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RAMANUJAN'S HARMONIC NUMBER EXPANSION INTO NEGATIVE POWERS OF A TRIANGULAR NUMBER

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ABSTRACT. An algebraic transformation of the *DeTemple–Wang* half-integer approximation to the harmonic series produces the general formula and error estimate for the *Ramanujan* expansion for the nth harmonic number into negative powers of the nth triangular number. We also discuss the history of the *Ramanujan* expansion for the nth harmonic number as well as sharp estimates of its accuracy, with complete proofs, and we compare it with other approximative formulas.

Key words and phrases: Inequalities for sums, series and integrals, Approximation to limiting values.

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1. INTRODUCTION

1.1. **The Harmonic Series.** In 1350, Nicholas Oresme proved that the celebrated *Harmonic Series*,

$$\boxed{1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}+\cdots,} \tag{1.1}$$

is *divergent*. (Note: we use boxes around some of the displayed formulas to emphasize their importance.) He actually proved a more precise result. If the n^{th} partial sum of the harmonic series, today called the n^{th} *harmonic number*, is denoted by the symbol H_n:

$$\boxed{H_n := 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n},} \tag{1.2}$$

then the inequality

$$H_{2^k} > \frac{k+1}{2} \tag{1.3}$$

holds for $k = 2, 3, \ldots$. This inequality gives a lower bound for the *speed* of divergence.

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Almost four hundred years passed until Leonhard Euler, in 1755 [3] applied the Euler–Maclaurin sum formula to find the famous standard Euler *asymptotic* expansion for H_n,

$$\begin{aligned} H_n &:= \sum_{k=1}^{n} \frac{1}{k} \sim \ln n + \gamma + \frac{1}{2n} - \frac{1}{12n^2} + \frac{1}{120n^4} - [\cdots] \\ &= \ln n + \gamma - \sum_{k=1}^{\infty} \frac{B_k}{n^k}, \end{aligned} \tag{1.4}$$

where B_k denotes the k^{th} Bernoulli number and $\gamma := 0.57721\cdots$ is Euler's constant. This gives a complete answer to the speed of divergence of H_n in powers of $\frac{1}{n}$.

Since then many mathematicians have contributed other approximative formulas for H_n and have studied the rate of divergence. We will present a detailed study of such a formula stated by Ramanujan, with complete proofs, as well as of some related formulas.

1.2. **Ramanujan's Formula.** Entry 9 of Chapter 38 of B. Berndt's edition of Ramanujan's Notebooks [2, p. 521] reads,

"*Let $m := \frac{n(n+1)}{2}$, where n is a positive integer. Then, as n approaches infinity,*

$$\begin{aligned} \sum_{k=1}^{n} \frac{1}{k} \sim \frac{1}{2}\ln(2m) + \gamma + \frac{1}{12m} - \frac{1}{120m^2} + \frac{1}{630m^3} - \frac{1}{1680m^4} + \frac{1}{2310m^5} \\ - \frac{191}{360360m^6} + \frac{29}{30030m^7} - \frac{2833}{1166880m^8} + \frac{140051}{17459442m^9} - [\cdots]." \end{aligned} \tag{1.5}$$

We note that $m := \frac{n(n+1)}{2}$ is the nth *triangular number*, so that Ramanujan's expansion of H_n is into *powers of the reciprocal of the n^{th} triangular number.*

Berndt's proof simply verifies (as he himself explicitly notes) that Ramanujan's expansion coincides with the standard Euler expansion (1.4).

However, Berndt does not give the *general formula* for the coefficient of $\frac{1}{m^k}$ in Ramanujan's expansion, nor does he prove that it is an *asymptotic* series in the sense that the error in the value obtained by stopping at any particular stage in Ramanujan's series is less than the next term in the series. Indeed we have been unable to find *any* error analysis of Ramanujan's series.

We will prove the following theorem.

Theorem 1.1. *For any integer $p \geq 1$ define*

$$\boxed{R_p := \frac{(-1)^{p-1}}{2p \cdot 8^p} \left\{ 1 + \sum_{k=1}^{p} \binom{p}{k} (-4)^k B_{2k}(\tfrac{1}{2}) \right\}} \tag{1.6}$$

where $B_{2k}(x)$ is the Bernoulli polynomial of order $2k$. Put

$$\boxed{m := \frac{n(n+1)}{2}} \tag{1.7}$$

where n is a positive integer. Then, for every integer $r \geq 1$, there exists a Θ_r, $0 < \Theta_r < 1$, for which the following equation is true:

$$\boxed{1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} = \frac{1}{2}\ln(2m) + \gamma + \sum_{p=1}^{r} \frac{R_p}{m^p} + \Theta_r \cdot \frac{R_{r+1}}{m^{r+1}}.} \tag{1.8}$$

We observe that the formula for R_p can be written symbolically as follows:

$$R_p = -\frac{1}{2p}\left(\frac{4B^2-1}{8}\right)^p, \tag{1.9}$$

where we write $B_{2m}(\frac{1}{2})$ in place of B^{2m} after carrying out the above expansion.

