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ON A REVERSE OF A HARDY-HILBERT TYPE INEQUALITY

BICHENG YANG

Department Of Mathematics
Guangdong Education College, Guangzhou
Guangdong 510303, People's Republic of China

*E*Mail: bcyang@pub.guangzhou.gd.cn

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Abstract

This paper deals with a reverse of the Hardy-Hilbert's type inequality with a best constant factor. The other reverse of the form is considered.

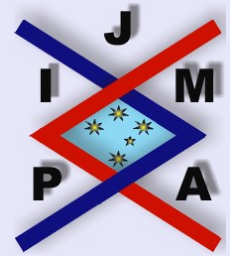
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1. Introduction

If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $a_n, b_n \geq 0$, such that $0 < \sum_{n=0}^{\infty} a_n^p < \infty$ and $0 < \sum_{n=0}^{\infty} b_n^q < \infty$, then we have the well known Hardy-Hilbert inequality (Hardy et al. [1]):

$$(1.1) \quad \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{a_m b_n}{m+n+1} < \frac{\pi}{\sin\left(\frac{\pi}{p}\right)} \left\{ \sum_{n=0}^{\infty} a_n^p \right\}^{\frac{1}{p}} \left\{ \sum_{n=0}^{\infty} b_n^q \right\}^{\frac{1}{q}},$$

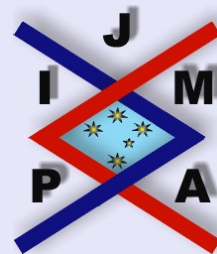
where the constant factor $\pi/\sin(\pi/p)$ is the best possible. The equivalent form is (see Yang et al. [8]):

$$(1.2) \quad \sum_{n=0}^{\infty} \left(\sum_{m=0}^{\infty} \frac{a_m}{m+n+1} \right)^p < \left[\frac{\pi}{\sin\left(\frac{\pi}{p}\right)} \right]^p \sum_{n=0}^{\infty} a_n^p,$$

where the constant factor $[\pi/\sin(\pi/p)]^p$ is still the best possible.

Inequalities (1.1) and (1.2) are important in analysis and its applications (see Mitrinović, et al. [3]). In recent years, inequality (1.1) had been strengthened by Yang [5] as

$$(1.3) \quad \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{a_m b_n}{m+n+1} < \left\{ \sum_{n=0}^{\infty} \left[\frac{\pi}{\sin\left(\frac{\pi}{p}\right)} - \frac{\ln 2 - \gamma}{(2n+1)^{1+\frac{1}{p}}} \right] a_n^p \right\}^{\frac{1}{p}} \\ \times \left\{ \sum_{n=0}^{\infty} \left[\frac{\pi}{\sin\left(\frac{\pi}{p}\right)} - \frac{\ln 2 - \gamma}{(2n+1)^{1+\frac{1}{q}}} \right] b_n^q \right\}^{\frac{1}{q}},$$



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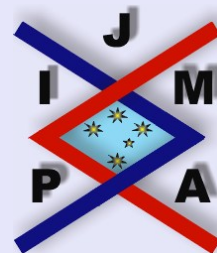
where $\ln 2 - \gamma = 0.11593^+$ (γ is Euler's constant). Another strengthened version of (1.1) was given in [6]. By introducing a parameter λ , two extensions of (1.1) were proved in [8, 7] as:

$$(1.4) \quad \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{a_m b_n}{(m+n+1)^\lambda} < k_\lambda(p) \left\{ \sum_{n=0}^{\infty} \left(n + \frac{1}{2}\right)^{1-\lambda} a_n^p \right\}^{\frac{1}{p}} \left\{ \sum_{n=0}^{\infty} \left(n + \frac{1}{2}\right)^{1-\lambda} b_n^q \right\}^{\frac{1}{q}};$$

$$(1.5) \quad \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{a_m b_n}{(m+n+1)^\lambda} < B\left(\frac{\lambda}{p}, \frac{\lambda}{q}\right) \times \left\{ \sum_{n=0}^{\infty} \left(n + \frac{1}{2}\right)^{(p-1)(1-\lambda)} a_n^p \right\}^{\frac{1}{p}} \left\{ \sum_{n=0}^{\infty} \left(n + \frac{1}{2}\right)^{(q-1)(1-\lambda)} b_n^q \right\}^{\frac{1}{q}},$$

