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OSTROWSKI TYPE INEQUALITIES FOR ISOTONIC LINEAR FUNCTIONALS

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ABSTRACT. Some inequalities of Ostrowski type for isotonic linear functionals defined on a linear class of function $L := \{f : [a,b] \to \mathbb{R}\}$ are established. Applications for integral and discrete inequalities are also given.

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1. INTRODUCTION

The following result is known in the literature as Ostrowski's inequality [13].

Theorem 1.1. *Let $f : [a,b] \to \mathbb{R}$ be a differentiable mapping on (a,b) with the property that $|f'(t)| \leq M$ for all $t \in (a,b)$. Then*

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)\,dt \right| \leq \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] (b-a) M \tag{1.1}$$

for all $x \in [a,b]$.
The constant $\frac{1}{4}$ is the best possible in the sense that it cannot be replaced by a smaller constant.

The following Ostrowski type result for absolutely continuous functions whose derivatives belong to the Lebesgue spaces $L_p[a,b]$ also holds (see [9], [10] and [11]).

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Theorem 1.2. *Let $f:[a,b]\to\mathbb{R}$ be absolutely continuous on $[a,b]$. Then, for all $x\in[a,b]$, we have:*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \tag{1.2}$$
$$\leq\begin{cases}\left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right](b-a)\|f'\|_\infty & \text{if } f'\in L_\infty[a,b];\\ \frac{1}{(p+1)^{\frac{1}{p}}}\left[\left(\frac{x-a}{b-a}\right)^{p+1}+\left(\frac{b-x}{b-a}\right)^{p+1}\right]^{\frac{1}{p}}(b-a)^{\frac{1}{p}}\|f'\|_q & \text{if } f'\in L_q[a,b],\\ & \frac{1}{p}+\frac{1}{q}=1,\ p>1;\\ \left[\frac{1}{2}+\left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right]\|f'\|_1;\end{cases}$$

where $\|\cdot\|_r$ ($r\in[1,\infty]$) are the usual Lebesgue norms on $L_r[a,b]$, i.e.,

$$\|g\|_\infty := ess\sup_{t\in[a,b]}|g(t)|$$

and

$$\|g\|_r := \left(\int_a^b |g(t)|^r\,dt\right)^{\frac{1}{r}},\ r\in[1,\infty).$$

The constants $\frac{1}{4}$, $\frac{1}{(p+1)^{\frac{1}{p}}}$ and $\frac{1}{2}$ respectively are sharp in the sense presented in Theorem 1.1.

The above inequalities can also be obtained from Fink's result in [12] on choosing $n=1$ and performing some appropriate computations.

If one drops the condition of absolute continuity and assumes that f is Hölder continuous, then one may state the result (see [7]):

Theorem 1.3. *Let $f:[a,b]\to\mathbb{R}$ be of $r-H-$Hölder type, i.e.,*

$$|f(x)-f(y)|\leq H|x-y|^r,\ \text{for all } x,y\in[a,b], \tag{1.3}$$

where $r\in(0,1]$ and $H>0$ are fixed. Then for all $x\in[a,b]$ we have the inequality:

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\leq\frac{H}{r+1}\left[\left(\frac{b-x}{b-a}\right)^{r+1}+\left(\frac{x-a}{b-a}\right)^{r+1}\right](b-a)^r. \tag{1.4}$$

The constant $\frac{1}{r+1}$ is also sharp in the above sense.

Note that if $r=1$, i.e., f is Lipschitz continuous, then we get the following version of Ostrowski's inequality for Lipschitzian functions (with L instead of H) (see [3])

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\leq\left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right](b-a)L. \tag{1.5}$$

Here the constant $\frac{1}{4}$ is also best.

Moreover, if one drops the continuity condition of the function, and assumes that it is of bounded variation, then the following result may be stated (see [4]).

Theorem 1.4. *Assume that $f:[a,b]\to\mathbb{R}$ is of bounded variation and denote by $\bigvee_a^b(f)$ its total variation. Then*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\leq\left[\frac{1}{2}+\left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right]\bigvee_a^b(f) \tag{1.6}$$

for all $x \in [a,b]$.
The constant $\frac{1}{2}$ *is the best possible.*

If we assume more about f, i.e., f is monotonically increasing, then the inequality (1.6) may be improved in the following manner [5] (see also [2]).

