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A NOTE ON MULTIVALUED QUASI VARIATIONAL INEQUALITIES IN BANACH SPACES

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ABSTRACT. Some iterative algorithms for quasi variational inequalities with noncompact sets in Banach spaces are suggested in Noor, Moudafi, and Xu [1] (*J. Inequal. Pure and Appl. Math.*, **3**(3) (2002), Art. 36). However, the convergence analysis for the algorithms in Noor, Moudafi, and Xu [1] is wrong. In this note, we correct the error in Noor, Moudafi, and Xu [1]. Our results improve and generalize many known corresponding algorithms and results.

Key words and phrases: Generalized set-valued quasi variational inclusion; Iterative algorithm with error; Banach space.

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1. INTRODUCTION

Multivalued quasi variational inequalities provide us with a unified, natural, innovative and general approach to study a wide class of problems arising in different branches of mathematics, physics and engineering science. Noor, Moudafi and Xu [1] suggest and analyze iterative methods for solving multivalued quasi variational inequalities in Banach spaces. However, the convergence analysis for the algorithms in Noor, Moudafi and Xu [1] is wrong. In this note, we correct the error in Noor, Moudafi and Xu [1]. Since multivalued quasi variational inequalities include quasi variational inequalities, complementarity problems and nonconvex programming problems as special cases, the results obtained in this paper continue to hold for these problems. Our results represent an improvement of previous results. In this note, the notions and concepts are just as ones in Noor, Moudafi and Xu [1], unless otherwise specified.

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2. ALGORITHM

For given point-to-set mapping $K : u \longrightarrow K(u)$, which associates a closed convex set of X with any element of X, and $N(\cdot,\cdot) : X \times X \longrightarrow X$, we consider the problem of finding $u \in X, w \in Tu, y \in Vu$ such that

$$\langle N(w,y), J(g(v)-g(u))\rangle \geq 0, \tag{2.1}$$

where $J : X \longrightarrow X^*$ is the normalized duality mapping.

Problem (2.1) is called multivalued quasi variational inequality in Banach spaces, which is introduced in Noor, Moudafi and Xu [1].

In order to suggest the algorithm for (2.1), we need the following lemma.

Lemma 2.1. [1]. *The multivalued quasi variational inequality (2.1) has a solution* $u \in X$, $w \in Tu$, $y \in Vu$ *if and only if* $u \in X$, $w \in Tu$, $y \in Vu$ *satisfies the relation*

$$g(u) = P_{K(u)}[g(u) - \rho N(w,y)], \tag{2.2}$$

where $\rho > 0$ *is a constant.*

Noor, Moudafi and Xu [1] use this alternative equivalent formation to suggest the following iterative algorithm for solving (2.1).

Algorithm 2.1. [1]. For given $u_0 \in X$, $w_0 \in Tu_0$, $y_0 \in Vu_0$ and $0 < \varepsilon < 1$, compute the sequences $\{u_n\}, \{w_n\}, \{y_n\}$ by the iterative schemes:

$$g(u_n) = P_{K(u_n)}[g(u_n) - N(w_n, y_n)], \tag{2.3}$$

$$w_n \in T(u_n) : \|w_{n+1} - w_n)\| \leq M(T(u_{n+1}), T(u_n)) + \varepsilon^{n+1}\|u_{n+1} - u_n\|, \tag{2.4}$$

$$y_n \in V(u_n) : \|y_{n+1} - y_n)\| \leq M(V(u_{n+1}), V(u_n)) + \varepsilon^{n+1}\|u_{n+1} - u_n\|, \tag{2.5}$$

where $M(\cdot,\cdot)$ is the Hausedorff metric defined on $CB(X)$.

3. CONVERGENCE ANALYSIS

In this section, we study the convergence analysis of Algorithm 2.1. For this purpose, we need the following condition and lemma.

Assumption 1. For all $u, v \in E$, the operator$P_{K(u)}$ satisfy the conditions

$$\|P_{K(u)}w - P_{K(v)}w\| \leq \nu\|u - v\|,$$

where ν are constants.