We will also trace the history of Ramanujan's expansion as well and discuss the relative accuracy of his approximation when compared to other approximative formulas proposed by mathematicians.

1.3. **History of Ramanujan's Formula.** In 1885, two years before Ramanujan was born, Cesàro [4] proved the following.

Theorem 1.2. *For every positive integer $n \geq 1$ there exists a number c_n, $0 < c_n < 1$, such that the following approximation is valid:*

$$\boxed{H_n = \frac{1}{2}\ln(2m) + \gamma + \frac{c_n}{12m}\,.}$$ □

This gives the first two terms of Ramanujan's expansion, with an error term. The method of proof, different from ours, does not lend itself to generalization. We believe Cesàro's paper to be the first appearance in the literature of Ramanujan's expansion.

Then, in 1904, Lodge, in a very interesting paper [8], which later mathematicians inexplicably (in our opinion) ignored, proved a version of the following two results.

Theorem 1.3. *For every positive integer n, define the quantity λ_n by the following equation:*

$$\boxed{1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} := \frac{1}{2}\ln(2m) + \gamma + \frac{1}{12m + \frac{6}{5}} + \lambda_n\,.} \tag{1.10}$$

Then

$$0 < \lambda_n < \frac{19}{25200m^3}.$$

In fact,

$$\lambda_n = \frac{19}{25200m^3} - \rho_n, \quad \text{where} \quad 0 < \rho_n < \frac{43}{84000m^4}.$$

The constants $\frac{19}{25200}$ and $\frac{43}{84000}$ are the best possible.

Theorem 1.4. *For every positive integer n, define the quantity Λ_n by the following equation:*

$$\boxed{1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} =: \frac{1}{2}\ln(2m) + \gamma + \frac{1}{12m + \Lambda_n}\,.} \tag{1.11}$$

Then

$$\Lambda_n = \frac{6}{5} - \frac{19}{175m} + \frac{13}{250m^2} - \frac{\delta_n}{m^3},$$

where $0 < \delta_n < \frac{187969}{4042500}$. The constants in the expansion of Λ_n all are the best possible.

These two theorems appeared, in much less precise form and *with no error estimates*, in Lodge [8]. Lodge gives some numerical examples of the error in the approximative equation

$$H_n \approx \frac{1}{2}\ln(2m) + \gamma + \frac{1}{12m + \frac{6}{5}}$$

in Theorem 1.3; he also presents the first two terms of Λ_n from Theorem 1.4. An asymptotic error estimate for Theorem 1.3 (with the incorrect constant $\frac{1}{150}$ instead of $\frac{1}{165\frac{15}{19}}$) appears as Exercise 19 on page 460 in Bromwich [3].

Theorem 1.3 and Theorem 1.4 are immediate corollaries of Theorem 1.1.

The next appearance of the expansion of H_n, into powers of the reciprocal of the n^{th} triangular number, $m = \frac{1}{\frac{n(n+1)}{2}}$, is Ramanujan's own expansion (1.5).

1.4. **Sharp Error Estimates.** Mathematicians have continued to offer alternate approximative formulas to Euler's. We cite the following formulas, which appear in order of increasing accuracy.

No.	Approximative Formula for H_n	Type	Asymptotic Error Estimate
1	$\ln n + \gamma + \dfrac{1}{2n}$	overestimates	$\dfrac{1}{12n^2}$
2	$\ln n + \gamma + \dfrac{1}{2n + \frac{1}{3}}$	underestimates	$\dfrac{1}{72n^3}$
3	$\ln \sqrt{n(n+1)} + \gamma + \dfrac{1}{6n(n+1) + \frac{6}{5}}$	overestimates	$\dfrac{1}{165\frac{15}{19}[n(n+1)]^3}$
4	$\ln(n + \frac{1}{2}) + \gamma + \dfrac{1}{24(n+\frac{1}{2})^2 + \frac{21}{5}}$	overestimates	$\dfrac{1}{389\frac{781}{2071}(n+\frac{1}{2})^6}$

Formula 1 is the original Euler approximation, and it *overestimates* the true value of H_n by terms of order $\frac{1}{12n^2}$.

