Journal of Homotopy and Related Structures, vol. 2(2), 2007, pp.187–216

THIRD MAC LANE COHOMOLOGY VIA CATEGORICAL RINGS

M. JIBLADZE AND T. PIRASHVILI

(*communicated by George Janelidze*)

Abstract

A notion of categorical ring is introduced. Categorical rings are proved to be classifiable by the third Mac Lane cohomology group.

Introduction

In the fifties, Saunders Mac Lane invented a cohomology theory of rings using the cubical construction introduced earlier by Eilenberg and himself to calculate stable homology of Eilenberg-Mac Lane spaces. As shown in [9], this theory coincides with the topological Hochschild cohomology for Eilenberg-Mac Lane ring spectra. In particular, the third dimensional cohomology group is expected to provide classification of 2-types of ring spectra. Some algebraic models for such 2-types have been constructed in [1]. In this paper we consider one such algebraic model of different kind which in our opinion is especially straightforwardly related to 3-cocycles in Mac Lane cohomology.

This is the notion of categorical ring—a category carrying the structure of a ring up to some natural isomorphisms satisfying certain coherence conditions. Our axioms for the categorical ring present a slightly modified version of the notion of Ann-category due to Quang [10]. Axioms we use reflect defining relations of Mac Lane 3-cocycles. Our main result is Theorem 4.4 which asserts that for any ring R and any R-bimodule B there is a bijection

$$H^3(R; B) \approx \mathrm{Crext}(R; B)$$

between the third Mac Lane cohomology group of R with coefficients in B and equivalence classes of categorical rings $\mathscr{R}$ with $\pi_0(\mathscr{R}) = R$, $\pi_1(\mathscr{R}) = B$, and the induced bimodule structure coinciding with the original one.

In [10], Ann-categories of a particular kind—the so called regular ones—are considered. These correspond to the ring spectra whose underlying spectrum splits into a product of Eilenberg-Mac Lane spectra. It is shown in [10] that regular Ann-categories are classified by the third Shukla cohomology group [11]. The latter is the Barr-Beck-Quillen cohomology group for the category of associative rings [2].

Difference between the above two cases is quite subtle. In fact, Shukla and Mac Lane cohomologies are isomorphic up to dimension 2 (in dimensions 0 and 1 both also coincide with the Hochschild cohomology; this coincidence extends to dimension 2 if the underlying abelian group of R is free). The third Shukla cohomology group embeds into the

Received August 31, 2006, revised March 3, 2007; published on December 27, 2007.
2000 Mathematics Subject Classification: 16E40, 18D10.
Key words and phrases: Mac Lane cohomology, topological Hochschild cohomology, categorification, categorical rings.

third Mac Lane cohomology group, and failure of isomorphism is measured by an explicit obstruction furnished by the following exact sequence (see [7, 8], [5], [2]):

$$0 \to H^3_{\text{Shukla}}(R;B) \to H^3(R;B) \to H^0(R;{}_2B).$$

It can be shown that in terms of categorical rings, the above map $H^3(R;B) \to H^0(R;{}_2B)$ sends an element of $H^3(R;B)$ represented by a categorical ring $\mathscr{R}$ to the element

$$0 \to 0+0 \xrightarrow{\{0,0\}} 0+0 \to 0$$

of $\mathrm{Aut}_{\mathscr{R}}(0) = B$, where 0 is the neutral object with respect to the additive structure, $\{x,y\} : x+y \to y+x$ is the commutativity constraint, and the isomorphisms between 0 and $0+0$ are the canonical ones.

1. Recollections on symmetric categorical groups

Let us begin by recalling

(1.1) Definition. A *categorical group* $\mathscr{A}$ is a groupoid equipped with a monoidal structure—i. e. a bifunctor $+ : \mathscr{A} \times \mathscr{A} \to \mathscr{A}$, an object $0 \in \mathscr{A}$ and natural isomorphisms $\langle a,b,c\rangle : (a+b)+c \to a+(b+c)$, $\lambda(a) : 0+a \to a$ and $\rho(a) : a+0 \to a$ satisfying the Mac Lane coherence conditions—together with a choice, for each object $a \in \mathscr{A}$, of another object $-a \in \mathscr{A}$ and of an isomorphism $\iota(a) : -a+a \to 0$.

A *braiding* on a categorical group $\mathscr{A}$ is a collection of isomorphisms $\{a,b\} : a+b \to b+a$ turning $\mathscr{A}$ into a braided monoidal category; a *symmetric* categorical group is a braided one whose monoidal structure is symmetric, i. e. the inverse of $\{a,b\}$ is equal to $\{b,a\}$ for any objects a,b.

Categorical groups are also known in the literature under the name of *Picard categories*. There is much literature on them, see e. g. references in [13]. The fact that categorical groups are classified by third group cohomology was discovered in [12]. As one more example of relatively early fundamental work on the topic one could name e. g. [4].

We will need some specific facts and auxiliary notation concerning symmetric categorical groups. More on braided and symmetric categorical groups can be found e. g. in [6].

It is easy to see that for any monoidal functor $\boldsymbol{f} = (f, f_+, f_0) : \mathscr{A} \to \mathscr{A}'$ to a categorical group, the canonical isomorphism $f_0 : f(0) \to 0$ is determined by the rest of the structure. Namely, f_0 is equal to the composite

$$\begin{aligned} f(0) &\xrightarrow{\lambda(f(0))^{-1}} 0+f(0) \xrightarrow{\iota(f(0))^{-1}+f(0)} (-f(0)+f(0))+f(0) \xrightarrow{\langle -f(0),f(0),f(0)\rangle} -f(0)+(f(0)+f(0)) \\ &\xrightarrow{-f(0)+f_+(0,0)} -f(0)+f(0+0) \xrightarrow{-f(0)+f(\lambda(0))} -f(0)+f(0) \xrightarrow{\iota(f(0))} 0 \end{aligned} \qquad (1.2)$$

We will use the well-known fact that a monoidal functor $\boldsymbol{f}:\mathscr{A}\to\mathscr{A}'$ between categorical groups is an equivalence if and only if it induces an isomorphism on π_0 and π_1. Here, π_1 of a categorical group is defined to be the automorphism group of its neutral object, and the homomorphism $f_\#:\pi_1(\mathscr{A})\to\pi_1(\mathscr{A}')$ induced by a monoidal functor $\boldsymbol{f}$ assigns to $\alpha:0_\mathscr{A}\to 0_\mathscr{A}$ the composite

$$0_{\mathscr{A}'}\xrightarrow{f_0^{-1}} f(0_\mathscr{A})\xrightarrow{f(\alpha)} f(0_\mathscr{A})\xrightarrow{f_0} 0_{\mathscr{A}'}.$$

It is equally well known that in a categorical group $\mathscr{A}$, hom(x,y) has a structure of a bitorsor under $\pi_1(\mathscr{A})$ for any isomorphic objects x, y of $\mathscr{A}$. In more detail, hom(x,y) has compatible left hom(y,y)- and right hom(x,x)-actions which are principal, i. e. for any $\varphi,\varphi':x\to y$ there is a unique $\chi:x\to x$ with $\varphi'=\varphi\chi$ (namely $\chi=\varphi^{-1}\varphi'$) and a unique $\upsilon:y\to y$ with $\varphi'=\upsilon\varphi$ (namely, $\upsilon=\varphi'\varphi^{-1}$). And on the other hand for any categorical group $\mathscr{A}$ the maps $\pi_1(\mathscr{A})\to$hom(x,x) sending $\alpha:0\to 0$ to the composite

$$x\xrightarrow{\lambda(x)^{-1}}0+x\xrightarrow{\alpha+x}0+x\xrightarrow{\lambda(x)}x$$

are group isomorphisms for any object x, which allows to transfer the above bitorsor structures from hom(x,x), resp. hom(y,y), to $\pi_1(\mathscr{A})$.

In particular, for any two parallel morphisms $\varphi,\varphi':x\to y$ there exists a *unique* $\alpha\in\pi_1(\mathscr{A})$ making the diagram

$$\begin{array}{ccc} 0+x & \xrightarrow{\alpha+\varphi} & 0+y \\ {\scriptstyle\lambda(x)}\downarrow & & \downarrow{\scriptstyle\lambda(y)} \\ x & \xrightarrow{\varphi'} & y \end{array}$$

commute. It will be more convenient for us to depict such circumstances by a diagram of the form

$$x\ \underset{\varphi'}{\overset{\varphi}{\rightrightarrows}}\ y \quad (\alpha).$$

In such cases we will also write $\alpha=\varphi'\div\varphi$.

Note that exchanging order of φ and φ' introduces a sign, i. e. one has

$$x\ \underset{\varphi'}{\overset{\varphi}{\rightrightarrows}}\ y\ (\alpha) \qquad\Longleftrightarrow\qquad x\ \underset{\varphi}{\overset{\varphi'}{\rightrightarrows}}\ y\ (-\alpha);$$

in the diagrams that we will encounter, this order is not specified as it can be unambiguously recovered from the context.

We will need the following simple fact concerning this formalism.

(1.3) Proposition. *Let $\boldsymbol{f}=(f,f_+):\mathscr{A}\to\mathscr{A}'$ be a monoidal functor between categorical* *groups. Then for any parallel arrows $\varphi, \varphi' : x \to y$ in $\mathscr{A}$ one has*

$$x \underset{\varphi'}{\overset{\varphi}{\underset{\alpha}{\rightrightarrows}}} y \qquad\Rightarrow\qquad f(x) \underset{f(\varphi')}{\overset{f(\varphi)}{\underset{f_\#(\alpha)}{\rightrightarrows}}} f(y).$$

Proof. There is a commutative diagram

$$\begin{array}{ccc}
0+f(x) & \xrightarrow{f_\#(\alpha)+f(\varphi)} & 0+f(y) \\
{\scriptstyle f_0^{-1}+f(x)}\downarrow & & \downarrow{\scriptstyle f_0^{-1}+f(y)} \\
f(0)+f(x) & \xrightarrow{f(\alpha)+f(\varphi)} & f(0)+f(y) \\
{\scriptstyle f_+(0,x)}\downarrow & & \downarrow{\scriptstyle f_+(0,y)} \\
f(0+x) & \xrightarrow{f(\alpha+\varphi)} & f(0+y) \\
{\scriptstyle f(\lambda(x))}\downarrow & & \downarrow{\scriptstyle f(\lambda(y))} \\
f(x) & \xrightarrow{f(\varphi')} & f(y).
\end{array}$$

By the aforementioned uniqueness, the proposition follows. □

(1.4) Corollary. *For any parallel arrows $\varphi_i, \varphi_i' : x_i \to y_i$, $i = 1, 2$, in a braided (in particular, symmetric) categorical group $\mathscr{A}$ one has*