where, the constant factors $k_\lambda(p) = B\left(\frac{p+\lambda-2}{p}, \frac{q+\lambda-2}{q}\right)$ ($2 - \min\{p, q\} < \lambda \leq 2$), and $B\left(\frac{\lambda}{p}, \frac{\lambda}{q}\right)$ ($0 < \lambda \leq \min\{p, q\}$) are the best possible ($B(u, v)$ is the β function). For $\lambda = 1$, both (1.4) and (1.5) reduce to (1.1). We call (1.4) and (1.5) Hardy-Hilbert type inequalities. Yang et al. [9] summarized the use of weight coefficients in research for Hardy-Hilbert type inequalities. But the problem on how to build the reverse of this type inequalities is unsolved.

The main objective of this paper is to deal with a reverse of inequality (1.4) for $\lambda = 2$. Another related reverse of the form is considered.



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2. Some Lemmas

We need the formula of the β function $B(p, q)$ as follows (see [4]):

$$(2.1) \quad B(p, q) = B(q, p) = \int_0^\infty \frac{t^{p-1}}{(1+t)^{p+q}} dt,$$

and the following inequality (see [5, 2]): If $f^4 \in C[0, \infty)$, $0 < \int_0^\infty f(t) dt < \infty$ and $(-1)^n f^{(n)}(x) > 0$, $f^{(n)}(\infty) = 0$ ($n = 0, 1, 2, 3, 4$), then

$$(2.2) \quad \int_0^\infty f(t) dt + \frac{1}{2}f(0) < \sum_{k=0}^\infty f(k) < \int_0^\infty f(t) dt + \frac{1}{2}f(0) - \frac{1}{12}f'(0).$$

Lemma 2.1. Define the weight function $\omega(n)$ as

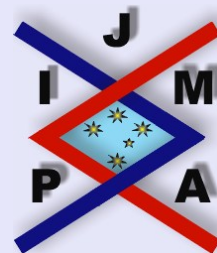
$$(2.3) \quad \omega(n) = \left(n + \frac{1}{2}\right) \sum_{k=0}^\infty \frac{1}{(k+n+1)^2}, \quad n \in \mathbb{N}_0 (= \mathbb{N} \cup \{0\}),$$

then we have

$$(2.4) \quad 1 - \frac{1}{4(n+1)^2} < \omega(n) < 1 - \frac{1}{6(n+1)(2n+1)} \quad (n \in \mathbb{N}_0).$$

Proof. For fixed n , setting $f(t) = \frac{1}{(t+n+1)^2}$ ($t \in [0, \infty)$), we obtain

$$f(0) = \frac{1}{(n+1)^2}, \quad f'(0) = -\frac{2}{(n+1)^3}, \quad \text{and} \quad \int_0^\infty f(t) dt = \frac{1}{n+1}.$$



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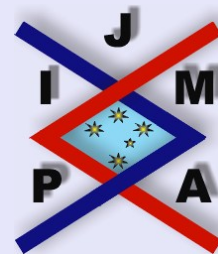
By (2.2), we find

$$\begin{aligned}
 (2.5) \quad \omega(n) &> \left[\int_0^\infty \frac{1}{(t+n+1)^2} dt + \frac{1}{2(n+1)^2} \right] \left(n + \frac{1}{2} \right) \\
 &= \left[\frac{1}{(n+1)} + \frac{1}{2(n+1)^2} \right] \left[(n+1) - \frac{1}{2} \right] \\
 &= 1 - \frac{1}{4(n+1)^2}.
 \end{aligned}$$

$$\begin{aligned}
 (2.6) \quad \omega(n) &< \left[\int_0^\infty \frac{1}{(t+n+1)^2} dt + \frac{1}{2(n+1)^2} + \frac{1}{6(n+1)^3} \right] \left(n + \frac{1}{2} \right) \\
 &= \left[\frac{1}{n+1} + \frac{1}{2(n+1)^2} + \frac{1}{6(n+1)^3} \right] \left[(n+1) - \frac{1}{2} \right] \\
 &= 1 - \left[\frac{1}{12(n+1)^2} + \frac{1}{12(n+1)^3} \right].
 \end{aligned}$$