Formula 2 is the Tóth–Mare approximation, see [9], and it *underestimates* the true value of H_n by terms of order $\frac{1}{72n^3}$.

Formula 3 is the Ramanujan–Lodge approximation, and it *overestimates* the true value of H_n by terms of order $\frac{19}{3150[n(n+1)]^3}$, see [10].

Formula 4 is the DeTemple–Wang approximation, and it *overestimates* the true value of H_n by terms of order $\frac{2071}{806400(n+\frac{1}{2})^6}$, see [6].

In 2003, Chao-Ping Chen and Feng Qi [5] gave a proof of the following sharp form of the Tóth–Mare approximation.

Theorem 1.5. *For any natural number $n \geq 1$, the following inequality is valid:*

$$\boxed{\frac{1}{2n + \frac{1}{1-\gamma} - 2} \leq H_n - \ln n - \gamma < \frac{1}{2n + \frac{1}{3}}}\,. \tag{1.12}$$

The constants $\frac{1}{1-\gamma} - 2 = .3652721\cdots$ and $\frac{1}{3}$ are the best possible, and equality holds only for $n = 1$.

The first *statement* of this theorem had been announced ten years earlier by the editors of the "Problems" section of the *American Mathematical Monthly*, **99** (1992), p. 685, as part of a commentary on the solution of Problem E 3432, but they did not publish the proof. So, the first published proof is apparently that of Chen and Qi.

In this paper we will prove *new and sharp forms* of the Ramanujan–Lodge approximation and the DeTemple–Wang approximation.

Theorem 1.6 (Ramanujan–Lodge). *For any natural number $n \geq 1$, the following inequality is valid:*

$$\boxed{\frac{1}{6n(n+1) + \frac{6}{5}} < H_n - \ln \sqrt{n(n+1)} - \gamma \leq \frac{1}{6n(n+1) + \frac{1}{1-\gamma-\ln\sqrt{2}} - 12}}\,. \tag{1.13}$$

The constants $\frac{1\ln 2}{1-\gamma-\ln\sqrt{2}} - 12 = 1.12150934\cdots$ *and* $\frac{6}{5}$ *are the best possible, and equality holds only for* $n = 1$*.*

Theorem 1.7 (DeTemple–Wang)**.** *For any natural number* $n \geq 1$*, the following inequality is valid:*

$$\boxed{\frac{1}{24(n+\frac{1}{2})^2 + \frac{21}{5}} \leq H_n - \ln(n+\tfrac{1}{2}) - \gamma < \frac{1}{24(n+\frac{1}{2})^2 + \frac{1}{1-\ln\frac{3}{2}-\gamma} - 54}}\,. \tag{1.14}$$

The constants $\frac{1}{1-\ln\frac{3}{2}-\gamma} - 54 = 3.73929752\cdots$ *and* $\frac{21}{5}$ *are the best possible, and equality holds only for* $n = 1$*.*

DeTemple and Wang never stated this approximation to H_n explicitly. They gave the asymptotic expansion of H_n, cited below in Proposition 3.1, and we developed the corresponding approximative formulas given above.

All three theorems are corollaries of the following stronger theorem.

Theorem 1.8. *For any natural number* $n \geq 1$*, define* f_n*,* λ_n*, and* d_n *by*

$$H_n =: \ln n + \gamma + \frac{1}{2n + f_n}$$

$$=: \ln\sqrt{n(n+1)} + \gamma + \frac{1}{6n(n+1) + \lambda_n} \tag{1.15}$$

$$=: \ln(n+\tfrac{1}{2}) + \gamma + \frac{1}{24(n+\frac{1}{2})^2 + d_n}, \tag{1.16}$$

respectively. Then for any natural number $n \geq 1$ *the sequence* $\{f_n\}$ *is* ***monotonically decreasing*** *while the sequences* $\{\lambda_n\}$ *and* $\{d_n\}$ *are* ***monotonically increasing****.*

Chen and Qi [5] proved that the sequence $\{f_n\}$ *decreases* monotonically. In this paper we will use their techniques to prove the monotonicity of the sequences $\{\lambda_n\}$ and $\{d_n\}$.