Theorem 1.5. *Let* $f : [a,b] \to \mathbb{R}$ *be monotonic nondecreasing. Then for all* $x \in [a,b]$, *we have the inequality:*

$$(1.7) \quad \left| f(x) - \frac{1}{b-a} \int_a^b f(t)\, dt \right| \le \frac{1}{b-a} \left\{ [2x - (a+b)] f(x) + \int_a^b sgn(t-x) f(t)\, dt \right\}$$
$$\le \frac{1}{b-a} \{ (x-a)[f(x) - f(a)] + (b-x)[f(b) - f(x)] \}$$
$$\le \left[\frac{1}{2} + \left| \frac{x - \frac{a+b}{2}}{b-a} \right| \right] [f(b) - f(a)].$$

All the inequalities in (1.7) are sharp and the constant $\frac{1}{2}$ *is the best possible.*

The version of Ostrowski's inequality for convex functions was obtained in [6] and is incorporated in the following theorem:

Theorem 1.6. *Let* $f : [a,b] \to \mathbb{R}$ *be a convex function on* $[a,b]$. *Then for any* $x \in (a,b)$ *we have the inequality*

$$(1.8) \quad \frac{1}{2} \left[(b-x)^2 f'_+(x) - (x-a)^2 f'_-(x) \right] \le \int_a^b f(t)\, dt - (b-a) f(x)$$
$$\le \frac{1}{2} \left[(b-x)^2 f'_-(b) - (x-a)^2 f'_+(a) \right].$$

In both parts of the inequality (1.8) the constant $\frac{1}{2}$ *is sharp.*

For other Ostrowski type inequalities, see [8].

In this paper we extend Ostrowski's inequality for arbitrary isotonic linear functionals $A : L \to \mathbb{R}$, where L is a linear class of absolutely continuous functions defined on $[a,b]$. Some applications for particular instances of linear functionals A are also provided.

2. PRELIMINARIES

Let L be a *linear class* of real-valued functions, $g : E \to \mathbb{R}$ having the properties

(L1) $f, g \in L$ imply $(\alpha f + \beta g) \in L$ for all $\alpha, \beta \in \mathbb{R}$;
(L2) $\mathbf{1} \in L$, i.e., if $f(t) = 1$, $t \in E$, then $f \in L$.

An *isotonic linear functional* $A : L \to \mathbb{R}$ is a functional satisfying

(A1) $A(\alpha f + \beta g) = \alpha A(f) + \beta A(g)$ for all $f, g \in L$ and $\alpha, \beta \in \mathbb{R}$;
(A2) If $f \in L$ and $f \ge 0$, then $A(f) \ge 0$.

The mapping A is said to be *normalised* if

(A3) $A(\mathbf{1}) = 1$.

Usual examples of isotonic linear functional that are normalised are the following ones

$$A(f) := \frac{1}{\mu(X)} \int_X f(x)\, d\mu(x), \quad \text{if } \mu(X) < \infty$$

or

$$A_w(f) := \frac{1}{\int_X w(x)\,d\mu(x)} \int_X w(x) f(x)\,d\mu(x),$$

where $w(x) \geq 0$, $\int_X w(x)\,d\mu(x) > 0$, X is a measurable space and μ is a positive measure on X.

In particular, for $\bar{x} = (x_1, \dots, x_n)$, $\bar{w} := (w_1, \dots, w_n) \in \mathbb{R}^n$ with $w_i \geq 0$, $W_n := \sum_{i=1}^n w_i > 0$ we have

$$A(\bar{x}) := \frac{1}{n} \sum_{i=1}^n x_i$$

and

$$A_{\bar{w}}(\bar{x}) := \frac{1}{W_n} \sum_{i=1}^n w_i x_i,$$

are normalised isotonic linear functionals on $\mathbb{R}^n$.

The following representation result for absolutely continuous functions holds.

Lemma 2.1. *Let $f : [a,b] \to \mathbb{R}$ be an absolutely continuous function on $[a,b]$ and define $e(t) = t$, $t \in [a,b]$, $g(t,x) = \int_0^1 f'[(1-\lambda)x + \lambda t]\,d\lambda$, $t \in [a,b]$ and $x \in [a,b]$. If $A : L \to \mathbb{R}$ is a normalised linear functional on a linear class L of absolutely continuous functions defined on $[a,b]$ and $(x-e) \cdot g(\cdot, x) \in L$, then we have the representation*

$$f(x) = A(f) + A[(x-e) \cdot g(\cdot, x)], \tag{2.1}$$

for $x \in [a,b]$.

