of Theorems	5	
4	Other Related Inequalities	10	



#### journal of inequalities in pure and applied mathematics

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## 1. Introduction

In 2006, A. Zeikii posted and proved on the Mathlinks Forum [1] the following inequality

$$(1.1) a^a + b^b \ge a^b + b^a$$

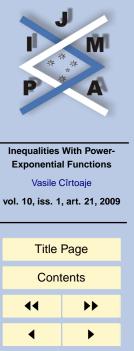
where a and b are positive real numbers less than or equal to 1. In addition, he conjectured that the following inequality holds under the same conditions:

(1.2) 
$$a^{2a} + b^{2b} \ge a^{2b} + b^{2a}.$$

Starting from this, we have conjectured in [1] that

$$(1.3) a^{ea} + b^{eb} \ge a^{eb} + b^{ea}$$

for all positive real numbers a and b.



mathematics issn: 1443-5756

Page 3 of 14

Go Back

Full Screen

Close

journal of inequalities in pure and applied

## 2. Main Results

In what follows, we will prove some relevant results concerning the power-exponential inequality

$$(2.1) a^{ra} + b^{rb} \ge a^{rb} + b^{ra}$$

for a, b and r positive real numbers. We will prove the following theorems.

**Theorem 2.1.** Let r, a and b be positive real numbers. If (2.1) holds for  $r = r_0$ , then it holds for any  $0 < r \le r_0$ .

**Theorem 2.2.** If a and b are positive real numbers such that  $\max\{a, b\} \ge 1$ , then (2.1) holds for any positive real number r.

**Theorem 2.3.** If  $0 < r \le 2$ , then (2.1) holds for all positive real numbers a and b.

**Theorem 2.4.** If a and b are positive real numbers such that either  $a \ge b \ge \frac{1}{r}$  or  $\frac{1}{r} \ge a \ge b$ , then (2.1) holds for any positive real number  $r \le e$ .

**Theorem 2.5.** If r > e, then (2.1) does not hold for all positive real numbers a and b.

From the theorems above, it follows that the inequality (2.1) continues to be an open problem only for  $2 < r \le e$  and  $0 < b < \frac{1}{r} < a < 1$ . For the most interesting value of r, that is r = e, only the case  $0 < b < \frac{1}{e} < a < 1$  is not yet proved.





#### journal of inequalities in pure and applied mathematics

## **3. Proofs of Theorems**

*Proof of Theorem 2.1.* Without loss of generality, assume that  $a \ge b$ . Let x = ra and y = rb, where  $x \ge y$ . The inequality (2.1) becomes

(3.1) 
$$x^{x} - y^{x} \ge r^{x-y}(x^{y} - y^{y})$$

By hypothesis,

$$x^{x} - y^{x} \ge r_{0}^{x-y}(x^{y} - y^{y}).$$

Since  $x - y \ge 0$  and  $x^y - y^y \ge 0$ , we have  $r_0^{x-y}(x^y - y^y) \ge r^{x-y}(x^y - y^y)$ , and hence

$$x^{x} - y^{x} \ge r_{0}^{x-y}(x^{y} - y^{y}) \ge r^{x-y}(x^{y} - y^{y})$$

*Proof of Theorem 2.2.* Without loss of generality, assume that  $a \ge b$  and  $a \ge 1$ . From  $a^{r(a-b)} \ge b^{r(a-b)}$ , we get  $b^{rb} \ge \frac{a^{rb}b^{ra}}{a^{ra}}$ . Therefore,

$$a^{ra} + b^{rb} - a^{rb} - b^{ra} \ge a^{ra} + \frac{a^{rb}b^{ra}}{a^{ra}} - a^{rb} - b^{ra}$$
$$= \frac{(a^{ra} - a^{rb})(a^{ra} - b^{ra})}{a^{ra}} \ge 0$$

because  $a^{ra} \ge a^{rb}$  and  $a^{ra} \ge b^{ra}$ .

*Proof of Theorem 2.3.* By Theorem 2.1 and Theorem 2.2, it suffices to prove (2.1) for r = 2 and 1 > a > b > 0. Setting  $c = a^{2b}$ ,  $d = b^{2b}$  and  $s = \frac{a}{b}$  (where c > d > 0 and s > 1), the desired inequality becomes

$$c^s - d^s \ge c - d.$$



Inequalities With Power- Exponential Functions			
Vasile (	Vasile Cîrtoaje		
vol. 10, iss. 1	vol. 10, iss. 1, art. 21, 2009		
Title Page			
Contents			
44	••		
•	•		
Page 5 of 14			
Go Back			
Full Screen			
Close			