Since for $n \in \mathbb{N}_0$,

$$\begin{aligned}
 &\left[\frac{1}{12(n+1)^2} + \frac{1}{12(n+1)^3} \right] 6(n+1)(2n+1) \\
 &= \frac{2(n+1)-1}{2(n+1)} + \frac{2(n+1)-1}{2(n+1)^2} \\
 &= 1 + \frac{1}{2(n+1)} - \frac{1}{2(n+1)^2} \geq 1,
 \end{aligned}$$



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then we find

$$\frac{1}{12(n+1)^2} + \frac{1}{12(n+1)^3} \geq \frac{1}{6(n+1)(2n+1)},$$

and in view of (2.6), it follows that

$$(2.7) \quad \omega(n) < 1 - \frac{1}{6(n+1)(2n+1)}.$$

In virtue of (2.5) and (2.7), we have (2.4). The lemma is proved. \square

Lemma 2.2. For $0 < \varepsilon < p$, we have

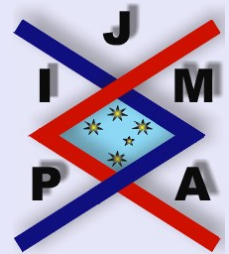
$$(2.8) \quad I := \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{1}{(m+n+1)^2} \left(m + \frac{1}{2}\right)^{-\frac{\varepsilon}{p}} \left(n + \frac{1}{2}\right)^{-\frac{\varepsilon}{q}} \\ < (1 + o(1)) \sum_{n=0}^{\infty} \frac{1}{\left(n + \frac{1}{2}\right)^{1+\varepsilon}} \quad (\varepsilon \rightarrow 0^+).$$

Proof. For fixed n , setting $f(t) = \frac{(t+\frac{1}{2})^{-\varepsilon/p}}{(t+n+1)^2}$ ($t \in (-\frac{1}{2}, \infty)$), then we have

$$f(0) = \frac{2^{\varepsilon/p}}{(n+1)^2}, \quad f'(0) = -\frac{2^{1+\varepsilon/p}}{(n+1)^3} - \frac{\varepsilon 2^{1+\varepsilon/p}}{p(n+1)^2}.$$

Setting $u = (t + \frac{1}{2}) / (n + \frac{1}{2})$ in the following integral, we obtain

$$\int_0^{\infty} f(t) dt < \int_{-1/2}^{\infty} f(t) dt = \frac{1}{(n + \frac{1}{2})^{1+\frac{\varepsilon}{p}}} \int_0^{\infty} \frac{u^{-\frac{\varepsilon}{p}}}{(1+u)^2} du.$$



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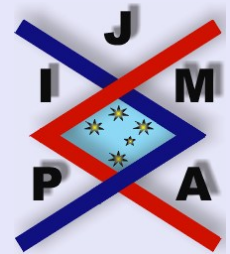
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Hence by (2.2) and (2.1), we find

$$\begin{aligned}
 I &= \sum_{n=0}^{\infty} \left(n + \frac{1}{2}\right)^{-\frac{\varepsilon}{q}} \left[\sum_{m=0}^{\infty} \frac{1}{(m+n+1)^2} \left(m + \frac{1}{2}\right)^{-\frac{\varepsilon}{p}} \right] \\
 &< \sum_{n=0}^{\infty} \left(n + \frac{1}{2}\right)^{-\frac{\varepsilon}{q}} \left[\frac{1}{\left(n + \frac{1}{2}\right)^{1+\frac{\varepsilon}{p}}} \int_0^{\infty} \frac{u^{-\frac{\varepsilon}{p}}}{(1+u)^2} du \right. \\
 &\quad \left. + \frac{2^{\varepsilon/p}}{2(n+1)^2} + \frac{2^{1+\varepsilon/p}}{12(n+1)^3} + \frac{\varepsilon 2^{1+\varepsilon/p}}{12p(n+1)^2} \right] \\
 &= \sum_{n=0}^{\infty} \frac{1}{\left(n + \frac{1}{2}\right)^{1+\varepsilon}} B\left(1 - \frac{\varepsilon}{p}, 1 + \frac{\varepsilon}{p}\right) + O(1) \quad (\varepsilon \rightarrow 0^+).
 \end{aligned}$$