Proof. For any $x, t \in [a,b]$ with $t \neq x$, one has

$$\frac{f(x) - f(t)}{x - t} = \frac{\int_t^x f'(u)}{x - t} = \int_0^1 f'[(1-\lambda)x + \lambda t]\,d\lambda = g(t,x),$$

giving the equality

$$f(x) = f(t) + (x-t)\,g(t,x) \tag{2.2}$$

for any $t, x \in [a,b]$.

Applying the functional A, we get

$$A(f(x) \cdot \mathbf{1}) = A(f + (x-e)\,g(\cdot,x)),$$

for any $x \in [a,b]$.

Since

$$A(f(x) \cdot \mathbf{1}) = f(x)\,A(\mathbf{1}) = f(x)$$

and

$$A(f + (x-e) \cdot g(\cdot,x)) = A(f) + A((x-e) \cdot g(\cdot,x)),$$

the equality (2.1) is obtained. □

The following particular cases are of interest:

Corollary 2.2. *Let $f : [a,b] \to \mathbb{R}$ be an absolutely continuous function on $[a,b]$. Then we have the representation:*

$$f(x) = \frac{1}{\int_a^b w(t)\,dt} \int_a^b w(t) f(t)\,dt + \frac{1}{\int_a^b w(t)\,dt} \int_a^b w(t)(x-t) \left(\int_0^1 f'[(1-\lambda)x + \lambda t]\,d\lambda \right) dt \tag{2.3}$$

for any $x \in [a,b]$, *where* $p : [a,b] \to \mathbb{R}$ *is a Lebesgue integrable function with* $\int_a^b w(t)\,dt \neq 0$.

In particular, we have

$$f(x) = \frac{1}{b-a}\int_a^b f(t)\,dt + \frac{1}{b-a}\int_a^b (x-t)\left(\int_0^1 f'[(1-\lambda)x + \lambda t]\,d\lambda\right)dt \tag{2.4}$$

for each $x \in [a,b]$.

The proof is obvious by Lemma 2.1 applied for the normalised linear functionals

$$A_w(f) := \frac{1}{\int_a^b w(t)\,dt}\int_a^b w(t) f(t)\,dt, \quad A(f) := \frac{1}{b-a}\int_a^b f(t)\,dt$$

defined on

$$L := \{f : [a,b] \to \mathbb{R},\ f \text{ is absolutely continuous on } [a,b]\}.$$

The following discrete case also holds.

Corollary 2.3. *Let* $f : [a,b] \to \mathbb{R}$ *be an absolutely continuous function on* $[a,b]$. *Then we have the representation:*

$$f(x) = \frac{1}{W_n}\sum_{i=1}^n w_i f(x_i) + \frac{1}{W_n}\sum_{i=1}^n w_i (x-x_i)\left(\int_0^1 f'[(1-\lambda)x + \lambda x_i]\,d\lambda\right) \tag{2.5}$$

for any $x \in [a,b]$, *where* $x_i \in [a,b]$, $w_i \in \mathbb{R}$ $(i = \{1,\dots,n\})$ *with* $W_n := \sum_{i=1}^n w_i \neq 0$.

In particular, we have

$$f(x) = \frac{1}{n}\sum_{i=1}^n f(x_i) + \frac{1}{n}\sum_{i=1}^n (x-x_i)\left(\int_0^1 f'[(1-\lambda)x + \lambda x_i]\,d\lambda\right) \tag{2.6}$$

for any $x \in [a,b]$.

3. OSTROWSKI TYPE INEQUALITIES

The following theorem holds.