## journal of inequalities in pure and applied mathematics

In order to prove this inequality, we show that

(3.2) 
$$c^s - d^s > s(cd)^{\frac{s-1}{2}}(c-d) > c-d.$$

The left side of the inequality in (3.2) is equivalent to f(c) > 0, where  $f(c) = c^s - d^s - s(cd)^{\frac{s-1}{2}}(c-d)$ . We have  $f'(c) = \frac{1}{2}sc^{\frac{s-3}{2}}g(c)$ , where

$$g(c) = 2c^{\frac{s+1}{2}} - (s+1)cd^{\frac{s-1}{2}} + (s-1)d^{\frac{s+1}{2}}$$

Since

$$g'(c) = (s+1)\left(c^{\frac{s-1}{2}} - d^{\frac{s-1}{2}}\right) > 0,$$

g(c) is strictly increasing, g(c) > g(d) = 0, and hence f'(c) > 0. Therefore, f(c) is strictly increasing, and then f(c) > f(d) = 0.

The right side of the inequality in (3.2) is equivalent to

$$\frac{a}{b}(ab)^{a-b} > 1.$$

Write this inequality as f(b) > 0, where

$$f(b) = \frac{1+a-b}{1-a+b} \ln a - \ln b.$$

In order to prove that f(b) > 0, it suffices to show that f'(b) < 0 for all  $b \in (0, a)$ ; then f(b) is strictly decreasing, and hence f(b) > f(a) = 0. Since

$$f'(b) = \frac{-2}{(1-a+b)^2} \ln a - \frac{1}{b},$$

the inequality f'(b) < 0 is equivalent to g(a) > 0, where

$$g(a) = 2\ln a + \frac{(1-a+b)^2}{b}$$



Inequalities With Power-**Exponential Functions** Vasile Cîrtoaje vol. 10, iss. 1, art. 21, 2009 **Title Page** Contents 44 ◀ Page 6 of 14 Go Back Full Screen Close

#### journal of inequalities in pure and applied mathematics

Since 0 < b < a < 1, we have

$$g'(a) = \frac{2}{a} - \frac{2(1-a+b)}{b} = \frac{2(a-1)(a-b)}{ab} < 0$$

Thus, g(a) is strictly decreasing on [b, 1], and therefore g(a) > g(1) = b > 0. This completes the proof. Equality holds if and only if a = b.

*Proof of Theorem 2.4.* Without loss of generality, assume that  $a \ge b$ . Let x = ra and y = rb, where either  $x \ge y \ge 1$  or  $1 \ge x \ge y$ . The inequality (2.1) becomes

$$x^x - y^x \ge r^{x-y}(x^y - y^y)$$

Since  $x \ge y$ ,  $x^y - y^y \ge 0$  and  $r \le e$ , it suffices to show that

(3.3) 
$$x^{x} - y^{x} \ge e^{x - y} (x^{y} - y^{y}).$$

For the nontrivial case x > y, using the substitutions  $c = x^y$  and  $d = y^y$  (where c > d), we can write (3.3) as

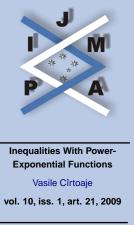
$$c^{\frac{x}{y}} - d^{\frac{x}{y}} \ge e^{x-y}(c-d).$$

In order to prove this inequality, we will show that

$$c^{\frac{x}{y}} - d^{\frac{x}{y}} > \frac{x}{y}(cd)^{\frac{x-y}{2y}}(c-d) > e^{x-y}(c-d)$$

The left side of the inequality is just the left hand inequality in (3.2) for  $s = \frac{x}{y}$ , while the right side of the inequality is equivalent to

$$\frac{x}{y}(xy)^{\frac{x-y}{2}} > e^{x-y}$$



## journal of inequalities in pure and applied mathematics

We write this inequality as f(x) > 0, where

$$f(x) = \ln x - \ln y + \frac{1}{2}(x - y)(\ln x + \ln y) - x + y.$$

We have

$$f'(x) = \frac{1}{x} + \frac{\ln(xy)}{2} - \frac{y}{2x} - \frac{1}{2}$$

and

$$f''(x) = \frac{x + y - 2}{2x^2}.$$

Case  $x > y \ge 1$ . Since f''(x) > 0, f'(x) is strictly increasing, and hence

$$f'(x) > f'(y) = \frac{1}{y} + \ln y - 1$$

Let  $g(y) = \frac{1}{y} + \ln y - 1$ . From  $g'(y) = \frac{y-1}{y^2} > 0$ , it follows that g(y) is strictly increasing,  $g(y) \ge g(1) = 0$ , and hence f'(x) > 0. Therefore, f(x) is strictly increasing, and then f(x) > f(y) = 0.