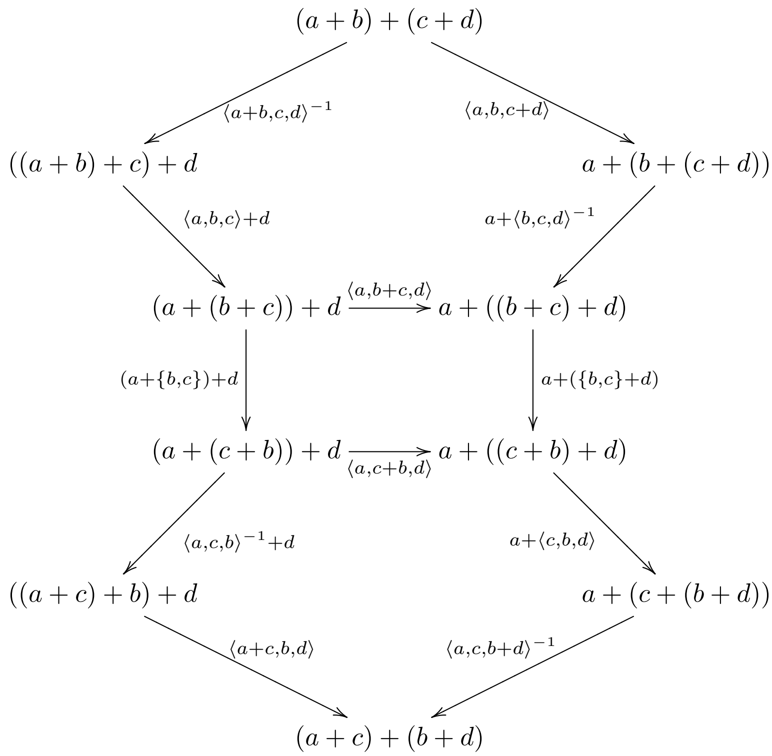
$$x_1 \underset{\varphi_1'}{\overset{\varphi_1}{\underset{\alpha_1}{\rightrightarrows}}} y_1\ , \quad x_2 \underset{\varphi_2'}{\overset{\varphi_2}{\underset{\alpha_2}{\rightrightarrows}}} y_2 \qquad\Rightarrow\qquad x_1+x_2 \underset{\varphi_1'+\varphi_2'}{\overset{\varphi_1+\varphi_2}{\underset{\alpha_1+\alpha_2}{\rightrightarrows}}} y_1+y_2.$$

Proof. It is well known that for a braided category $\mathscr{A}$ the functor $+ : \mathscr{A} \times \mathscr{A} \to \mathscr{A}$ acquires a monoidal structure (in fact it is known that for any monoidal category there is a one-to-one correspondence between braidings and monoidal functor structures on $+$). Since obviously $+_\#(\alpha_1, \alpha_2) = \alpha_1 + \alpha_2$ for any $\alpha_1, \alpha_2 \in \pi_1(\mathscr{A})$, the statement follows from (1.3). □

For any four objects a, b, c, d of a symmetric categorical group $\mathscr{A}$, by

$$\left\langle \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right\rangle : (a+b)+(c+d) \to (a+c)+(b+d)$$

will be denoted the composite canonical isomorphism in the commutative diagram



2. Categorical rings

Our algebraic models for 2-types of ring spectra are certain bimonoidal categories which we call categorical rings. They can be called "rings up to coherent isomorphisms", in the sense that (a) isomorphism classes of objects of a categorical ring form an associative ring; and (b) the structure of a categorical ring on a category is an equivalence invariant, i. e. any equivalence between a categorical ring and another category allows one to transfer the categorical ring structure along it.

(2.1) Definition. A *categorical ring* is a symmetric categorical group $\mathscr{R}$ together with a bifunctor (denoted by juxtaposition) $\mathscr{R} \times \mathscr{R} \to \mathscr{R}$, an object $1 \in \mathscr{R}$, and natural isomorphisms

$$[r, s, t] : (rs)t \to r(st)$$

(associativity),

$$\lambda.(r) : 1r \to r, \ \ \rho.(r) : r1 \to r$$

(left and right unitality),

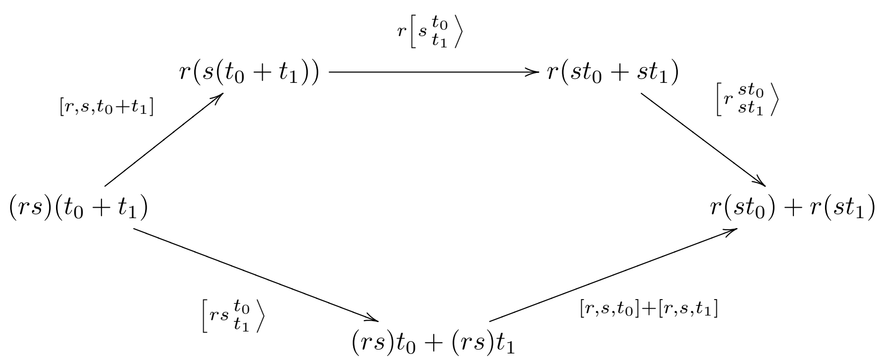
$$[r \,{}^{s_0}_{s_1}\rangle : r(s_0 + s_1) \to rs_0 + rs_1$$

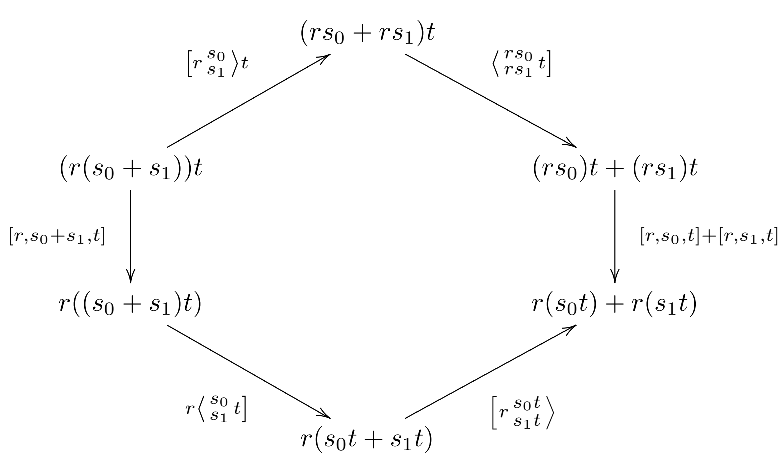
(left distributivity),

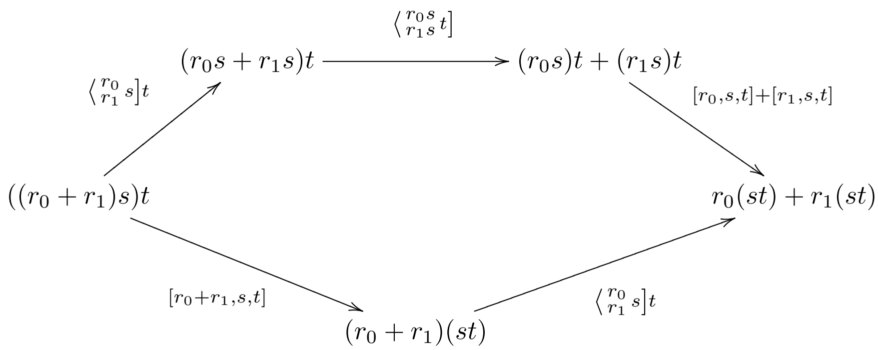
$$\left\langle {r_0 \atop r_1} s \right] : (r_0 + r_1)s \to r_0 s + r_1 s$$

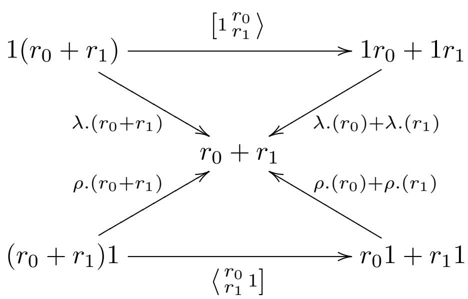
(right distributivity).

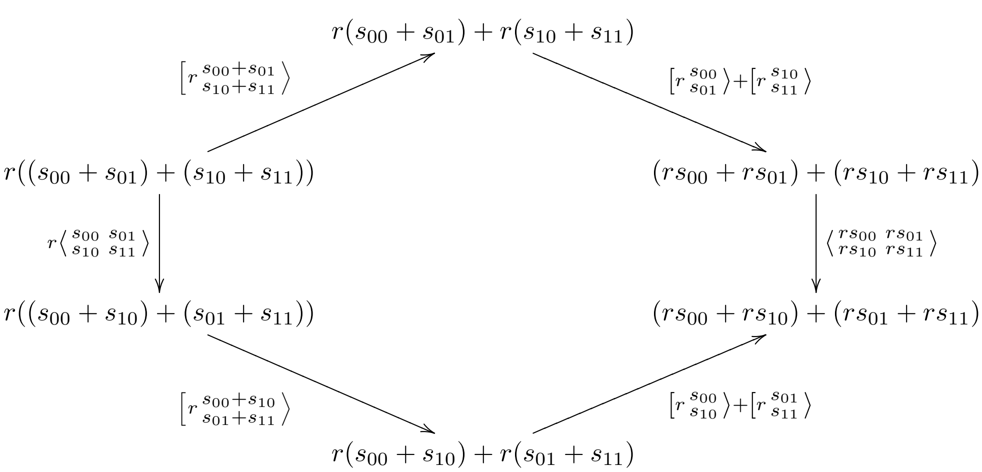
It is required that the $[\,,\,,\,]$ together with $\lambda.$ and $\rho.$ constitute a monoidal structure (i. e. the appropriate pentagonal and triangular coherence diagrams commute for it) and moreover the following diagrams commute for all possible objects of $\mathscr{R}$:

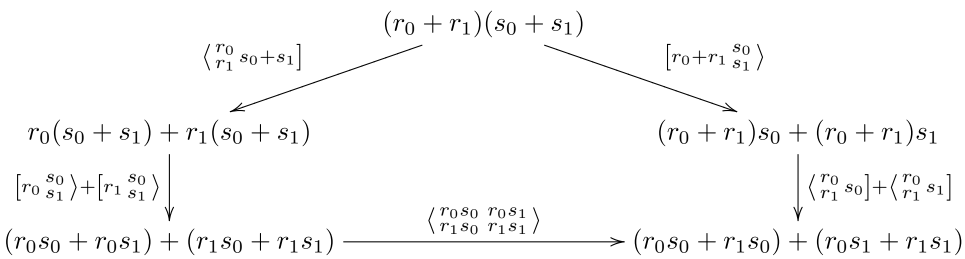


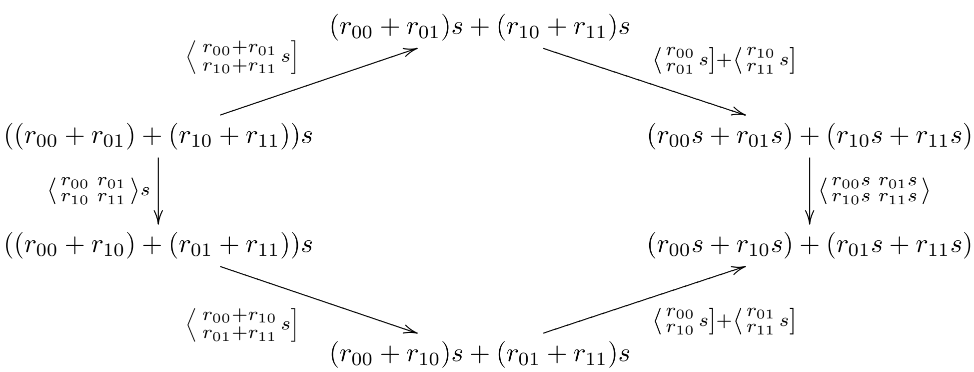



$((r_0+r_1)s)t$
$(r_0s+r_1s)t$
$(r_0s)t+(r_1s)t$
$r_0(st)+r_1(st)$
$(r_0+r_1)(st)$
$\langle {}^{r_0}_{r_1} s]t$
$\langle {}^{r_0 s}_{r_1 s} t]$
$[r_0,s,t]+[r_1,s,t]$
$[r_0+r_1,s,t]$
$\langle {}^{r_0}_{r_1} s]t$