Theorem 3.1. *With the assumptions of Lemma 2.1, and assuming that* $A : L \to \mathbb{R}$ *is isotonic, then we have the inequalities*

$$|f(x) - A(f)| \le \begin{cases} A\left(|x-e|\,\|f'\|_{[x,\cdot],\infty}\right) & \text{if } |x-e|\,\|f'\|_{[x,\cdot],\infty} \in L,\ f' \in L_\infty[a,b]; \\ A\left(|x-e|^{\frac{1}{q}}\,\|f'\|_{[x,\cdot],p}\right) & \text{if } |x-e|^{\frac{1}{q}}\,\|f'\|_{[x,\cdot],p} \in L,\ f' \in L_p[a,b], \\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ A\left(\|f'\|_{[x,\cdot],1}\right) & \text{if } \|f'\|_{[x,\cdot],1} \in L, \end{cases} \tag{3.1}$$

where

$$\|h\|_{[m,n],\infty} := ess \sup_{\substack{t\in[m,n] \\ (t\in[n,m])}} |h(t)| \quad \textit{and}$$

$$\|h\|_{[m,n],p} := \left|\int_m^n |h(t)|^p\,dt\right|^{\frac{1}{p}},\ p \ge 1.$$

If we denote

$$M_{\infty}(x) := A\left(|x-e|\,\|f'\|_{[x,\cdot],\infty}\right),$$
$$M_{p}(x) := A\left(|x-e|^{\frac{1}{q}}\,\|f'\|_{[x,\cdot],p}\right),$$
$$M_{1}(x) := A\left(\|f'\|_{[x,\cdot],1}\right),$$

then we have the inequalities:

$$M_{\infty}(x) \tag{3.2}$$

$$\leq \begin{cases} \|f'\|_{[a,b],\infty} A(|x-e|) & \text{if } |x-e| \in L,\ f' \in L_{\infty}[a,b]; \\ \left[A\left(\|f'\|_{[x,\cdot],\infty}^{\beta}\right)\right]^{\frac{1}{\beta}} \left[A\left(|x-e|^{\alpha}\right)\right]^{\frac{1}{\alpha}} & \text{if } \|f'\|_{[x,\cdot],\infty}^{\beta}, |x-e|^{\alpha} \in L, \\ & f' \in L_{\infty}[a,b],\ \alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1; \\ \left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right] A\left(\|f'\|_{[x,\cdot],\infty}\right) & \text{if } \|f'\|_{[x,\cdot],\infty} \in L,\ f' \in L_{\infty}[a,b]. \end{cases}$$

$$M_{p}(x) \tag{3.3}$$

$$\leq \begin{cases} \max\left\{\|f'\|_{[a,x],p}, \|f'\|_{[x,b],p}\right\} A\left(|x-e|^{\frac{1}{q}}\right) & \text{if } |x-e|^{\frac{1}{q}} \in L,\ f' \in L_{p}[a,b]; \\ \left[A\left(\|f'\|_{[x,\cdot],p}^{\beta}\right)\right]^{\frac{1}{\beta}} \left[A\left(|x-e|^{\frac{\alpha}{q}}\right)\right]^{\frac{1}{\alpha}} & \text{if } \|f'\|_{[x,\cdot],p}^{\beta}, |x-e|^{\frac{\alpha}{q}} \in L, \\ & f' \in L_{p}[a,b],\ \alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1; \\ \left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]^{\frac{1}{q}} A\left(\|f'\|_{[x,\cdot],p}\right) & \text{if } \|f'\|_{[x,\cdot],p} \in L,\ f' \in L_{p}[a,b] \end{cases}$$

and

$$M_{1}(x) \leq \begin{cases} \frac{1}{2}\|f'\|_{[a,b],1} + \frac{1}{2}\left|\|f'\|_{[a,x],1} - \|f'\|_{[x,b],1}\right|, \\ \left[A\left(\|f'\|_{[x,\cdot],1}^{\beta}\right)\right]^{\frac{1}{\beta}}, \quad \beta>1. \end{cases} \tag{3.4}$$

Proof. Using (2.1) and taking the modulus, we have

$$\begin{aligned} |f(x) - A(f)| &= |A((x-e)\cdot g(\cdot,x))| \\ &\leq A(|(x-e)\cdot g(\cdot,x)|) \\ &= A(|x-e|\,|g(\cdot,x)|). \end{aligned} \tag{3.5}$$

For $t \neq x$ $(t,x \in [a,b])$ we may state

$$\begin{aligned} |g(t,x)| &\leq \int_0^1 |f'((1-\lambda)x + \lambda t)|\, d\lambda \\ &\leq ess \sup_{\lambda\in[0,1]} |f'((1-\lambda)x+\lambda t)| \\ &= \|f'\|_{[x,t],\infty}. \end{aligned}$$