2. PROOF OF THE SHARP ERROR ESTIMATES

2.1. **A Few Lemmas.**

Our proof is based on inequalities satisfied by the *digamma* function $\Psi(x)$,

$$\boxed{\Psi(x) := \frac{d}{dx}\ln\Gamma(x) \equiv \frac{\Gamma'(x)}{\Gamma(x)} \equiv -\gamma - \frac{1}{x} + x\sum_{n=1}^{\infty}\frac{1}{n(x+n)}}\,, \tag{2.1}$$

which is the generalization of H_n to the real variable x since $\Psi(x)$ and H_n satisfy the equation [1, (6.3.2), p. 258]:

$$\Psi(n+1) = H_n - \gamma. \tag{2.2}$$

Lemma 2.1. *For every* $x > 0$ *there exist numbers* θ_x *and* Θ_x*, with* $0 < \theta_x < 1$ *and* $0 < \Theta_x < 1$*, for which the following equations are true:*

$$\Psi(x+1) = \ln x + \frac{1}{2x} - \frac{1}{12x^2} + \frac{1}{120x^4} - \frac{1}{252x^6} + \frac{1}{240x^8}\theta_x, \tag{2.3}$$

$$\Psi'(x+1) = \frac{1}{x} - \frac{1}{2x^2} + \frac{1}{6x^3} - \frac{1}{30x^5} + \frac{1}{42x^7} - \frac{1}{30x^9}\Theta_x. \tag{2.4}$$

Proof. Both formulas are well known. See, for example, [7, pp. 124–125]. ☐

Lemma 2.2. *The following inequalities are true for $x > 0$:*

$$\begin{aligned}\frac{1}{3x^2}-\frac{1}{3x^3}+\frac{4}{15x^4}-\frac{1}{5x^5}+\frac{10}{63x^6}-\frac{1}{7x^7} &< 2\Psi(x+1)-\ln\{x(x+1)\}\\ &< \frac{1}{3x^2}-\frac{1}{3x^3}+\frac{4}{15x^4}-\frac{1}{5x^5}+\frac{10}{63x^6},\end{aligned}\tag{2.5}$$

$$\begin{aligned}\frac{2}{3x^3}-\frac{1}{4x^4}+\frac{16}{15x^5}-\frac{1}{x^6}+\frac{20}{21x^7}-\frac{1}{x^8} &< \frac{1}{x}+\frac{1}{x+1}-2\Psi'(x+1)\\ &< \frac{2}{3x^3}-\frac{1}{4x^4}+\frac{16}{15x^5}-\frac{1}{x^6}+\frac{20}{21x^7}.\end{aligned}\tag{2.6}$$

Proof. The inequalities (2.5) are an immediate consequence of (2.3) and the Taylor expansion of

$$-\ln x(x+1) = -2\ln x - \ln\left(1+\frac{1}{x}\right) = 2\ln\left(\frac{1}{x}\right) - \frac{1}{x} + \frac{1}{2x^2} - \frac{1}{3x^3} + [\cdots]$$

which is an alternating series with the property that its sum is bracketed by two consecutive partial sums.

For (2.6) we start with (2.4). We conclude that

$$\frac{1}{2x^2}-\frac{1}{6x^3}+\frac{1}{30x^5}-\frac{1}{36x^7} < \frac{1}{x}-\Psi'(x+1) < \frac{1}{2x^2}-\frac{1}{6x^3}+\frac{1}{30x^5}.$$

Now we multiply all three components of the inequality by 2 and add $\frac{1}{x+1}-\frac{1}{x}$ to them. □

Lemma 2.3. *The following inequalities are true for $x > 0$:*

$$\begin{aligned}&\frac{1}{(x+\frac{1}{2})}-\frac{1}{x}+\frac{1}{2x^2}-\frac{1}{6x^3}+\frac{1}{30x^5}-\frac{1}{42x^7}\\ &\qquad< \frac{1}{x+\frac{1}{2}}-\Psi'(x+1)\\ &\qquad< \frac{1}{(x+\frac{1}{2})}-\frac{1}{x}+\frac{1}{2x^2}-\frac{1}{6x^3}+\frac{1}{30x^5}\,,\end{aligned}$$

$$\begin{aligned}&\frac{1}{24x^2}-\frac{1}{24x^3}+\frac{23}{960x^4}-\frac{1}{160x^5}-\frac{11}{8064x^6}-\frac{1}{896x^7}\\ &< \Psi(x+1)-\ln(x+\tfrac{1}{2})\\ &< \frac{1}{24x^2}-\frac{1}{24x^3}+\frac{23}{960x^4}-\frac{1}{160x^5}-\frac{11}{8064x^6}-\frac{1}{896x^7}+\frac{143}{30720x^8}\,.\end{aligned}$$

Proof. Similar to the proof of Lemma 2.2. □

2.2. Proof for the Ramanujan–Lodge approximation.

Proof of Theorem 1.8 for $\{\lambda_n\}$. We solve (1.15) for λ_n and use (2.2) to obtain

$$\lambda_n = \frac{1}{\Psi(n+1)-\ln\sqrt{n(n+1)}} - 6n(n+1).$$

Define

$$\Lambda_x := \frac{1}{2\Psi(x+1)-\ln x(x+1)} - 3x(x+1),$$

for all $x > 0$. Observe that $2\Lambda_n = \lambda_n$.