$1(r_0+r_1)$
$1r_0+1r_1$
r_0+r_1
$(r_0+r_1)1$
r_01+r_11
$[1 {}^{r_0}_{r_1}\rangle$
$\lambda.(r_0+r_1)$
$\lambda.(r_0)+\lambda.(r_1)$
$\rho.(r_0+r_1)$
$\rho.(r_0)+\rho.(r_1)$
$\langle {}^{r_0}_{r_1} 1]$


$r(s_{00}+s_{01})+r(s_{10}+s_{11})$
$r((s_{00}+s_{01})+(s_{10}+s_{11}))$
$(rs_{00}+rs_{01})+(rs_{10}+rs_{11})$
$r((s_{00}+s_{10})+(s_{01}+s_{11}))$
$(rs_{00}+rs_{10})+(rs_{01}+rs_{11})$
$r(s_{00}+s_{10})+r(s_{01}+s_{11})$
$[r\, {}^{s_{00}+s_{01}}_{s_{10}+s_{11}}\rangle$
$[r\, {}^{s_{00}}_{s_{01}}\rangle+[r\, {}^{s_{10}}_{s_{11}}\rangle$
$r\langle {}^{s_{00}\ s_{01}}_{s_{10}\ s_{11}}\rangle$
$\langle {}^{rs_{00}\ rs_{01}}_{rs_{10}\ rs_{11}}\rangle$
$[r\, {}^{s_{00}+s_{10}}_{s_{01}+s_{11}}\rangle$
$[r\, {}^{s_{00}}_{s_{10}}\rangle+[r\, {}^{s_{01}}_{s_{11}}\rangle$





It can be shown that the notion of categorical ring is essentially the same as the one of Ann-category from [10], in the sense that a categorical ring satisfies coherence conditions from [10] and vice versa. Note however that the notion of *regular* Ann-category is strictly stronger; it corresponds to categorical rings whose symmetry morphisms $\{a, a\}$ are equal to the identity of a for all objects a.

Morphisms of categorical rings are defined as follows:

(2.2) Definition. A *2-homomorphism* $\boldsymbol{f} : \mathscr{R} \to \mathscr{R}'$ is a quadruple $(f, f_+, f_\cdot, f_1)$ where f is a functor from $\mathscr{R}$ to $\mathscr{R}'$, f_+, $f_\cdot$ are natural morphisms of the form

$$f_+(r_0, r_1) : f(r_0) + f(r_1) \to f(r_0 + r_1),$$
$$f_\cdot(r, s) : f(r)f(s) \to f(rs)$$

and $f_1 : f(1_{\mathscr{R}}) \to 1_{\mathscr{R}'}$ is a morphism such that (f, f_+, f_0) and $(f, f_\cdot, f_1)$ are monoidal functor structures with respect to the monoidal structures corresponding to $+$ and $\cdot$ respectively, with f_0 as in (1.2), and moreover the diagrams

$$\begin{array}{ccc}
 & f(r)f(s_0)+f(r)f(s_1) \xrightarrow{f.(r,s_0)+f.(r,s_1)} f(rs_0)+f(rs_1) & \\
\left[f(r)\,{}^{f(s_0)}_{f(s_1)}\right\rangle \nearrow & & \searrow f_+(rs_0,rs_1) \\
f(r)(f(s_0)+f(s_1)) & & f(rs_0+rs_1) \\
f(r)f_+(s_0,s_1) \searrow & & \nearrow f(\left[r\,{}^{s_0}_{s_1}\right\rangle) \\
 & f(r)f(s_0+s_1) \xrightarrow[f.(r,s_0+s_1)]{} f(r(s_0+s_1)) &
\end{array}$$

and

$$\begin{array}{ccc}
 & f(r_0)f(s)+f(r_1)f(s) \xrightarrow{f.(r_0,s)+f.(r_1,s)} f(r_0s)+f(r_1s) & \\
\left\langle {}^{f(r_0)}_{f(r_1)}\,f(s)\right] \nearrow & & \searrow f_+(r_0s,r_1s) \\
(f(r_0)+f(r_1))f(s) & & f(r_0s+r_1s) \\
f_+(r_0,r_1)f(s) \searrow & & \nearrow f(\left\langle {}^{r_0}_{r_0}\,s\right]) \\
 & f(r_0+r_1)f(s) \xrightarrow[f.(r_0+r_1,s)]{} f((r_0+r_1)s) &
\end{array}$$

commute for all possible objects involved.

We will need the following fact in what follows:

(2.3) Proposition. *In any categorical ring $\mathscr{R}$ one has*

$$x \overset{\varphi}{\underset{\varphi'}{\Downarrow\alpha}} y \quad \Rightarrow \quad rx \overset{r\varphi}{\underset{r\varphi'}{\Downarrow r\alpha}} ry, \qquad xr \overset{\varphi r}{\underset{\varphi' r}{\Downarrow \alpha r}} yr.$$

Proof. It follows from the definition of categorical ring that for any $r \in \mathscr{R}$ the morphisms $\left[r\,{}^{-}_{-}\right\rangle$, resp. $\left\langle {}^{-}_{-}\,r\right]$, constitute a structure of a monoidal functor on the endofunctor $r\cdot$, resp. $\cdot r : \mathscr{R} \to \mathscr{R}$ of $\mathscr{R}$ (with respect to the additive monoidal structure). The proposition is thus particular case of (1.3). $\square$

3. Third Mac Lane cohomology group

We refer to [7, 8] for the original construction of Mac Lane cohomology. Here we will only explicitate the definition of the third cohomology group as this is all that we need. To make expressions shorter, we will need the cross-effect notation. Recall that for a map $f : A \to B$ between abelian groups, its first cross-effect is a map

$$(-\mid-)_f : A \times A \to B$$

given by

$$(x \mid y)_f = f(x) + f(y) - f(x+y).$$

(3.1) Definition. For a ring R and an R-bimodule B, the group $C^3(R;B)$ of Mac Lane 3-cochains of R with coefficients in B consists of quadruples $(\varphi_{\cdot}, \varphi_{\cdot+}, \varphi_{+\cdot}, \varphi_{+})$, of maps

$$\varphi_{\cdot}, \varphi_{\cdot+}, \varphi_{+\cdot} : R^3 \to B$$

and

$$\varphi_{+} : R^4 \to B$$

which are *normalized* in the sense that $\varphi_{\cdot}$, $\varphi_{\cdot+}$ and $\varphi_{+\cdot}$ take zero values if one of their arguments is zero, and moreover φ_+ satisfies

$$\varphi_+\left(\begin{smallmatrix} r_0 & r_1 \\ 0 & 0 \end{smallmatrix}\right) = \varphi_+\left(\begin{smallmatrix} 0 & 0 \\ r_0 & r_1 \end{smallmatrix}\right) = \varphi_+\left(\begin{smallmatrix} r_0 & 0 \\ r_1 & 0 \end{smallmatrix}\right) = \varphi_+\left(\begin{smallmatrix} 0 & r_0 \\ 0 & r_1 \end{smallmatrix}\right) = \varphi_+\left(\begin{smallmatrix} r_0 & 0 \\ 0 & r_1 \end{smallmatrix}\right) = 0$$

for all $r_0, r_1 \in R$. The group structure is given by valuewise addition of functions.

The subgroup $Z^3(R;B) \subseteq C^3(R;B)$ of 3-cocycles is singled out by the following equations:

$$\begin{aligned}
r\varphi_{\cdot}(s,t,u) - \varphi_{\cdot}(rs,t,u) + \varphi_{\cdot}(r,st,u)& \\
-\varphi_{\cdot}(r,s,tu) + \varphi_{\cdot}(r,s,t)u &= 0, \\
r\varphi_{\cdot+}(s,t_0,t_1) - \varphi_{\cdot+}(rs,t_0,t_1) + \varphi_{\cdot+}(r,st_0,st_1) &= (t_0 \mid t_1)_{\varphi_{\cdot}(r,s,-)}, \\
\varphi_{\cdot+}(r,s_0t,s_1t) - \varphi_{\cdot+}(r,s_0,s_1)t &= \varphi_{+\cdot}(rs_0,rs_1,t) - r\varphi_{+\cdot}(s_0,s_1,t), \\
\varphi_{+\cdot}(r_0s,r_1s,t) - \varphi_{+\cdot}(r_0,r_1,st) + \varphi_{+\cdot}(r_0,r_1,s)t &= -(r_0 \mid r_1)_{\varphi_{\cdot}(-,s,t)}, \\
\varphi_+\left(\begin{smallmatrix} rs_{00} & rs_{01} \\ rs_{10} & rs_{11} \end{smallmatrix}\right) - r\varphi_+\left(\begin{smallmatrix} s_{00} & s_{01} \\ s_{10} & s_{11} \end{smallmatrix}\right) &= ((s_{00},s_{10}) \mid (s_{01},s_{11}))_{\varphi_{\cdot+}(r,-,-)} \\
&\quad - ((s_{00},s_{01}) \mid (s_{10},s_{11}))_{\varphi_{\cdot+}(r,-,-)}, \\
\varphi_+\left(\begin{smallmatrix} r_0s_0 & r_0s_1 \\ r_1s_0 & r_1s_1 \end{smallmatrix}\right) &= (r_0 \mid r_1)_{\varphi_{\cdot+}(-,s_0,s_1)} \\
&\quad - (s_0 \mid s_1)_{\varphi_{+\cdot}(r_0,r_1,-)}, \\
\varphi_+\left(\begin{smallmatrix} r_{00}s & r_{01}s \\ r_{10}s & r_{11}s \end{smallmatrix}\right) - \varphi_+\left(\begin{smallmatrix} r_{00} & r_{01} \\ r_{10} & r_{11} \end{smallmatrix}\right)s &= ((r_{00},r_{10}) \mid (r_{01},r_{11}))_{\varphi_{+\cdot}(-,-,s)} \\
&\quad - ((r_{00},r_{01}) \mid (r_{10},r_{11}))_{\varphi_{+\cdot}(-,-,s)}, \\
\left(\left(\begin{smallmatrix} r_{000} & r_{001} \\ r_{010} & r_{011} \end{smallmatrix}\right) \mid \left(\begin{smallmatrix} r_{100} & r_{101} \\ r_{110} & r_{111} \end{smallmatrix}\right)\right)_{\varphi_+}& \\
-\left(\left(\begin{smallmatrix} r_{000} & r_{001} \\ r_{100} & r_{101} \end{smallmatrix}\right) \mid \left(\begin{smallmatrix} r_{010} & r_{011} \\ r_{110} & r_{111} \end{smallmatrix}\right)\right)_{\varphi_+}& \\
+\left(\left(\begin{smallmatrix} r_{000} & r_{010} \\ r_{100} & r_{110} \end{smallmatrix}\right) \mid \left(\begin{smallmatrix} r_{001} & r_{011} \\ r_{101} & r_{111} \end{smallmatrix}\right)\right)_{\varphi_+} &= 0.
\end{aligned}$$

The subgroup $B^3(R;B) \subseteq Z^3(R;B)$ of 3-coboundaries consists of those quadruples $(\varphi_{\cdot}, \varphi_{\cdot+}, \varphi_{+\cdot}, \varphi_{+})$ for which there exist maps $\gamma_{\cdot}, \gamma_{+} : R^2 \to B$ such that

$$\begin{aligned}
\varphi_{\cdot}(r,s,t) &= r\gamma_{\cdot}(s,t) - \gamma_{\cdot}(rs,t) + \gamma_{\cdot}(r,st) - \gamma_{\cdot}(r,s)t, \\
\varphi_{\cdot+}(r,s_0,s_1) &= r\gamma_+(s_0,s_1) - \gamma_+(rs_0,rs_1) + (s_0 \mid s_1)_{\gamma_{\cdot}(r,-)}, \\
\varphi_{+\cdot}(r_0,r_1,s) &= \gamma_+(r_0s,r_1s) - \gamma_+(r_0,r_1)s - (r_0 \mid r_1)_{\gamma_{\cdot}(-,s)}, \\
\varphi_+\left(\begin{smallmatrix} r_{00} & r_{01} \\ r_{10} & r_{11} \end{smallmatrix}\right) &= ((r_{00},r_{01}) \mid (r_{10},r_{11}))_{\gamma_+} - ((r_{00},r_{10}) \mid (r_{01},r_{11}))_{\gamma_+}
\end{aligned}$$

for all $r, \ldots \in R$.