Case  $1 \ge x > y$ . Since f''(x) < 0, f(x) is strictly concave on [y, 1]. Then, it suffices to show that  $f(y) \ge 0$  and f(1) > 0. The first inequality is trivial, while the second inequality is equivalent to g(y) > 0 for 0 < y < 1, where

$$g(y) = \frac{2(y-1)}{y+1} - \ln y.$$

From

$$g'(y) = \frac{-(y-1)^2}{y(y+1)^2} < 0,$$



Inequalities With Power-**Exponential Functions** Vasile Cîrtoaje vol. 10, iss. 1, art. 21, 2009 **Title Page** Contents 44 Þ Page 8 of 14 Go Back Full Screen Close

## journal of inequalities in pure and applied mathematics

it follows that g(y) is strictly decreasing, and hence g(y) > g(1) = 0. This completes the proof.

Equality holds if and only if a = b.

Proof of Theorem 2.5. (after an idea of Wolfgang Berndt [1]). We will show that

 $a^{ra} + b^{rb} < a^{rb} + b^{ra}$ 

for r = (x+1)e,  $a = \frac{1}{e}$  and  $b = \frac{1}{r} = \frac{1}{(x+1)e}$ , where x > 0; that is

$$xe^x + \frac{1}{(x+1)^x} > x+1.$$

Since  $e^x > 1 + x$ , it suffices to prove that

$$\frac{1}{(x+1)^x} > 1 - x^2.$$

For the nontrivial case 0 < x < 1, this inequality is equivalent to f(x) < 0, where

$$f(x) = \ln(1 - x^2) + x\ln(x + 1).$$

We have

$$f'(x) = \ln(x+1) - \frac{x}{1-x}$$

and

$$f''(x) = \frac{x(x-3)}{(1+x)(1-x)^2}.$$

Since f''(x) < 0, f'(x) is strictly decreasing for 0 < x < 1, and then f'(x) < f'(0) = 0. Therefore, f(x) is strictly decreasing, and hence f(x) < f(0) = 0.  $\Box$ 



# in pure and applied mathematics

## 4. Other Related Inequalities

**Proposition 4.1.** If a and b are positive real numbers such that  $\min\{a, b\} \le 1$ , then the inequality

(4.1) 
$$a^{-ra} + b^{-rb} \le a^{-rb} + b^{-ra}$$

holds for any positive real number r.

*Proof.* Without loss of generality, assume that  $a \leq b$  and  $a \leq 1$ . From  $a^{r(b-a)} \leq b^{r(b-a)}$  we get  $b^{-rb} \leq \frac{a^{-rb}b^{-ra}}{a^{-ra}}$ , and

$$a^{-ra} + b^{-rb} - a^{-rb} - b^{-ra} \le a^{-ra} + \frac{a^{-rb}b^{-ra}}{a^{-ra}} - a^{-rb} - b^{-ra}$$
$$= \frac{(a^{-ra} - a^{-rb})(a^{-ra} - b^{-ra})}{a^{-ra}} \le 0,$$

because  $b^{-ra} \leq a^{-ra} \leq a^{-rb}$ .

**Proposition 4.2.** If a, b, c are positive real numbers, then

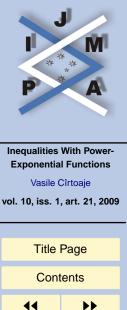
(4.2)  $a^a + b^b + c^c \ge a^b + b^c + c^a.$ 

This inequality, with  $a, b, c \in (0, 1)$ , was posted as a conjecture on the Mathlinks Forum by Zeikii [1].

*Proof.* Without loss of generality, assume that  $a = \max\{a, b, c\}$ . There are three cases to consider:  $a \ge 1$ ,  $c \le b \le a < 1$  and  $b \le c \le a < 1$ .

Case  $a \ge 1$ . By Theorem 2.3, we have  $b^b + c^c \ge b^c + c^b$ . Thus, it suffices to prove that

$$a^a + c^b \ge a^b + c^a$$



vol. 10, iss. 1, art. 21, 2009		
Title Page		
Contents		
••	••	
•		
Page 10 of 14		
Go Back		
Full Screen		
Close		
ournal of inequalities		

#### journal of inequalities in pure and applied mathematics

For a = b, this inequality is an equality. Otherwise, for a > b, we substitute  $x = a^b$ ,  $y = c^b$  and  $s = \frac{a}{b}$  (where  $x \ge 1$ ,  $x \ge y$  and s > 1) to rewrite the inequality as  $f(x) \ge 0$ , where

$$f(x) = x^s - x - y^s + y.$$

Since

$$f'(x) = sx^{s-1} - 1 \ge s - 1 > 0,$$

f(x) is strictly increasing for  $x \ge y$ , and therefore  $f(x) \ge f(y) = 0$ .