Since $B\left(1 - \frac{\varepsilon}{p}, 1 + \frac{\varepsilon}{p}\right) \rightarrow B(1, 1) = 1$ ($\varepsilon \rightarrow 0^+$), then (2.8) is valid. The lemma is proved. \square



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3. Main Results

Theorem 3.1. If $0 < p < 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $a_n, b_n \geq 0$, such that $0 < \sum_{n=0}^{\infty} \frac{a_n^p}{2n+1} < \infty$ and $0 < \sum_{n=0}^{\infty} \frac{b_n^q}{2n+1} < \infty$, then

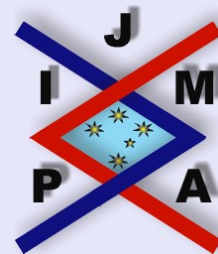
$$(3.1) \quad \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{a_m b_n}{(m+n+1)^2} > 2 \left\{ \sum_{n=0}^{\infty} \left[1 - \frac{1}{4(n+1)^2} \right] \frac{a_n^p}{2n+1} \right\}^{\frac{1}{p}} \\ \times \left\{ \sum_{n=0}^{\infty} \left[1 - \frac{1}{6(n+1)(2n+1)} \right] \frac{b_n^q}{2n+1} \right\}^{\frac{1}{q}},$$

where the constant factor 2 is the best possible. In particular, one has

$$(3.2) \quad \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{a_m b_n}{(m+n+1)^2} \\ > 2 \left\{ \sum_{n=0}^{\infty} \left[1 - \frac{1}{4(n+1)^2} \right] \frac{a_n^p}{2n+1} \right\}^{\frac{1}{p}} \left\{ \sum_{n=0}^{\infty} \frac{b_n^q}{2n+1} \right\}^{\frac{1}{q}}.$$

Proof. By the reverse of Hölder's inequality and (2.3), one has

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{a_m b_n}{(m+n+1)^2} \\ = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \left[\frac{a_m}{(m+n+1)^{\frac{2}{p}}} \right] \left[\frac{b_n}{(m+n+1)^{\frac{2}{q}}} \right]$$



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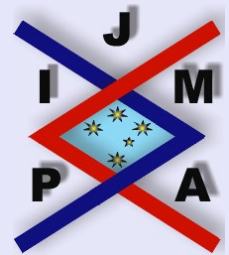
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$$\begin{aligned}
&\geq \left\{ \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{a_m^p}{(m+n+1)^2} \right\}^{\frac{1}{p}} \left\{ \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{b_n^q}{(m+n+1)^2} \right\}^{\frac{1}{q}} \\
&= \left\{ \sum_{m=0}^{\infty} \left[\sum_{n=0}^{\infty} \frac{m+\frac{1}{2}}{(m+n+1)^2} \right] \frac{2a_m^p}{2m+1} \right\}^{\frac{1}{p}} \left\{ \sum_{n=0}^{\infty} \left[\sum_{m=0}^{\infty} \frac{n+\frac{1}{2}}{(m+n+1)^2} \right] \frac{2b_n^q}{2n+1} \right\}^{\frac{1}{q}} \\
&= 2 \left\{ \sum_{m=0}^{\infty} \omega(m) \frac{a_m^p}{2m+1} \right\}^{\frac{1}{p}} \left\{ \sum_{n=0}^{\infty} \omega(n) \frac{b_n^q}{2n+1} \right\}^{\frac{1}{q}}.
\end{aligned}$$

Since $0 < p < 1$ and $q < 0$, by (2.4), it follows that (3.1) is valid.

For $0 < \varepsilon < p$, setting $\tilde{a}_n = (n + \frac{1}{2})^{-\varepsilon/p}$, $\tilde{b}_n = (n + \frac{1}{2})^{-\varepsilon/q}$, $n \in \mathbb{N}_0$, we find