Finally, we define

$$H^3(R; B) := Z^3(R; B)/B^3(R; B).$$

4. Characteristic class of a categorical ring

Suppose given a categorical ring $\mathscr{R}$ as in 2.1. Then the set $R = \pi_0(\mathscr{R})$ of isomorphism classes of objects of $\mathscr{R}$ is a ring, and the group $B = \pi_1(\mathscr{R})$ of automorphisms of the zero object of $\mathscr{R}$ has a canonical structure of a $\pi_0(\mathscr{R})$-bimodule, with $r\alpha$, resp. αr, for any $r \in \pi_0(\mathscr{R})$ and any $\alpha : 0 \to 0$, given by

$$0 \xrightarrow{\bar{r}\cdot_0^{-1}} \bar{r}0 \xrightarrow{\bar{r}\alpha} \bar{r}0 \xrightarrow{\bar{r}\cdot_0} 0,$$

resp.

$$0 \xrightarrow{\cdot\bar{r}_0^{-1}} 0\bar{r} \xrightarrow{\alpha\bar{r}} 0\bar{r} \xrightarrow{\cdot\bar{r}_0} 0,$$

where $\bar{r}$ is any object from the isomorphism class r, and the morphisms $\bar{r}\cdot_0$, resp. $\cdot\bar{r}_0$, come from the monoidal functor structures on $\bar{r}\cdot$, resp. $\cdot\bar{r}$ indicated in the proof of 2.3.

We are going to assign to $\mathscr{R}$ a cohomology class

$$\langle\mathscr{R}\rangle \in H^3(\pi_0(\mathscr{R}); \pi_1(\mathscr{R})).$$

For that, we arbitrarily choose an object $\bar{r}$ of $\mathscr{R}$ in each isomorphism class $r \in \pi_0(\mathscr{R})$; moreover we arbitrarily choose morphisms

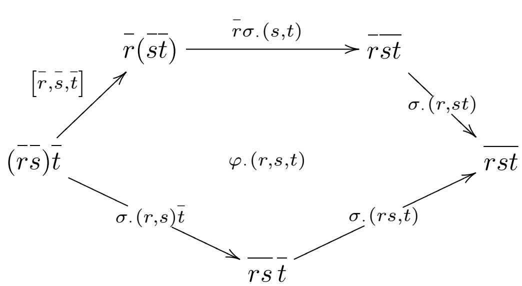
$$\sigma_\cdot(r, s) : \bar{r}\bar{s} \to \overline{rs}$$

and

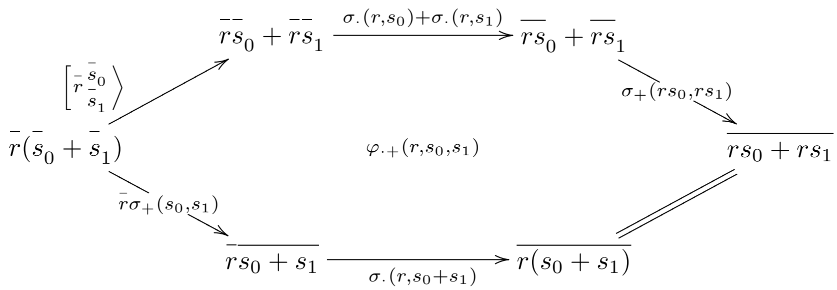
$$\sigma_+(r_0, r_1) : \bar{r}_0 + \bar{r}_1 \to \overline{r_0 + r_1}.$$

These morphisms give rise to several not necessarily commutative diagrams. They define elements of $\pi_1(\mathscr{R})$ as in section 1.

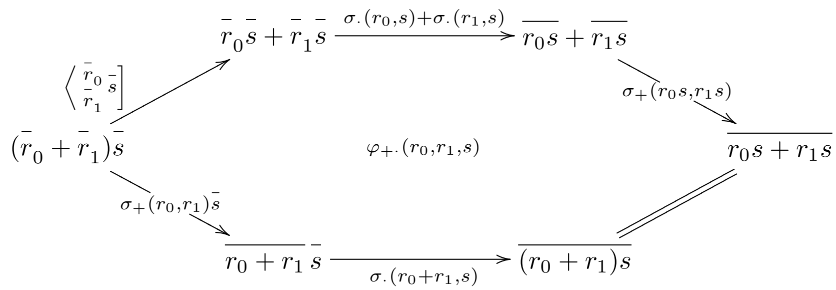
In particular, for any $r, s, t \in \pi_0(\mathscr{R})$ the diagram



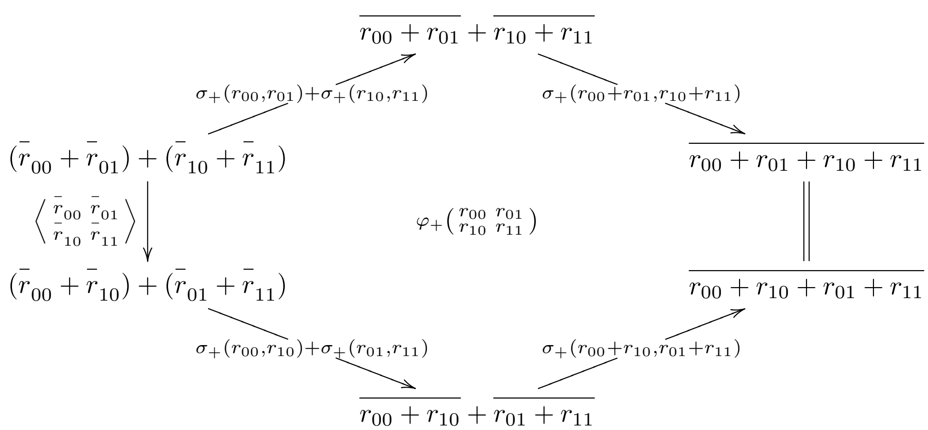
produces an element $\varphi_\cdot(r, s, t) \in \pi_1(\mathscr{R})$ measuring deviation from its commutativity. Similarly for any $r, s_0, s_1 \in \pi_0(\mathscr{R})$ deviation from commutativity of the diagram



is measured by an element $\varphi_{\cdot+}(r, s_0, s_1) \in \pi_1(\mathscr{R})$; for any $r_0, r_1, s \in \pi_0(\mathscr{R})$ the diagram



gives $\varphi_{+\cdot}(r_0, r_1, s) \in \pi_1(\mathscr{R})$; and for any $r_{00}, r_{01}, r_{10}, r_{11} \in \pi_0(\mathscr{R})$ the diagram



gives $\varphi_{+}\left(\begin{smallmatrix}r_{00} & r_{01}\\ r_{10} & r_{11}\end{smallmatrix}\right) \in \pi_1(\mathscr{R})$.

Thus the above diagrams give rise to a 3-cochain φ in the Mac Lane complex of $\pi_0(\mathscr{R})$

with coefficients in $\pi_1(\mathscr{R})$. Explicitly, it is defined by

$$\varphi_{\cdot}(r,s,t) = \sigma_{\cdot}(rs,t)\circ\sigma_{\cdot}(r,s)\bar{t} \;\div\; \sigma_{\cdot}(r,st)\circ\bar{r}\sigma_{\cdot}(s,t)\circ\left[\bar{r},\bar{s},\bar{t}\right],$$

$$\varphi_{\cdot+}(r,s_0,s_1) = \sigma_{\cdot}(r,s_0+s_1)\circ\bar{r}\sigma_{+}(s_0,s_1) \;\div\; \sigma_{+}(rs_0,rs_1)\circ(\sigma_{\cdot}(r,s_0)+\sigma_{\cdot}(r,s_1))\circ\left[\bar{r}\,{\bar{s}_0 \atop \bar{s}_1}\right\rangle,$$

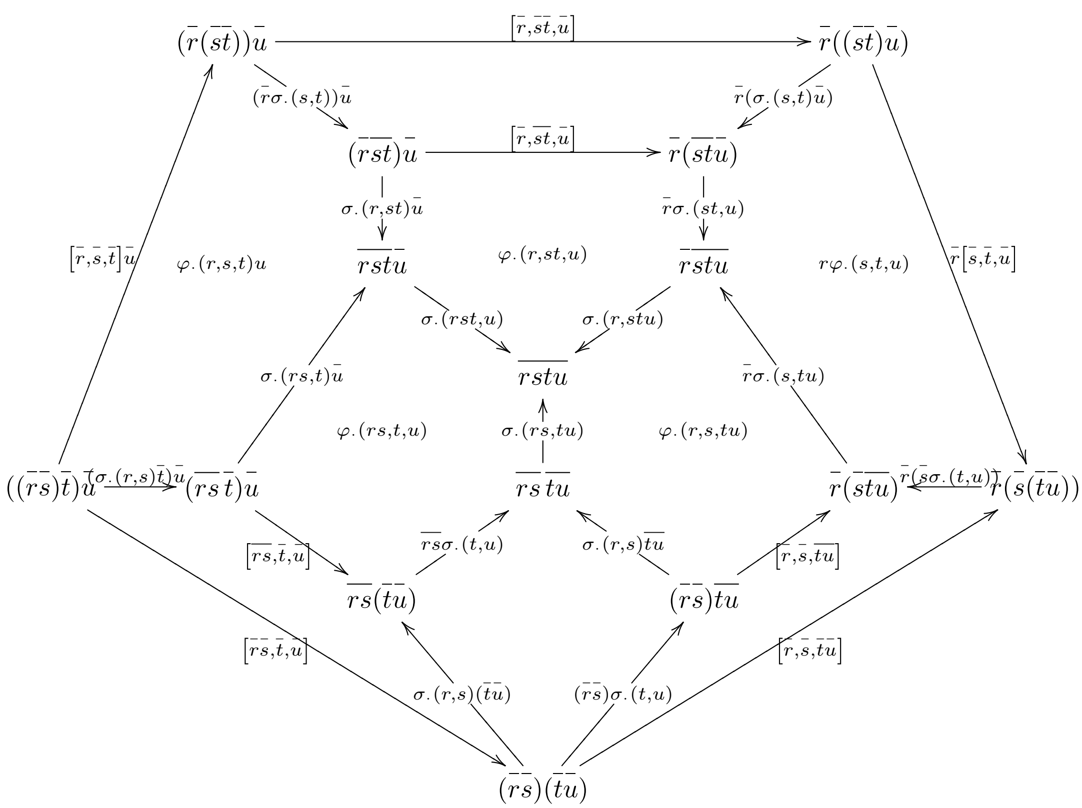
$$\varphi_{+\cdot}(r_0,r_1,s) = \sigma_{\cdot}(r_0+r_1,s)\circ\sigma_{+}(r_0,r_1)\bar{s} \;\div\; \sigma_{+}(r_0s,r_1s)\circ(\sigma_{\cdot}(r_0,s)+\sigma_{\cdot}(r_1,s))\circ\left\langle {\bar{r}_0 \atop \bar{r}_1}\,\bar{s}\right],$$