Case  $c \le b \le a < 1$ . By Theorem 2.3, we have  $a^a + b^b \ge a^b + b^a$ . Thus, it suffices to show that

$$b^a + c^c \ge b^c + c^a,$$

which is equivalent to  $f(b) \ge f(c)$ , where  $f(x) = x^a - x^c$ . This inequality is true if  $f'(x) \ge 0$  for  $c \le x \le b$ . From

$$f'(x) = ax^{a-1} - cx^{c-1}$$
  
=  $x^{c-1}(ax^{a-c} - c)$   
 $\ge x^{c-1}(ac^{a-c} - c) = x^{c-1}c^{a-c}(a - c^{1-a+c}),$ 

we need to show that  $a - c^{1-a+c} \ge 0$ . Since  $0 < 1 - a + c \le 1$ , by Bernoulli's inequality we have

$$c^{1-a+c} = (1+(c-1))^{1-a+c}$$
  

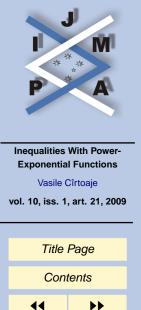
$$\leq 1+(1-a+c)(c-1) = a - c(a-c) \leq a.$$

Case  $b \le c \le a < 1$ . The proof of this case is similar to the previous case. So the proof is completed.

Equality holds if and only if a = b = c.

**Conjecture 4.3.** If a, b, c are positive real numbers, then

(4.3)  $a^{2a} + b^{2b} + c^{2c} \ge a^{2b} + b^{2c} + c^{2a}.$ 



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Page 11 of 14

Go Back

Full Screen

Close

journal of inequalities

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in pure and applied

mathematics

**Conjecture 4.4.** *Let r be a positive real number. The inequality* 

(4.4) 
$$a^{ra} + b^{rb} + c^{rc} \ge a^{rb} + b^{rc} + c^{ra}$$

holds for all positive real numbers a, b, c with  $a \le b \le c$  if and only if  $r \le e$ .

We can prove that the condition  $r \le e$  in Conjecture 4.4 is necessary by setting c = b and applying Theorem 2.5.

**Proposition 4.5.** If a and b are nonnegative real numbers such that a + b = 2, then

(4.5) 
$$a^{2b} + b^{2a} \le 2.$$

*Proof.* We will show the stronger inequality

$$a^{2b} + b^{2a} + \left(\frac{a-b}{2}\right)^2 \le 2$$

Without loss of generality, assume that  $a \ge b$ . Since  $0 \le a - 1 < 1$  and  $0 < b \le 1$ , by Bernoulli's inequality we have

$$a^b \le 1 + b(a-1) = 1 + b - b^2$$

and

$$b^{a} = b \cdot b^{a-1} \le b[1 + (a-1)(b-1)] = b^{2}(2-b).$$

Therefore,

$$a^{2b} + b^{2a} + \left(\frac{a-b}{2}\right)^2 - 2 \le (1+b-b^2)^2 + b^4(2-b)^2 + (1-b)^2 - 2$$
$$= b^3(b-1)^2(b-2) \le 0.$$



Exponential Functions Vasile Cîrtoaje		
vol. 10, iss. 1, art. 21, 2009		
Title Page		
Contents		
••	••	
•	►	
Page 12 of 14		
Go Back		
Full Screen		
Close		

#### journal of inequalities in pure and applied mathematics

issn: 1443-5756

**Conjecture 4.6.** *Let r be a positive real number. The inequality* 

$$(4.6) a^{rb} + b^{ra} \le 2$$

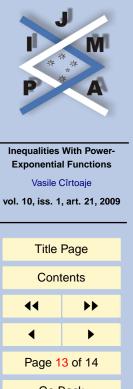
holds for all nonnegative real numbers a and b with a + b = 2 if and only if  $r \leq 3$ .

**Conjecture 4.7.** If a and b are nonnegative real numbers such that a + b = 2, then

(4.7) 
$$a^{3b} + b^{3a} + \left(\frac{a-b}{2}\right)^4 \le 2$$

**Conjecture 4.8.** If a and b are nonnegative real numbers such that a + b = 1, then

(4.8) 
$$a^{2b} + b^{2a} \le 1.$$



Go Back

Full Screen

Close

#### journal of inequalities in pure and applied mathematics

## References

[1] A. ZEIKII, V. CÎRTOAJE AND W. BERNDT, Mathlinks Forum, Nov. 2006, [ONLINE: http://www.mathlinks.ro/Forum/viewtopic. php?t=118722].



Exponential Functions Vasile Cîrtoaje vol. 10, iss. 1, art. 21, 2009		
Title Page		
Contents		
44	••	
•		
Page 14 of 14		
Go Back		
Full Screen		
Close		

#### journal of inequalities in pure and applied mathematics