Lemma 3.1. [1]. *Let* E *be an arbitrary real Banach space and* $J : E \to 2^{E^*}$ *a normalized duality mapping, then for any* $x, y \in E$,

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, j(x+y)\rangle$$

for all $j(x+y) \in J(x+y)$.

Theorem 3.2. *Let* X *be a real uniformly smooth Banach space. Let the operator* $N(\cdot,\cdot)$ *be* $\beta-$ *Lipschitz and* $\gamma-$ *Lipschitz continuous with respect to the first argument and the second argument, respectively. Let the operator* g *be Lipschitz continuous with a constant* $\delta > 0$ *and strongly accretive with a constant* $k > 1/2$. *Assume that* $g - \rho N$ *is contractive with a constant*

$c > 0$ and the operators $T, V : X \to CB(X)$ are $M-$Lipschitz continuous with constants $\mu > 0$ and $\eta > 0$. If Assumption 1 holds and

$$\frac{c+\nu}{\sqrt{2k-1}} < 1, \tag{3.1}$$

then there exists $u \in X, w \in T(u), y \in V(u)$ satisfy (2.1) and the iterative sequences $\{u_n\}, \{w_n\}$ and $\{y_n\}$ generated by Algorithm 2.1 converge to u, w and y strongly in X, respectively.

Proof. From Lemma 3.1 and Algorithm 2.1, it follows that there exists $j(u_{n+1}-u_n) \in J(u_{n+1}-u_n)$ such that

$$\begin{aligned}\|u_{n+1}-u_n\|^2 &= \|g(u_{n+1}) - g(u_n) + u_{n+1} - u_n - (g(u_{n+1}) - g(u_n))\|^2 \\ &\le \|g(u_{n+1}) - g(u_n)\|^2 + 2\left\langle u_{n+1} - u_n - (g(u_{n+1}) - g(u_n)), j(u_{n+1}-u_n)\right\rangle \\ &\le \|g(u_{n+1}) - g(u_n)\|^2 + 2\|u_{n+1}-u_n\|^2 - 2k\|u_{n+1}-u_n\|^2,\end{aligned}$$

which implies that

$$\|u_{n+1}-u_n\|^2 \le \frac{1}{2k-1}\|g(u_{n+1}) - g(u_n)\|^2,$$

that is

$$\|u_{n+1}-u_n\| \le \frac{1}{\sqrt{2k-1}}\|g(u_{n+1}) - g(u_n)\|. \tag{3.2}$$

Now using Assumption 1, we have

$$\begin{aligned}&\|g(u_{n+1}) - g(u_n)\| \\ &= \|P_{K(u_n)}[g(u_n) - \rho N(w_n, y_n)] - P_{K(u_{n-1})}[g(u_{n-1}) - \rho N(w_{n-1}, y_{n-1})]\| \\ &\le \|P_{K(u_n)}[g(u_n) - \rho N(w_n, y_n)] - P_{K(u_n)}[g(u_{n-1}) - \rho N(w_{n-1}, y_{n-1})]\| \\ &\qquad + \|P_{K(u_n)}[g(u_{n-1}) - \rho N(w_{n-1}, y_{n-1})] \\ &\qquad - P_{K(u_{n-1})}[g(u_{n-1}) - \rho N(w_{n-1}, y_{n-1})]\| \\ &\le \|g(u_n) - \rho N(w_n, y_n) - [g(u_{n-1}) - \rho N(w_{n-1}, y_{n-1})]\| + \nu\|u_n - u_{n-1}\| \\ &\le (c+\nu)\|u_n - u_{n-1}\|.\end{aligned} \tag{3.3}$$

From (3.2) and (3.3), we have

$$\begin{aligned}\|u_{n+1}-u_n\| &\le \frac{c+\nu}{\sqrt{2k-1}}\|u_n - u_{n-1}\| \\ &= \theta\|u_n - u_{n-1}\|,\end{aligned}$$

where

$$\theta = \frac{c+\nu}{\sqrt{2k-1}}.$$

From (3.1), we have $\theta < 1$. Consequently, the sequence $\{u_n\}$ is a Cauchy sequence in X. Since X is a Banach space, there exists $u \in X$, such that $u_n \to u$ as $n \to \infty$.