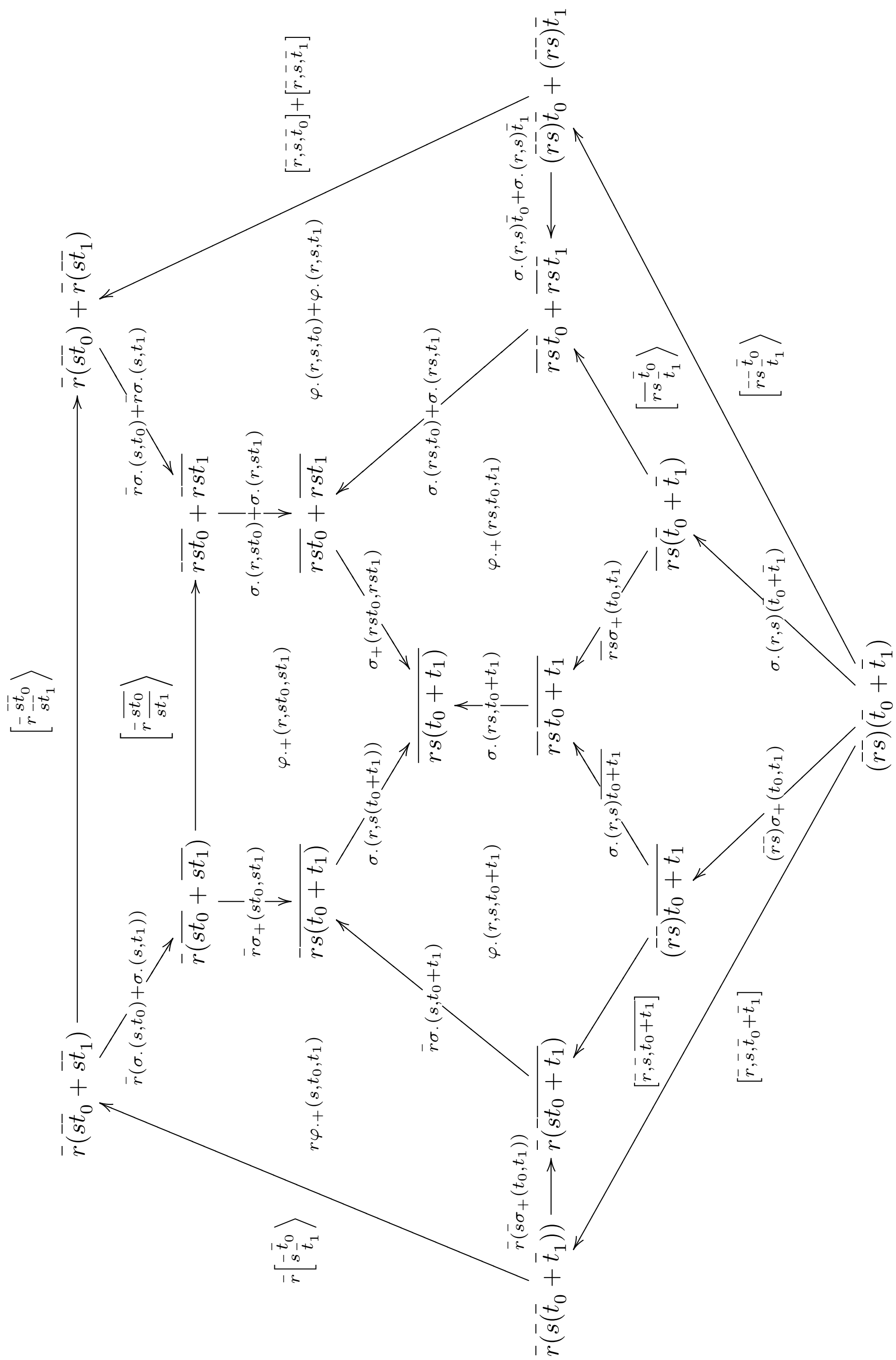
$$\varphi_{+}\left(\begin{smallmatrix} r_{00} & r_{01} \\ r_{10} & r_{11}\end{smallmatrix}\right) = \sigma_{+}(r_{00}+r_{10},r_{01}+r_{11})\circ(\sigma_{+}(r_{00},r_{10})+\sigma_{+}(r_{01},r_{11}))\circ\left\langle\begin{smallmatrix} \bar{r}_{00} & \bar{r}_{01} \\ \bar{r}_{10} & \bar{r}_{11}\end{smallmatrix}\right\rangle$$
$$\div\; \sigma_{+}(r_{00}+r_{01},r_{10}+r_{11})\circ(\sigma_{+}(r_{00},r_{01})+\sigma_{+}(r_{10},r_{11})).$$

We have

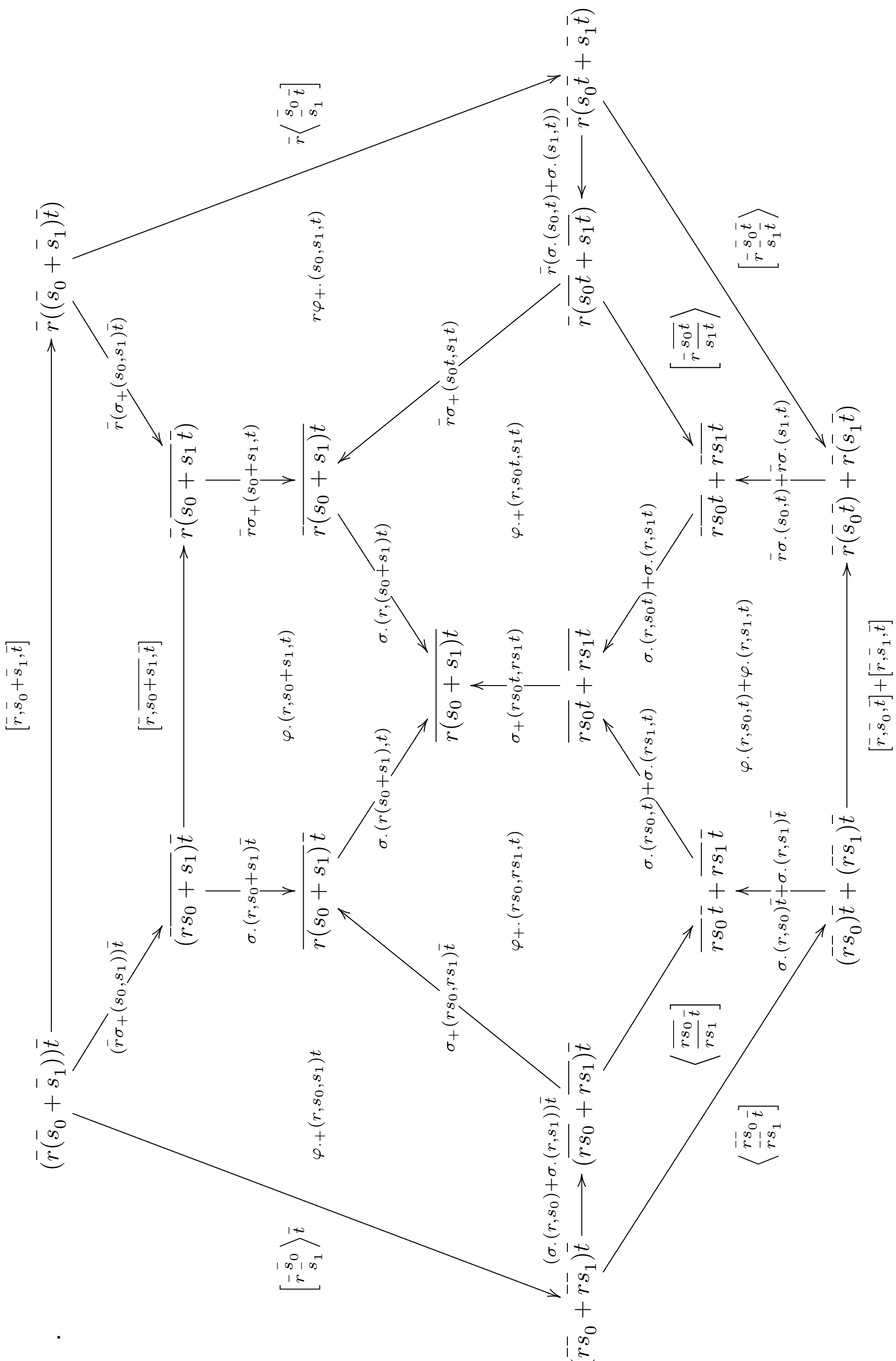
(4.1) Proposition. *The above cochain is a cocycle.*

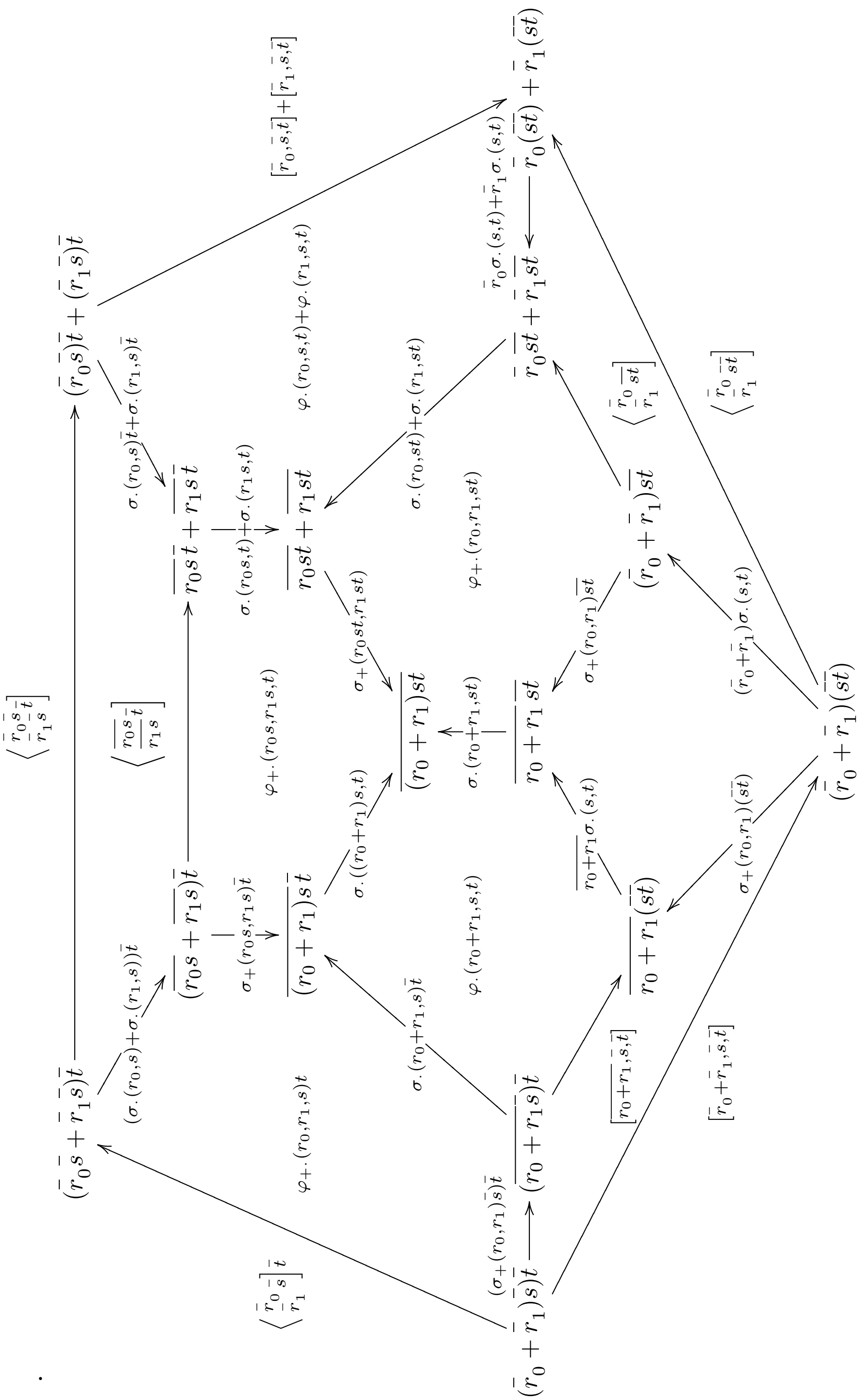
Proof. We have to check the eight equalities from (3.1). These equalities are deducible from considering eight diagrams below. In all of these diagrams, all quadrangles commute by naturality, the inner pentagons are filled by the indicated elements of $\pi_1(\mathscr{R})$ using (1.4) and (2.3) as needed, whereas the outer perimeters commute since each of them coincides with a coherence diagram from the definition of categorical ring.