We will show that that the derivative $\Lambda'_x > 0$ *for* $x > 28$. Computing the derivative we obtain

$$\Lambda'_x = \frac{\frac{1}{x} + \frac{1}{x+1} - \Psi'(x+1)}{\{2\Psi(x+1) - \ln x(x+1)\}^2} - (6x+3),$$

and therefore

$$\{2\Psi(x+1) - \ln x(x+1)\}^2 \Lambda'_x = \frac{1}{x} + \frac{1}{x+1} - \Psi'(x+1) - (6x+3)\{2\Psi(x+1) - \ln x(x+1)\}^2.$$

By Lemma 2.2, this is greater than

$$\begin{aligned} &\frac{2}{3x^3} - \frac{1}{4x^4} + \frac{16}{15x^5} - \frac{1}{x^6} + \frac{20}{21x^7} - \frac{1}{x^8} \\ &\qquad - (6x+3)\left\{\frac{1}{3x^2} - \frac{1}{3x^3} + \frac{4}{15x^4} - \frac{1}{5x^5} + \frac{10}{63x^6}\right\}^2 \\ &= \frac{798x^5 - 21693x^4 - 3654x^3 + 231x^2 + 1300x - 2500}{33075x^{12}} \\ &= \frac{(x-28)(798x^4 + 651x^3 + 14574x^2 + 408303x + 11433784) + 320143452}{33075x^{12}} \end{aligned}$$

(by the remainder theorem), which is obviously *positive* for $x > 28$. Thus, the sequence $\{\Lambda_n\}$, $n \geq 29$, is strictly *increasing*. Therefore, so is the sequence $\{\lambda_n\}$.

For $n = 1, 2, 3, \ldots, 28$, we compute λ_n directly:

$\lambda_1 = 1.1215093$	$\lambda_2 = 1.1683646$	$\lambda_3 = 1.1831718$	$\lambda_4 = 1.1896217$
$\lambda_5 = 1.1929804$	$\lambda_6 = 1.1949431$	$\lambda_7 = 1.1961868$	$\lambda_8 = 1.1970233$
$\lambda_9 = 1.1976125$	$\lambda_{10} = 1.1980429$	$\lambda_{11} = 1.1983668$	$\lambda_{12} = 1.1986165$
$\lambda_{13} = 1.1988131$	$\lambda_{14} = 1.1989707$	$\lambda_{15} = 1.1990988$	$\lambda_{16} = 1.1992045$
$\lambda_{17} = 1.1992926$	$\lambda_{18} = 1.1993668$	$\lambda_{19} = 1.1994300$	$\lambda_{20} = 1.1994842$
$\lambda_{21} = 1.1995310$	$\lambda_{22} = 1.1995717$	$\lambda_{23} = 1.1996073$	$\lambda_{24} = 1.1996387$
$\lambda_{25} = 1.1996664$	$\lambda_{26} = 1.1996911$	$\lambda_{27} = 1.1997131$	$\lambda_{28} = 1.1997329.$

Therefore, the sequence $\{\lambda_n\}$, $n \geq 1$, is a *strictly increasing sequence*.

Moreover, in Theorem 1.3, we proved that

$$\lambda_n = \frac{6}{5} - \Delta_n,$$

where $0 < \Delta_n < \frac{38}{175n(n+1)}$. Therefore

$$\lim_{n\to\infty} \lambda_n = \frac{6}{5}.$$

2.3. **Proof for the DeTemple–Wang Approximation.**

Proof of Theorem 1.8 for $\{d_n\}$. Following the idea in the proof of the Lodge–Ramanujan approximation, we solve (1.16) for d_n and define the corresponding real-variable version. Let

$$d_x := \frac{1}{\Psi(x+1) - \ln(x + \frac{1}{2})} - 24(x + \tfrac{1}{2})^2.$$

We compute the derivative, ask when is it *positive*, clear the denominator and observe that we have to solve the inequality:

$$\left\{\frac{1}{x+\frac{1}{2}} - \Psi'(x+1)\right\} - 48(x+\tfrac{1}{2})\big\{\Psi(x+1) - \ln(x+\tfrac{1}{2})\big\}^2 > 0.$$