$$\begin{aligned}
(3.3) \quad &\left\{ \sum_{n=0}^{\infty} \left[1 - \frac{1}{4(n+1)^2} \right] \frac{\tilde{a}_n^p}{2n+1} \right\}^{\frac{1}{p}} \left\{ \sum_{n=0}^{\infty} \frac{\tilde{b}_n^q}{2n+1} \right\}^{\frac{1}{q}} \\
&= \left\{ \sum_{n=0}^{\infty} \left[1 - \frac{1}{4(n+1)^2} \right] \frac{1}{2(n+\frac{1}{2})^{1+\varepsilon}} \right\}^{\frac{1}{p}} \left\{ \sum_{n=0}^{\infty} \frac{1}{2(n+\frac{1}{2})^{1+\varepsilon}} \right\}^{\frac{1}{q}} \\
&> \frac{1}{2} \left\{ \sum_{n=0}^{\infty} \frac{1}{(n+\frac{1}{2})^{1+\varepsilon}} - \sum_{n=0}^{\infty} \frac{1}{2(n+1)^2(2n+1)} \right\}^{\frac{1}{p}} \\
&\quad \times \left\{ \sum_{n=0}^{\infty} \frac{1}{(n+\frac{1}{2})^{1+\varepsilon}} \right\}^{\frac{1}{q}}
\end{aligned}$$



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$$= \frac{1}{2} \{1 - \tilde{o}(1)\}^{\frac{1}{p}} \sum_{n=0}^{\infty} \frac{1}{(n + \frac{1}{2})^{1+\varepsilon}} \quad (\varepsilon \rightarrow 0^+).$$

If the constant factor 2 in (3.1) is not the best possible, then there exists a real number k with $k > 2$, such that (3.1) is still valid if one replaces 2 by k . In particular, one has

$$(3.4) \quad \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\tilde{a}_m \tilde{b}_n}{(m+n+1)^2} > k \left\{ \sum_{n=0}^{\infty} \left[1 - \frac{1}{4(n+1)^2} \right] \frac{\tilde{a}_n^p}{2n+1} \right\}^{\frac{1}{p}} \left\{ \sum_{n=0}^{\infty} \frac{\tilde{b}_n^q}{2n+1} \right\}^{\frac{1}{q}}.$$

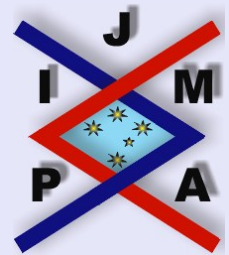
Hence by (2.8) and (3.3), it follows that

$$(1 + o(1)) \sum_{n=0}^{\infty} \frac{1}{(n + \frac{1}{2})^{1+\varepsilon}} > \frac{k}{2} \{1 - \tilde{o}(1)\}^{\frac{1}{p}} \sum_{n=0}^{\infty} \frac{1}{(n + \frac{1}{2})^{1+\varepsilon}},$$

and then $2 \geq k(\varepsilon \rightarrow 0^+)$. This contradicts the fact that $k > 2$. Hence the constant factor 2 in (3.1) is the best possible. The theorem is proved. \square

Theorem 3.2. *If $0 < p < 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $a_n \geq 0$, such that $0 < \sum_{n=0}^{\infty} \frac{a_n^p}{2n+1} < \infty$, then*

$$(3.5) \quad \sum_{n=0}^{\infty} \left(n + \frac{1}{2}\right)^{p-1} \left[\sum_{m=0}^{\infty} \frac{a_m}{(m+n+1)^2} \right]^p > 2 \sum_{n=0}^{\infty} \left[1 - \frac{1}{4(n+1)^2} \right] \frac{a_n^p}{2n+1},$$



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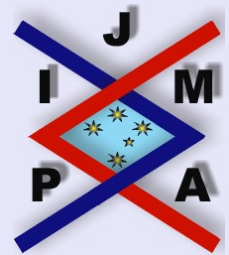
where the constant factor 2 is the best possible.

Proof. By the reverse of Hölder's inequality, (2.3) and (2.4), one has $\omega(n) < 1$ and

$$\begin{aligned}
 (3.6) \quad & \left[\sum_{m=0}^{\infty} \frac{a_m}{(m+n+1)^2} \right]^p \\
 &= \left\{ \sum_{m=0}^{\infty} \left[\frac{a_m}{(m+n+1)^{\frac{2}{p}}} \right] \left[\frac{1}{(m+n+1)^{\frac{2}{q}}} \right] \right\}^p \\
 &\geq \left\{ \sum_{m=0}^{\infty} \frac{a_m^p}{(m+n+1)^2} \right\} \left\{ \sum_{m=0}^{\infty} \frac{1}{(m+n+1)^2} \right\}^{p-1} \\
 &= \left\{ \sum_{m=0}^{\infty} \frac{a_m^p}{(m+n+1)^2} \right\} \left\{ \omega(n) \left(n + \frac{1}{2} \right)^{-1} \right\}^{p-1} \\
 &> \left(n + \frac{1}{2} \right)^{1-p} \sum_{m=0}^{\infty} \frac{a_m^p}{(m+n+1)^2}.
 \end{aligned}$$