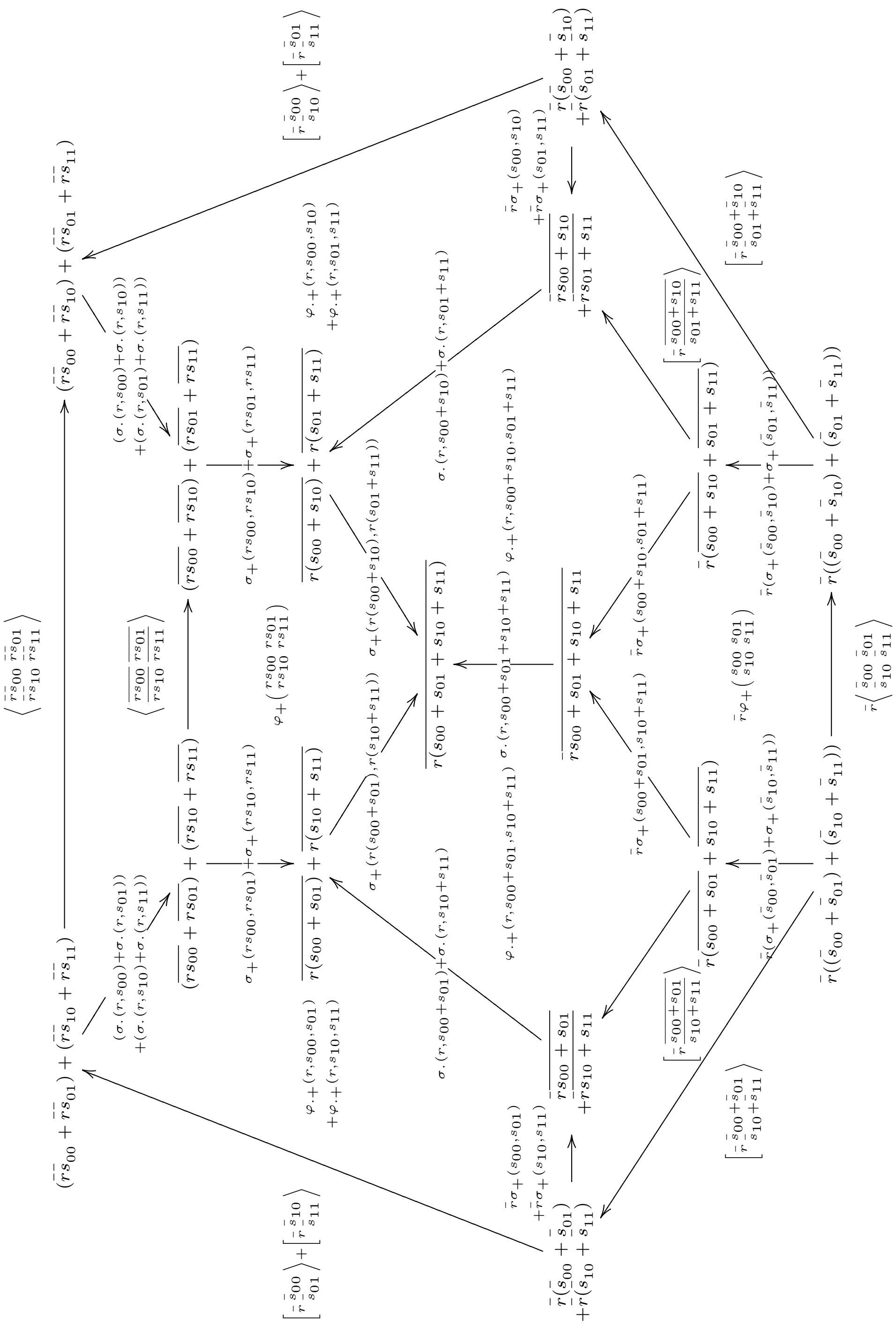




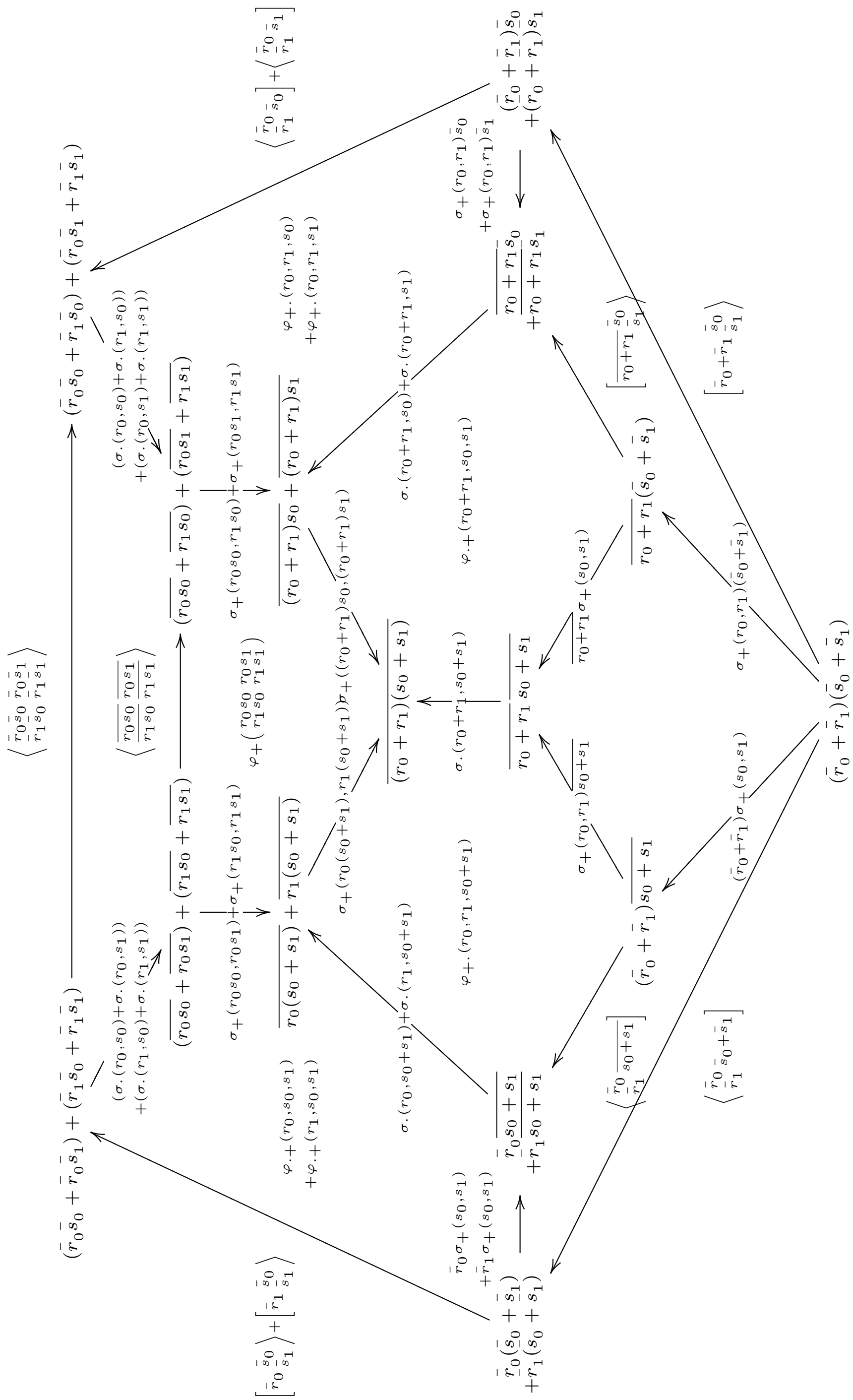
.