Using the Lipschitz continuity of $N(\cdot,\cdot)$ with respect to the first argument and $M-$Lipschitz continuity of T, we have

$$\begin{aligned}\|N(w_n, y_n) - N(w_{n-1}, y_n)\| &\le \beta\|w_n - w_{n-1}\| \\ &\le \beta(M(T(u_n), T(u_{n-1})) + \varepsilon^n\|u_n - u_{n-1}\|) \\ &\le \beta(\mu + \varepsilon^n)\|u_n - u_{n-1}\|.\end{aligned} \tag{3.4}$$

In a similar way,

$$\begin{aligned} \|N(w_{n-1}, y_n) - N(w_{n-1}, y_{n-1})\| &\leq \gamma\|y_n - y_{n-1}\| \\ &\leq \gamma(M(V(u_n), V(u_{n-1})) + \varepsilon^n\|u_n - u_{n-1}\|) \\ &\leq \gamma(\eta + \varepsilon^n)\|u_n - u_{n-1}\|. \end{aligned} \tag{3.5}$$

From (3.4) and (3.5), we see that $\{w_n\}, \{y_n\}$ are Cauchy sequences in X, that is, there exist $w, y \in E$ such that $w_n \to w, y_n \to y$. Now by using the continuity of the operators $N, T, V, g, P_{K(u)}$ and Algorithm 2.1, we have

$$g(u) = P_{K(u)}[g(u) - \rho N(w, y)].$$

Finally, we prove that $w \in T(u)$ and $y \in V(u)$. In fact, since $w \in T(u_n)$, we have

$$\begin{aligned} d(w, T(u)) &\leq \|w - w_n\| + d(w_n, T(u)) \\ &\leq \|w - w_n\| + M(T(u_n), T(u)) \\ &\leq \|w - w_n\| + \mu\|u_n - u\| \to 0, \quad \text{as} \quad n \to \infty, \end{aligned}$$

which implies that $d(w, T(u)) = 0$, and since $T(u)$ is a closed bounded subset of X, it follows that $w \in T(u)$. In a similar way, we can also prove that $y \in V(u)$.

By Lemma 2.1, it follows that (u, w, y) is a solution of the multivalued quasi variational inequalities problem (2.1), and $u_n \to u, w_n \to w, y_n \to y$ strongly in X, the required result. □

Remark 3.3. Theorem 3.2 in Noor [1] is wrong. In fact, we have that there exist $j(u - v) \in J(u - v)$,

$$k\|u - v\|^2 \leq \langle g(u) - g(v), j(u - v)\rangle \leq \delta\|u - v\|^2,$$

which implies $\delta \geq k$, thus $0 < \frac{\sqrt{2k-1} - (\delta+\nu)}{\beta\mu + \gamma\eta} < 0$. It is a contradiction! Theorem 3.2 corrects the error in the Theorem 3.2 in Noor [1], and improves many recent results.

Remark 3.4. Let $g(x) = \frac{2}{3}x, N \equiv g, \nu = 0$, then if $1 - \frac{\sqrt{3}}{2} < \rho < 1$, one can take $c = \frac{2(1-\rho)}{3}$ such that

$$\frac{c + \nu}{\sqrt{2k - 1}} = \frac{2(1 - \rho)}{3} \times \frac{1}{\sqrt{1/3}} < 1,$$

whereas in such case, according to the Remark 3.3, the conditions in Theorem 3.2 in [1] do not hold.

Remark 3.5. By using the technique in this paper, one can easily obtain the convergence result for the multivalued variational inclusions.

REFERENCES

[1] M.A. NOOR, A. MOUDAFI AND B.L. XU, Multivalued quasi variational inequalities in Banach spaces, *J. Inequal. Pure and Appl. Math.*, **3**(3) (2002), Art. 36. [ONLINE `http://jipam.vu.edu.au/article.php?sid=188`].