Hölder's inequality will produce

$$\begin{aligned}|g(t,x)| &\le \int_0^1 |f'((1-\lambda)x+\lambda t)|\,d\lambda \\ &\le \left[\int_0^1 |f'((1-\lambda)x+\lambda t)|^p\,d\lambda\right]^{\frac{1}{p}} \\ &= \left(\frac{1}{x-t}\int_t^x |f'(u)|^p\,du\right)^{\frac{1}{p}} \\ &= |x-t|^{-\frac{1}{p}}\,\|f'\|_{[x,t],p}\,, \quad p>1,\ \frac{1}{p}+\frac{1}{q}=1;\end{aligned}$$

and finally

$$|g(t,x)| \le \int_0^1 |f'((1-\lambda)x+\lambda t)|\,d\lambda = \frac{1}{t-x}\,\|f'\|_{[x,t],1}\,.$$

Consequently

$$|(x-e)|\,|g(\cdot,x)| \le \begin{cases} |x-e|\,\|f'\|_{[x,\cdot],\infty} & \text{if } f'\in L_\infty[a,b]\,; \\ |x-e|^{\frac{1}{q}}\,\|f'\|_{[x,\cdot],p} & \text{if } f'\in L_p[a,b]\,, \\ \|f'\|_{[x,\cdot],1} & \end{cases} \tag{3.6}$$

for any $x\in[a,b]$.

Applying the functional A to (3.6) and using (3.5) we deduce the inequality (3.1).

We have

$$\begin{aligned}M_\infty(x) &\le \sup_{t\in[a,b]}\left\{\|f'\|_{[x,t],\infty}\right\} A(|x-e|) \\ &= \max\left\{\|f'\|_{[a,x],\infty}\,,\|f'\|_{[x,b],\infty}\right\} A(|x-e|) \\ &= \|f'\|_{[a,b],\infty}\,A(|x-e|)\end{aligned}$$

and the first inequality in (3.2) is proved.

Using Hölder's inequality for the functional A, i.e.,

$$|A(hg)| \le [A(|h|^\alpha)]^{\frac{1}{\alpha}}\left[A\left(|g|^\beta\right)\right]^{\frac{1}{\beta}}, \quad \alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1, \tag{3.7}$$

where hg, $|h|^\alpha$, $|g|^\beta \in L$, we have

$$M_\infty(x) \le [A(|x-e|^\alpha)]^{\frac{1}{\alpha}}\left[A\left(\|f'\|^\beta_{[x,\cdot],\infty}\right)\right]^{\frac{1}{\beta}}$$

and the second part of (3.2) is proved.

In addition,

$$\begin{aligned}M_\infty(x) &\le \sup_{t\in[a,b]}|x-t|\,A\left(\|f'\|_{[x,\cdot],\infty}\right) \\ &= \max\{x-a,b-x\}\,A\left(\|f'\|_{[x,\cdot],\infty}\right) \\ &= \left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right] A\left(\|f'\|_{[x,\cdot],\infty}\right)\end{aligned}$$

and the inequality (3.2) is completely proved.

We also have

$$\begin{aligned} M_p(x) &\leq \sup_{t\in[a,b]} \left\{ \|f'\|_{[x,t],p} \right\} A\left(|x-e|^{\frac{1}{q}} \right) \\ &= \max\left\{ \|f'\|_{[a,x],p}, \|f'\|_{[x,b],p} \right\} A\left(|x-e|^{\frac{1}{q}} \right). \end{aligned}$$

Using Hölder's inequality (3.7) one has

$$M_p(x) \leq \left[A\left(|x-e|^{\frac{\alpha}{q}} \right) \right]^{\frac{1}{\alpha}} \left[A\left(\|f'\|_{[x,\cdot],p}^{\beta} \right) \right]^{\frac{1}{\beta}}, \ \alpha > 1, \ \frac{1}{\alpha} + \frac{1}{\beta} = 1$$

and

$$\begin{aligned} M_p(x) &\leq \sup_{t\in[a,b]} \left\{ |x-t|^{\frac{1}{q}} \right\} A\left(\|f'\|_{[x,\cdot],p} \right) \\ &= \max\left\{ (x-a)^{\frac{1}{q}}, (b-x)^{\frac{1}{q}} \right\} A\left(\|f'\|_{[x,\cdot],p} \right) \\ &= \left[\frac{1}{2}(b-a) + \left| x - \frac{a+b}{2} \right| \right]^{\frac{1}{q}} A\left(\|f'\|_{[x,\cdot],p} \right), \end{aligned}$$

proving the inequality (3.3).