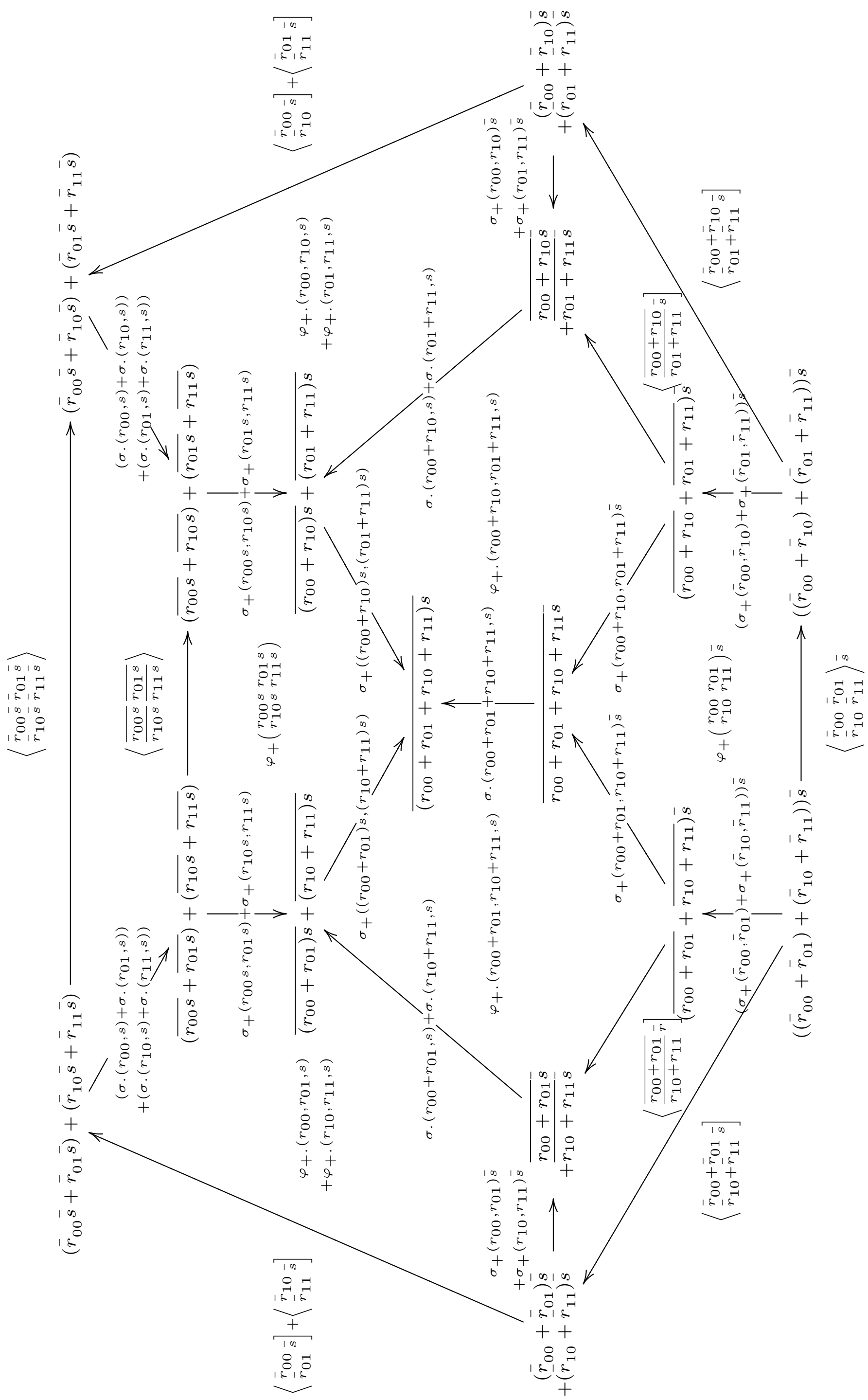




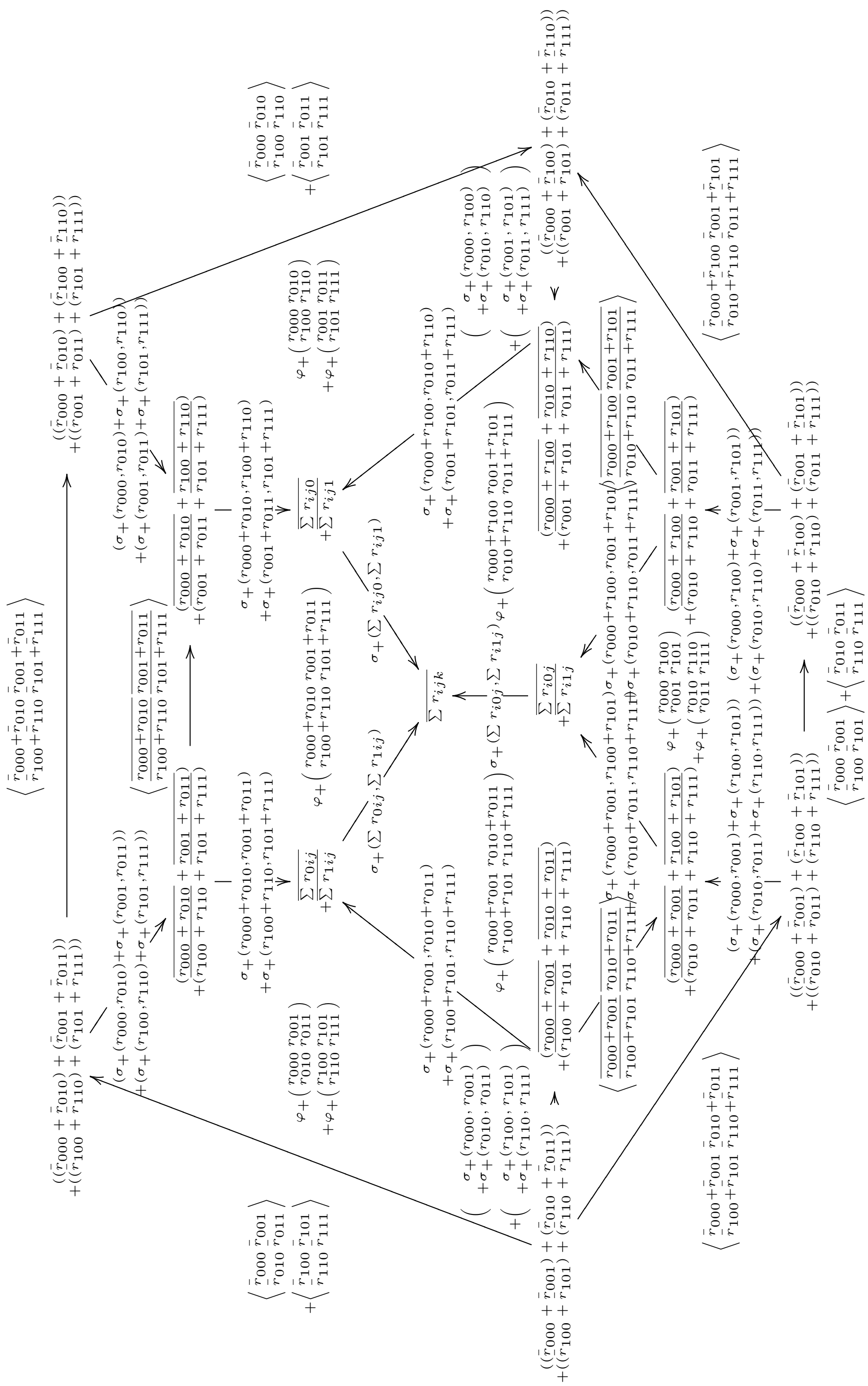
.



.



.



.

□

To show that the above rule indeed determines an assignment of a cohomology class to each categorical ring, we must also show

(4.2) Proposition. *A different choice of representative objects $r \mapsto \tilde{r}$ and morphisms*

$$\sigma'_{\cdot}(r,s) : \tilde{r}\tilde{s} \to \widetilde{rs},$$
$$\sigma'_{+}(r_0,r_1) : \tilde{r}_0 + \tilde{r}_1 \to \widetilde{r_0+r_1}$$

leads to a cocycle φ' which is cohomologous to φ.

Proof. By (3.1), 3-cocycles φ and φ' are cohomologous if and only if there exist maps $\gamma_{\cdot}, \gamma_{+} : R \times R \to B$ such that the following four equalities

$$\begin{aligned}
\varphi'_{\cdot}(r,s,t) =& \varphi_{\cdot}(r,s,t) + r\gamma_{\cdot}(s,t) - \gamma_{\cdot}(rs,t) + \gamma_{\cdot}(r,st) - \gamma_{\cdot}(r,s)t,\\
\varphi'_{\cdot+}(r,s_0,s_1) =& \varphi_{\cdot+}(r,s_0,s_1) + r\gamma_{+}(s_0,s_1) - \gamma_{+}(rs_0,rs_1) + (s_0 \mid s_1)_{\gamma_{\cdot}(r,-)},\\
\varphi'_{+\cdot}(r_0,r_1,s) =& \varphi_{+\cdot}(r_0,r_1,s) + \gamma_{+}(r_0s,r_1s) - \gamma_{+}(r_0,r_1)s - (r_0 \mid r_1)_{\gamma_{\cdot}(-,s)},\\
\varphi'_{+}\left(\begin{smallmatrix} r_{00} & r_{01} \\ r_{10} & r_{11}\end{smallmatrix}\right) =& \varphi_{+}\left(\begin{smallmatrix} r_{00} & r_{01} \\ r_{10} & r_{11}\end{smallmatrix}\right) + ((r_{00},r_{01}) \mid (r_{10},r_{11}))_{\gamma_{+}} - ((r_{00},r_{10}) \mid (r_{01},r_{11}))_{\gamma_{+}}
\end{aligned}$$

are satisfied for all possible elements $r, \ldots$ of R.

Let us then choose arbitrary morphisms

$$\hat{r} : \bar{r} \to \tilde{r}$$

for all $r \in R$ and define the maps γ_{+} and $\gamma_{\cdot}$, as above for φ, to measure deviation from commutativity of the diagrams

$$\begin{array}{ccc}
\bar{r}\bar{s} & \xrightarrow{\sigma_{\cdot}(r,s)} & \overline{rs} \\
{\scriptstyle \hat{r}\hat{s}}\downarrow & \gamma_{\cdot}(r,s) & \downarrow{\scriptstyle \widehat{rs}} \\
\tilde{r}\tilde{s} & \xrightarrow[\sigma'_{\cdot}(r,s)]{} & \widetilde{rs}
\end{array}$$

and

$$\begin{array}{ccc}
\bar{r}_0 + \bar{r}_1 & \xrightarrow{\sigma_{+}(r_0,r_1)} & \overline{r_0+r_1} \\
{\scriptstyle \hat{r}_0+\hat{r}_1}\downarrow & \gamma_{+}(r_0,r_1) & \downarrow{\scriptstyle \widehat{r_0+r_1}} \\
\tilde{r}_0 + \tilde{r}_1 & \xrightarrow[\sigma'_{+}(r_0,r_1)]{} & \widetilde{r_0+r_1}.
\end{array}$$

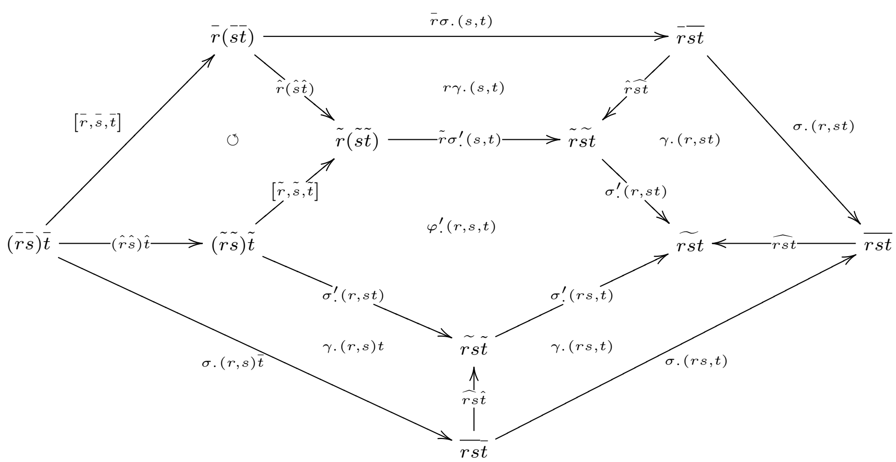
That is, we define

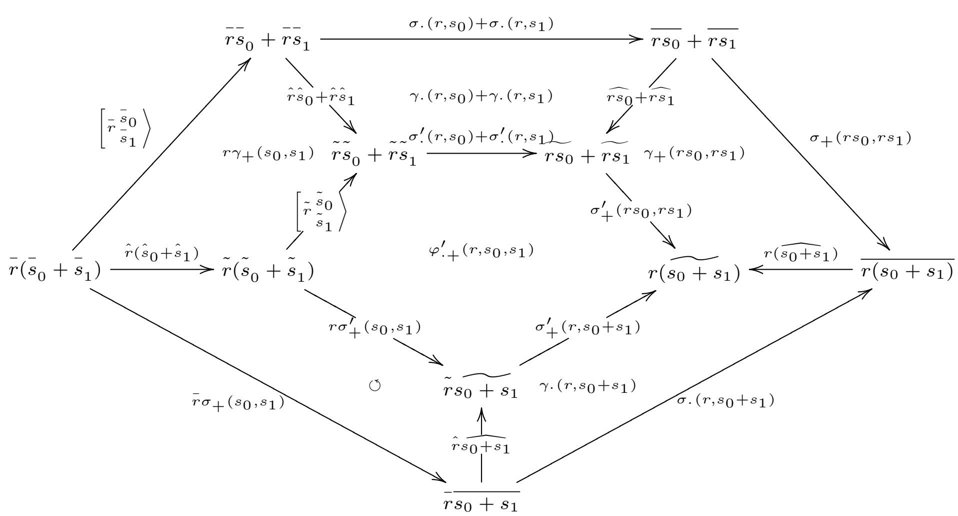
$$\gamma_{\cdot}(r,s) = \sigma'_{\cdot}(r,s) \circ \hat{r}\hat{s} \;\div\; \widehat{rs} \circ \sigma_{\cdot}(r,s)$$