By Lemma 2.3, the left hand side of this inequality is

$$\begin{aligned}&> \frac{1}{x+\frac{1}{2}} - \frac{1}{x} + \frac{1}{2x^2} - \frac{1}{6x^3} + \frac{1}{30x^5} - \frac{1}{42x^7} \\ &\qquad - 48(x+\tfrac{1}{2})\left(\frac{1}{24x^2} - \frac{1}{24x^3} + \frac{23}{960x^4} - \frac{1}{160x^5} - \frac{11}{8064x^6} - \frac{1}{896x^7} + \frac{143}{30720x^8}\right)^2\end{aligned}$$

for all $x > 0$. This last quantity is equal to

$$\frac{\begin{gathered}(-9018009 - 31747716x - 14007876x^2 + 59313792x^3 + 11454272x^4 - 129239296x^5 \\ + 119566592x^6 + 65630208x^7 - 701008896x^8 - 534417408x^9 + 178139136x^{10})\end{gathered}}{17340825600x^{16}(1+2x)}.$$

The denominator is evidently positive for $x > 0$ and the numerator can be written in the form

$$p(x)(x-4) + r,$$

where

$$\begin{aligned}p(x) &= 548963242092 + 137248747452x + 34315688832x^2 \\ &\quad + 8564093760x^3 + 2138159872x^4 + 566849792x^5 + 111820800x^6 \\ &\quad + 11547648x^7 + 178139136x^8 + 178139136x^9,\end{aligned}$$

with remainder $r = 2195843950359$.

Therefore, the numerator is clearly positive for $x > 4$, and therefore, the derivative d'_x is also positive for $x > 4$. Finally,

$$\begin{aligned}d_1 &= 3.73929752\cdots, \\ d_2 &= 4.08925414\cdots, \\ d_3 &= 4.13081174\cdots, \\ d_4 &= 4.15288035\cdots.\end{aligned}$$

Therefore $\{d_n\}$ is an *increasing* sequence for $n \geq 1$.

Now, if we expand the formula for d_n into an asymptotic series in powers of $\frac{1}{n+\frac{1}{2}}$, we obtain

$$d_n \sim \frac{21}{5} - \frac{1400}{2071(n+\frac{1}{2})} + \cdots$$

(this is an immediate consequence of Proposition 3.1 below) and we conclude that

$$\lim_{n\to\infty} d_n = \frac{21}{5}.$$

3. PROOF OF THE GENERAL RAMANUJAN–LODGE EXPANSION

Proof of Theorem 1.6. Our proof is founded on the half-integer approximation to H_n due to DeTemple and Wang [6]:

Proposition 3.1. *For any positive integer r there exists a θ_r, with $0 < \theta_r < 1$, for which the following equation is true:*

$$H_n = \ln(n + \tfrac{1}{2}) + \gamma + \sum_{p=1}^{r} \frac{D_p}{(n+\frac{1}{2})^{2p}} + \theta_r \cdot \frac{D_{r+1}}{(n+\frac{1}{2})^{2r+2}}, \tag{3.1}$$

where

$$D_p := -\frac{B_{2p}(\frac{1}{2})}{2p}, \tag{3.2}$$

and where $B_{2p}(x)$ is the Bernoulli polynomial of order $2p$.

Since $(n+\frac{1}{2})^2 = 2m + \frac{1}{4}$, we obtain

$$\begin{aligned}
\sum_{p=1}^{r} \frac{D_p}{(n+\frac{1}{2})^{2p}} &= \sum_{p=1}^{r} \frac{D_p}{(2m)^p\left(1+\frac{1}{8m}\right)^p} = \sum_{p=1}^{r} \frac{D_p}{(2m)^p}\left(1+\frac{1}{8m}\right)^{-p} \\
&= \sum_{p=1}^{r} \frac{D_p}{(2m)^p} \sum_{k=0}^{\infty} \binom{-p}{k} \frac{1}{8^k m^k} \\
&= \sum_{p=1}^{r} \frac{D_p}{2^p} \sum_{k=0}^{\infty} (-1)^k \binom{k+p-1}{k} \frac{1}{8^k} \cdot \frac{1}{m^{p+k}} \\
&= \sum_{p=1}^{r} \left\{ \sum_{s=0}^{p-1} \frac{D_s}{2^s} (-1)^{p-s} \binom{p-1}{p-s} \frac{1}{8^{p-s}} \right\} \cdot \frac{1}{m^p} + E_r.
\end{aligned}$$