Finally,

$$\begin{aligned} A\left(\|f'\|_{[x,\cdot],1} \right) &\leq \sup_{t\in[a,b]} \left\{ \|f'\|_{[x,t],1} \right\} A(\mathbf{1}) \\ &= \max\left\{ \|f'\|_{[a,x],1}, \|f'\|_{[x,b],1} \right\} \\ &= \frac{1}{2}\|f'\|_{[a,b],1} + \frac{1}{2}\left| \|f'\|_{[a,x],1} - \|f'\|_{[x,b],1} \right|. \end{aligned}$$

By Hölder's inequality, we have

$$A\left(\|f'\|_{[x,\cdot],1} \right) \leq \left[A\left(\|f'\|_{[x,\cdot],1}^{\beta} \right) \right]^{\frac{1}{\beta}}, \quad \beta > 1,$$

and the last part of (3.4) is also proved. □

4. THE CASE WHERE $|f'|$ IS CONVEX

The following theorem also holds.

Theorem 4.1. *Let $f : [a,b] \to \mathbb{R}$ be an absolutely continuous function such that $f' : (a,b) \to \mathbb{R}$ is convex in absolute value, i.e., $|f'|$ is convex on (a,b). If $A : L \to \mathbb{R}$ is a normalised isotonic linear functional and $|x-e|, |x-e||f'| \in L$, then*

$$|f(x) - A(f)| \leq \frac{1}{2}\left[|f'(x)| A(|x-e|) + A(|x-e||f'|) \right] \tag{4.1}$$

$$\leq \begin{cases} \frac{1}{2}\left[\|f'\|_{[a,b],\infty} + |f'(x)| \right] A(|x-e|), & \text{if } f' \in L_\infty[a,b]; \\ \frac{1}{2}\left[|f'(x)| A(|x-e|) + \left[A(|x-e|^\alpha) \right]^{\frac{1}{\alpha}} \left[A\left(|f'|^\beta \right) \right]^{\frac{1}{\beta}} \right] & \text{if } |x-e|^\alpha, |f'|^\beta \in L, \\ & \alpha > 1, \ \frac{1}{\alpha} + \frac{1}{\beta} = 1; \\ \frac{1}{2}\left[|f'(x)| A(|x-e|) + \left[\frac{1}{2}(b-a) + \left| x - \frac{a+b}{2} \right| \right] A(|f'|) \right] & \text{if } |f'| \in L. \end{cases}$$

Proof. Since $|f'|$ is convex, we have

$$\begin{aligned}|g(t,x)| &\le \int_0^1 |f'((1-\lambda)x+\lambda t)|\,d\lambda \\ &= |f'(x)|\int_0^1 (1-\lambda)\,d\lambda + |f'(t)|\int_0^1 \lambda d\lambda \\ &= \frac{|f'(x)|+|f'(t)|}{2}.\end{aligned}$$

Thus,

$$\begin{aligned}|f(x)-A(f)| &\le A\left(|x-e|\cdot\frac{|f'(x)|+|f'(t)|}{2}\right) \\ &= \frac{1}{2}\left[|f'(x)|\,A(|x-e|)+A(|x-e|\,|f'|)\right]\end{aligned}$$

and the first part of (4.1) is proved.

We have

$$\begin{aligned}A(|x-e|\,|f'|) &\le ess \sup_{t\in[a,b]}\{|f'(t)|\}\cdot A(|x-e|) \\ &= \|f'\|_{[a,b],\infty}\,A(|x-e|).\end{aligned}$$

By Hölder's inequality for isotonic linear functionals, we have

$$A(|x-e|\,|f'|) \le \left[A\left(|x-e|^{\alpha}\right)\right]^{\frac{1}{\alpha}}\left[A\left(|f'|^{\beta}\right)\right]^{\frac{1}{\beta}},\ \alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1$$

and finally,

$$\begin{aligned}A(|x-e|\,|f'|) &\le \sup_{t\in[a,b]}|x-t|\cdot A(|f'|) \\ &= \max(x-a,b-x)\cdot A(|f'|) \\ &= \left(\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right)A(|f'|).\end{aligned}$$