Hence we find

$$\begin{aligned}
 (3.7) \quad & \sum_{n=0}^{\infty} \left(n + \frac{1}{2} \right)^{p-1} \left[\sum_{m=0}^{\infty} \frac{a_m}{(m+n+1)^2} \right]^p \\
 &> \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{a_m^p}{(m+n+1)^2}
 \end{aligned}$$



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$$\begin{aligned}
&= \sum_{m=0}^{\infty} \left[\sum_{n=0}^{\infty} \frac{m + \frac{1}{2}}{(m+n+1)^2} \right] \frac{2a_m^p}{2m+1} \\
&= 2 \sum_{m=0}^{\infty} \omega(m) \frac{a_m^p}{2m+1}.
\end{aligned}$$

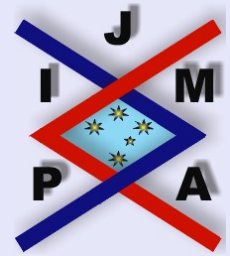
By (2.4), we have (3.5).

Setting $b_n \geq 0$ and $0 < \sum_{n=0}^{\infty} \frac{b_n^q}{2n+1} < \infty$, by the reverse of Hölder's inequality, one has

$$\begin{aligned}
(3.8) \quad &\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{a_m b_n}{(m+n+1)^2} \\
&= \sum_{n=0}^{\infty} \left[\left(n + \frac{1}{2} \right)^{\frac{1}{q}} \sum_{m=0}^{\infty} \frac{a_m}{(m+n+1)^2} \right] \left[\left(n + \frac{1}{2} \right)^{-\frac{1}{q}} b_n \right] \\
&\geq \left\{ \sum_{n=0}^{\infty} \left(n + \frac{1}{2} \right)^{p-1} \left[\sum_{m=0}^{\infty} \frac{a_m}{(m+n+1)^2} \right]^p \right\}^{\frac{1}{p}} \left\{ \sum_{n=0}^{\infty} \frac{2b_n^q}{2n+1} \right\}^{\frac{1}{q}}.
\end{aligned}$$

If the constant factor 2 in (3.5) is not the best possible, then by (3.6), we can get a contradiction that the constant factor 2 in (3.1) is not the best possible. The theorem is proved. \square

Remark 1. If a_n, b_n satisfy the conditions of (1.4) for $\lambda = 2, r > 1, \frac{1}{r} + \frac{1}{s} = 1$, and (3.2) for $0 < p < 1, \frac{1}{p} + \frac{1}{q} = 1$, then one can get the following two-sided



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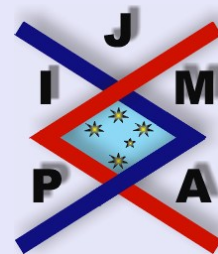
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inequality:

$$(3.9) \quad 0 < \left\{ \frac{3}{4} \sum_{n=0}^{\infty} \frac{a_n^p}{2n+1} \right\}^{\frac{1}{p}} \left\{ \sum_{n=0}^{\infty} \frac{b_n^q}{2n+1} \right\}^{\frac{1}{q}} \\ < \frac{1}{2} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{a_m b_n}{(m+n+1)^2} \\ < \left\{ \sum_{n=0}^{\infty} \frac{a_n^r}{2n+1} \right\}^{\frac{1}{r}} \left\{ \sum_{n=0}^{\infty} \frac{b_n^s}{2n+1} \right\}^{\frac{1}{s}} < \infty.$$



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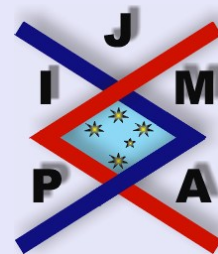
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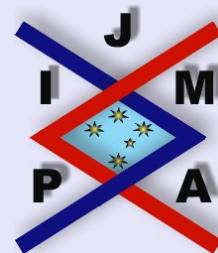
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