Substituting the right hand side of the last equation into the right hand side of (3.1) we obtain

$$\begin{aligned}
H_n = \ln(n+\tfrac{1}{2}) + \gamma + \sum_{p=1}^{r} &\left\{ \sum_{s=0}^{p-1} \frac{D_s}{2^s} (-1)^{p-s} \binom{p-1}{p-s} \frac{1}{8^{p-s}} \right\} \cdot \frac{1}{m^p} \\
&+ E_r + \theta_r \cdot \frac{D_{r+1}}{(n+\frac{1}{2})^{2r+2}}.
\end{aligned} \tag{3.3}$$

Moreover,

$$\begin{aligned}
\ln(n+\tfrac{1}{2}) &= \frac{\ln(n+\frac{1}{2})^2}{2} = \frac{1}{2}\ln(2m+\tfrac{1}{4}) \\
&= \frac{1}{2}\ln(2m) + \frac{1}{2}\ln\left(1+\frac{1}{8m}\right) \\
&= \frac{1}{2}\ln(2m) + \frac{1}{2}\sum_{l=1}^{\infty} (-1)^{l-1} \frac{1}{l\,8^l m^l}.
\end{aligned}$$

Substituting the right-hand side of this last equation into (3.3), we obtain

$$\begin{aligned}
H_n &= \frac{1}{2}\ln(2m) + \frac{1}{2}\sum_{l=1}^{r}(-1)^{l-1}\frac{1}{l\,8^l m^l} \\
&\quad + \gamma + \sum_{p=1}^{r}\left\{\sum_{s=0}^{p-1}\frac{D_s}{2^s}(-1)^{p-s}\binom{p-1}{p-s}\frac{1}{8^{p-s}}\right\}\cdot\frac{1}{m^p} \\
&\quad + \epsilon_r + E_r + \theta_r \cdot \frac{D_{r+1}}{(n+\frac{1}{2})^{2r+2}} \\
&= \frac{1}{2}\ln(2m) + \gamma + \sum_{p=1}^{r}\left\{(-1)^{p-1}\frac{1}{2p\,8^p} + \sum_{s=0}^{p-1}\frac{D_s}{2^s}(-1)^{p-s}\binom{p-1}{p-s}\frac{1}{8^{p-s}}\right\}\cdot\frac{1}{m^p} \\
&\quad + \epsilon_r + E_r + \theta_r \cdot \frac{D_{r+1}}{(n+\frac{1}{2})^{2r+2}}.
\end{aligned}$$

Therefore, we have obtained Ramanujan's expansion in powers of $\frac{1}{m}$, and the coefficient of $\frac{1}{m^p}$ is

$$R_p = (-1)^{p-1}\frac{1}{2p\,8^p} + \sum_{s=0}^{p-1}\frac{D_s}{2^s}(-1)^{p-s}\binom{p-1}{p-s}\frac{1}{8^{p-s}}\,. \tag{3.4}$$

But,

$$\begin{aligned}
\frac{D_s}{2^s}(-1)^{p-s}\binom{p-1}{p-s}\frac{1}{8^{p-s}} &= -\frac{B_{2s}(\frac{1}{2})/2s}{2^s}(-1)^{p-s}\binom{p-1}{p-s}\frac{1}{8^{p-s}} \\
&= (-1)^{p-s-1}\frac{B_{2s}(\frac{1}{2})}{2s\,2^s}\binom{p-1}{p-s}\frac{1}{8^{p-s}},
\end{aligned}$$

and therefore

$$\begin{aligned}
R_p &= (-1)^{p-1}\frac{1}{2p\,8^p} + \sum_{s=0}^{p-1}\frac{D_s}{2^s}(-1)^{p-s}\binom{p-1}{p-s}\frac{1}{8^{p-s}} \\
&= (-1)^{p-1}\frac{1}{2p\,8^p} + \sum_{s=0}^{p-1}(-1)^{p-s-1}\frac{B_{2s}(\frac{1}{2})}{2s\,2^s}\binom{p-1}{p-s}\frac{1}{8^{p-s}} \\
&= (-1)^{p-1}\left\{\frac{1}{2p\,8^p} + \sum_{s=1}^{p}(-1)^s\frac{B_{2s}(\frac{1}{2})}{2s\,2^s}\binom{p-1}{p-s}\frac{1}{8^{p-s}}\right\} \\
&= (-1)^{p-1}\left\{\frac{1}{2p\,8^p} + \sum_{s=1}^{p}(-1)^s\frac{B_{2s}(\frac{1}{2})}{2\cdot 2^s}\cdot\frac{1}{p}\binom{p}{s}\frac{1}{8^{p-s}}\right\} \\
&= \frac{(-1)^{p-1}}{2p\,8^p}\left\{1 + \sum_{s=1}^{p}\binom{p}{s}(-4)^s B_{2s}(\tfrac{1}{2})\right\}.
\end{aligned}$$