The theorem is thus proved. □

5. SOME INTEGRAL INEQUALITIES

If we consider the normalised isotonic linear functional $A(f)=\frac{1}{b-a}\int_a^b f$, then by Theorem 3.1 for $f:[a,b]\to\mathbb{R}$ an absolutely continuous function, we may state the following integral inequalities

(5.1)
$$\begin{aligned}&\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \\ &\qquad\le \frac{1}{b-a}\int_a^b |x-t|\,\|f'\|_{[x,t],\infty}\,dt\end{aligned}$$

$$\leq \begin{cases} \|f'\|_{[a,b],\infty}\left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right](b-a) & \text{(Ostrowski's inequality)} \\ & \text{provided } f'\in L_\infty[a,b]; \\ \left[\frac{1}{b-a}\int_a^b \|f'\|_{[x,t],\infty}^{\beta}\,dt\right]^{\frac{1}{\beta}}\left[\frac{(b-x)^{\alpha+1}+(x-a)^{\alpha+1}}{(\alpha+1)(b-a)}\right]^{\frac{1}{\alpha}} & \\ \quad \text{if } f'\in L_\infty[a,b],\ \|f'\|_{[x,\cdot],\infty}\in L_\beta[a,b],\ \alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1; & \\ \left[\frac{1}{2}+\frac{\left|x-\frac{a+b}{2}\right|}{b-a}\right]\int_a^b \|f'\|_{[x,t],\infty}\,dt & \\ \quad \text{if } f'\in L_\infty[a,b],\ \text{and if } \|f'\|_{[x,\cdot],\infty}\in L_1[a,b], & \end{cases}$$

for each $x \in [a,b]$;

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \tag{5.2}$$

$$\leq \frac{1}{b-a}\int_a^b |x-t|^{\frac{1}{q}}\|f'\|_{[x,t],p}\,dt$$

$$\leq \begin{cases} q\max\left\{\|f'\|_{[a,x],p},\|f'\|_{[x,b],p}\right\}\left[\frac{(b-x)^{\frac{1}{q}+1}+(x-a)^{\frac{1}{q}+1}}{(b-a)(q+1)}\right], \\ \quad p>1,\ \frac{1}{p}+\frac{1}{q}=1 \text{ and } f'\in L_p[a,b]; \\ q^{\frac{1}{\alpha}}\left(\frac{1}{b-a}\int_a^b \|f'\|_{[x,t],p}^{\beta}\,dt\right)^{\frac{1}{\beta}}\left[\frac{(b-x)^{\frac{\alpha}{q}+1}+(x-a)^{\frac{\alpha}{q}+1}}{(b-a)(q+\alpha)}\right]^{\frac{1}{\alpha}} \text{ if } f'\in L_p[a,b], \\ \quad \text{and } \|f'\|_{[x,\cdot],p}\in L_\beta[a,b],\ \text{where } \alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1 \\ \left[\frac{1}{2}+\frac{\left|x-\frac{a+b}{2}\right|}{b-a}\right]^{\frac{1}{q}}\frac{1}{b-a}\int_a^b \|f'\|_{[x,t],p}\,dt \\ \quad \text{if } f'\in L_p[a,b],\ \text{and } \|f'\|_{[x,\cdot],p}\in L_1[a,b], \end{cases}$$

for each $x \in [a,b]$ and

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \tag{5.3}$$

$$\leq \frac{1}{b-a}\int_a^b \|f'\|_{[x,t],1}\,dt$$

$$\leq \begin{cases} \frac{1}{2}\|f'\|_{[a,b],1}+\frac{1}{2}\left|\|f'\|_{[a,x],1}-\|f'\|_{[x,b],1}\right| \quad \text{if } f'\in L_1[a,b]; \\ \left(\frac{1}{b-a}\int_a^b \|f'\|_{[x,t],1}^{\beta}\,dt\right)^{\frac{1}{\beta}} \\ \text{if } f'\in L_1[a,b],\ \|f'\|_{[x,.],1}\in L_\beta[a,b],\ \text{where } \beta>1, \end{cases}$$

for each $x \in [a,b]$.