Therefore, the formula for H_n takes the form

$$H_n = \frac{1}{2}\ln(2m) + \gamma + \sum_{p=1}^{r}\frac{(-1)^{p-1}}{2p\,8^p}\left\{1 + \sum_{s=1}^{p}\binom{p}{s}(-4)^s B_{2s}(\tfrac{1}{2})\right\}\cdot\frac{1}{m^p} + \mathcal{E}_r, \tag{3.5}$$

where

$$\mathcal{E}_r := \epsilon_r + E_r + \theta_r \cdot \frac{D_{r+1}}{(n+\frac{1}{2})^{2r+2}}. \tag{3.6}$$

We see *that* (3.5) *is the Ramanujan expansion with the general formula* as given in the statement of the theorem, while (3.6) is a form of the *error term*.

We will now estimate the error, (3.6).

To do so, we will use the fact that the sum of a convergent alternating series, whose terms (taken with positive sign) decrease monotonically to zero, is equal to any partial sum *plus a positive fraction of the first neglected term (with sign)*.

Thus,

$$\epsilon_r := \sum_{l=r+1}^{\infty} (-1)^{l-1} \frac{1}{2l\, 8^l m^l} = \alpha_r (-1)^r \frac{1}{2(r+1)8^{r+1}m^{r+1}},$$

where $0 < \alpha_r < 1$.

Moreover,

$$\begin{aligned}
E_r &:= \frac{D_2}{2^1} \sum_{k=r}^{\infty} (-1)^k \binom{k}{k} \frac{1}{8^k} \cdot \frac{1}{m^{1+k}} + \frac{D_4}{2^2} \sum_{k=r-1}^{\infty} (-1)^k \binom{k+1}{k} \frac{1}{8^k} \cdot \frac{1}{m^{2+k}} + \cdots \\
&\quad + \frac{D_{2r}}{2^r} \sum_{k=1}^{\infty} (-1)^k \binom{k+r-1}{k} \frac{1}{8^k} \cdot \frac{1}{m^{r+k}} + \theta_r \cdot \frac{D_{2r+2}}{(2m)^{r+1}\left(1+\frac{1}{8m}\right)^{r+1}} \\
&= \left\{ \delta_1 \frac{D_2}{2^1} (-1)^r \binom{r}{r} \frac{1}{8^r} + \delta_2 \frac{D_4}{2^2} (-1)^{r-1} \binom{r}{r-1} \frac{1}{8^{r-1}} + \cdots \right. \\
&\quad \left. + \delta_r \frac{D_{2r}}{2^r} (-1)^1 \binom{r}{1} \frac{1}{8^1} + \delta_{r+1} \frac{D_{2r+2}}{2^{r+1}} \right\} \frac{1}{m^{r+1}} \\
&= \Delta_r \left\{ \frac{D_2}{2^1} (-1)^r \binom{r}{r} \frac{1}{8^r} + \frac{D_4}{2^2} (-1)^{r-1} \binom{r}{r-1} \frac{1}{8^{r-1}} + \cdots \right. \\
&\quad \left. + \frac{D_{2r}}{2^r} (-1)^1 \binom{r}{1} \frac{1}{8^1} + \frac{D_{2r+2}}{2^{r+1}} \right\} \frac{1}{m^{r+1}},
\end{aligned}$$

where $0 < \delta_k < 1$ for $k = 1, 2, \ldots, r+1$ and $0 < \Delta_r < 1$. Thus, *the error is equal to*

$$\begin{aligned}
\mathcal{E}_r &= \Theta_r \cdot \left\{ (-1)^r \frac{1}{2(r+1)8^{(r+1)}} + \sum_{q=1}^{r+1} \frac{D_{2q}}{2^q} (-1)^{r-q+1} \binom{r}{r-q+1} \frac{1}{8^{r-q+1}} \right\} \frac{1}{m^{r+1}} \\
&= \Theta_r \cdot R_{r+1},
\end{aligned}$$

by (1.6), where $0 < \Theta_r < 1$, which is of the required form. This completes the proof. □

The origin of Ramanujan's formula is mysterious. Berndt notes that in his remarks. Our analysis of it is *a posteriori* and, although it is full and complete, it does not shed light on how Ramanujan came to think of his expansion. It would also be interesting to develop an expansion for $n!$ into powers of m, a new Stirling expansion, as it were.

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