If we assume now that $f:[a,b]\to\mathbb{R}$ is absolutely continuous and such that $|f'|$ is convex on (a,b), then by Theorem 4.1 we obtain the following integral inequalities established in [1]

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \tag{5.4}$$
$$\le \frac{1}{2}\left[|f'(x)|\left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right](b-a)+\frac{1}{b-a}\int_a^b |x-t|\,|f'(t)|\,dt\right]$$
$$\le \begin{cases} \frac{1}{2}\left[\|f'\|_{[a,b],\infty}+|f'(x)|\right]\left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right](b-a) \ \text{ if } f'\in L_\infty[a,b]; \\ \frac{1}{2}\left\{|f'(x)|\left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right](b-a) \right. \\ \qquad \left. +\left[\frac{(b-x)^{\alpha+1}+(x-a)^{\alpha+1}}{(\alpha+1)(b-a)}\right]^{\frac{1}{\alpha}}\left[\frac{1}{b-a}\int_a^b |f'(t)|^\beta\,dt\right]^{\frac{1}{\beta}}\right\} \\ \qquad \text{if } \ f'\in L_\beta[a,b],\ \alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1; \\ \frac{1}{2}\left\{|f'(x)|\left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right](b-a)+\left[\frac{1}{2}+\frac{\left|x-\frac{a+b}{2}\right|}{b-a}\right]\int_a^b |f'(t)|\,dt\right\} \\ \qquad \text{if } f'\in L_1[a,b], \end{cases}$$

for each $x\in[a,b]$.

6. SOME DISCRETE INEQUALITIES

For a given interval $[a,b]$, consider the division

$$I_n: a=x_0<x_1<\cdots<x_{n-1}<x_n=b$$

and the intermediate points $\xi_i\in[x_i,x_{i+1}]$, $i=\overline{0,n-1}$. If $h_i:=x_{i+1}-x_i>0$ $\left(i=\overline{0,n-1}\right)$ we may define the following functionals

$$A(f;I_n,\xi):=\frac{1}{b-a}\sum_{i=0}^{n-1} f(\xi_i)\,h_i \qquad \text{(Riemann Rule)}$$
$$A_T(f;I_n):=\frac{1}{b-a}\sum_{i=0}^{n-1}\frac{f(x_i)+f(x_{i+1})}{2}\cdot h_i \qquad \text{(Trapezoid Rule)}$$
$$A_M(f;I_n):=\frac{1}{b-a}\sum_{i=0}^{n-1} f\left(\frac{x_i+x_{i+1}}{2}\right)\cdot h_i \qquad \text{(Mid-point Rule)}$$
$$A_S(f;I_n):=\frac{1}{3}A_T(f;I_n)+\frac{2}{3}A_M(f;I_n). \qquad \text{(Simpson Rule)}$$

We observe that, all the above functionals are obviously linear, isotonic and normalised.

Consequently, all the inequalities obtained in Sections 2 and 3 may be applied for these functionals.

If, for example, we use the following inequality (see Theorem 3.1)

$$|f(x)-A(f)|\le \|f'\|_{[a,b]}\,A(|x-e|),\quad x\in[a,b], \tag{6.1}$$

provided $f : [a,b] \to \mathbb{R}$ is absolutely continuous and $f' \in L_\infty [a,b]$, then we get the inequalities

$$\left| f(x) - \frac{1}{b-a} \sum_{i=0}^{n-1} f(\xi_i) h_i \right| \le \|f'\|_{[a,b],\infty} \frac{1}{b-a} \sum_{i=0}^{n-1} |x - \xi_i| h_i, \tag{6.2}$$

$$\left| f(x) - \frac{1}{b-a} \sum_{i=0}^{n-1} \frac{f(x_i) + f(x_{i+1})}{2} \cdot h_i \right| \le \|f'\|_{[a,b],\infty} \cdot \frac{1}{b-a} \sum_{i=0}^{n-1} \frac{|x - x_i| + |x - x_{i+1}|}{2} h_i, \tag{6.3}$$

$$\left| f(x) - \frac{1}{b-a} \sum_{i=0}^{n-1} f\left(\frac{x_i + x_{i+1}}{2} \right) \cdot h_i \right| \le \|f'\|_{[a,b],\infty} \frac{1}{b-a} \sum_{i=0}^{n-1} \left| x - \frac{x_i + x_{i+1}}{2} \right| h_i, \tag{6.4}$$

for each $x \in [a,b]$.

Similar results may be stated if one uses for example Theorem 4.1. We omit the details.

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