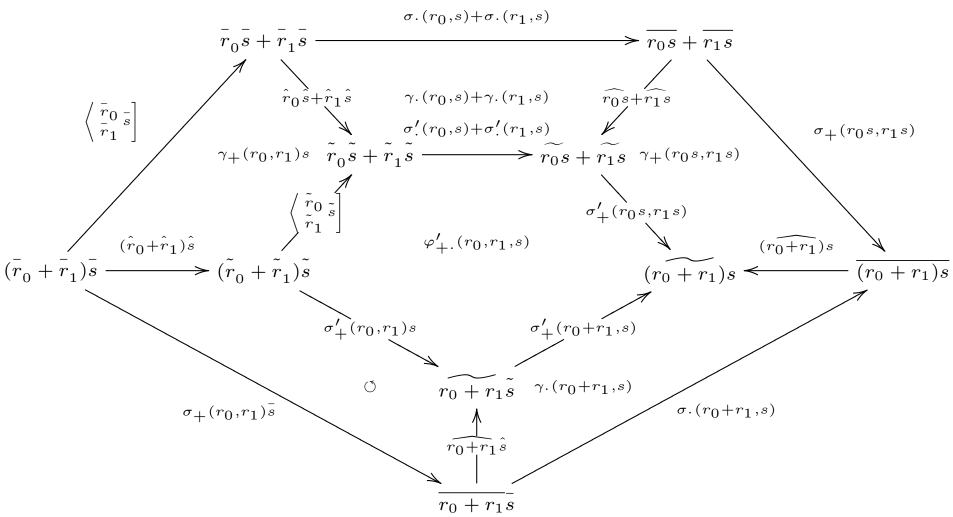
and

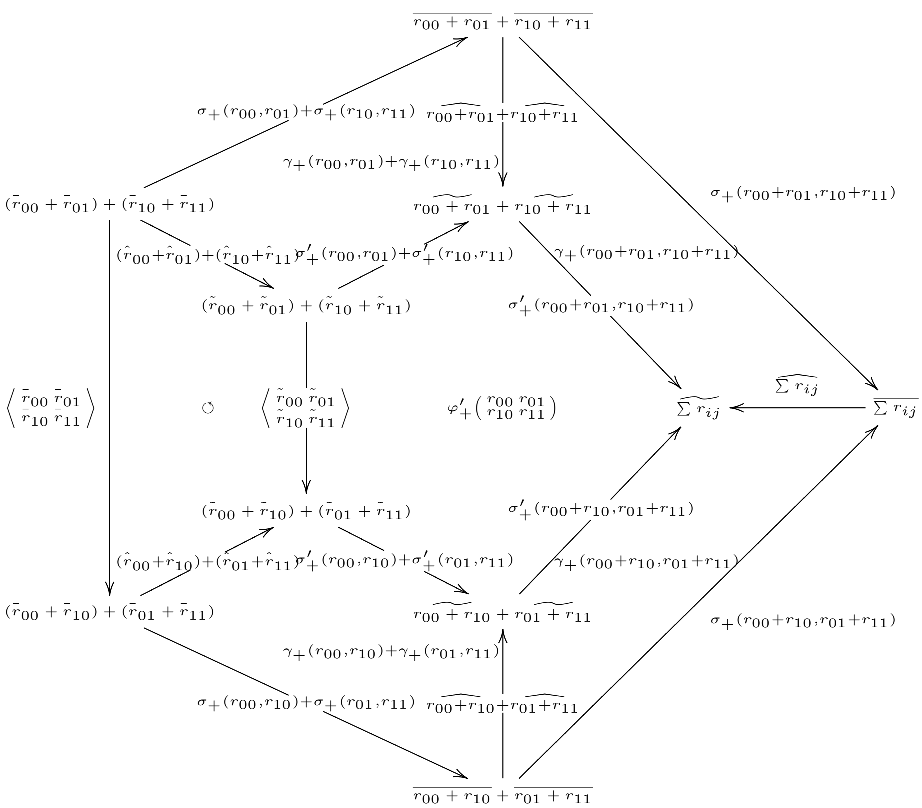
$$\gamma_+(r_0, r_1) = \sigma'_+(r_0, r_1) \circ (\hat{r}_0 + \hat{r}_1) \;\div\; \widehat{r_0 + r_1} \circ \sigma_+(r_0, r_1).$$

The above four equalities then follow from considering the following four diagrams, in view, as before, of (1.4) and (2.3), where "$\circlearrowleft$" marks the strictly commuting quadrangles:






σ.(r0,s)+σ.(r1,s)
γ.(r0,s)+γ.(r1,s)
σ'.(r0,s)+σ'.(r1,s)
γ+(r0,r1)s
γ+(r0s,r1s)
σ+(r0s,r1s)
σ'+(r0s,r1s)
φ'+.(r0,r1,s)
σ'+(r0,r1)s
σ'+(r0+r1,s)
γ.(r0+r1,s)
σ+(r0,r1)s
σ.(r0+r1,s)


σ+(r00,r01)+σ+(r10,r11)
γ+(r00,r01)+γ+(r10,r11)
σ+(r00+r01,r10+r11)
σ'+(r00,r01)+σ'+(r10,r11)
γ+(r00+r01,r10+r11)
σ'+(r00+r01,r10+r11)
σ'+(r00+r10,r01+r11)
σ'+(r00,r10)+σ'+(r01,r11)
γ+(r00+r10,r01+r11)
σ+(r00+r10,r01+r11)
γ+(r00,r10)+γ+(r01,r11)
σ+(r00,r10)+σ+(r01,r11)

□

We have thus obtained a map from the set of all categorical rings $\mathscr{R}$ with $\pi_0(\mathscr{R}) = R$, $\pi_1(\mathscr{R}) = B$ and matching bimodule structure to the group $H^3(R; B)$. Let us next show that this map factors through a quotient of the former set to yield a map

$$\langle - \rangle : \mathrm{Crext}(R; B) \to H^3(R; B),$$

where $\mathrm{Crext}(R; B)$ denotes the set of equivalence classes of categorical rings with π_0 equal to R and π_1 equal to B, two such being considered equivalent if there exists a 2-homomorphism between them inducing identities on R and B.

Indeed, in the same way as in (4.2) we more generally have:

(4.3) Proposition. *For any categorical rings $\mathscr{R}$ and $\mathscr{R}'$ such that there exists a 2-homomorphism $\mathscr{R} \to \mathscr{R}'$ inducing identity maps on π_0 and π_1, one has $\langle\mathscr{R}\rangle = \langle\mathscr{R}'\rangle$.*

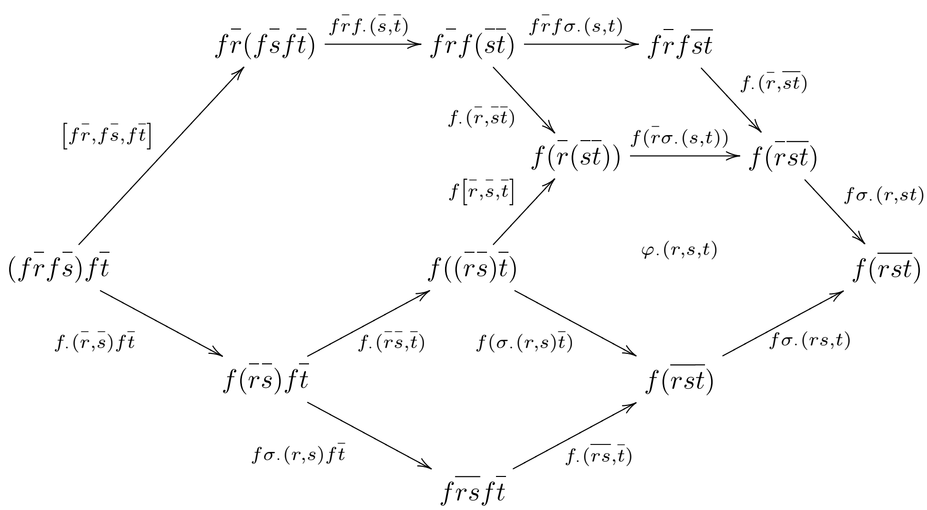
Proof. Given a 2-homomorphism $\boldsymbol{f} : \mathscr{R} \to \mathscr{R}'$, let us choose $^-$ and $\sigma_\cdot$, σ_+ for $\mathscr{R}$ as above, and then choose the corresponding maps for $\mathscr{R}'$ as follows:

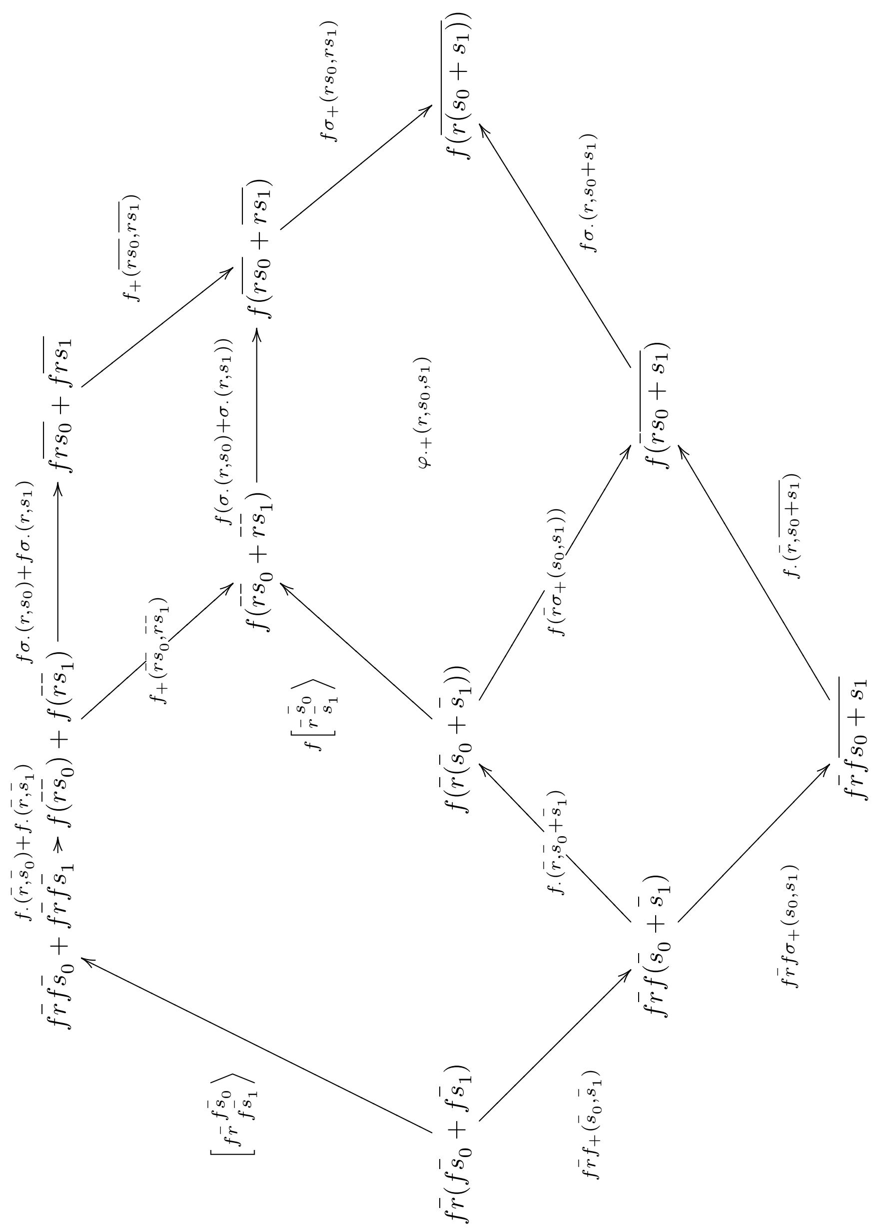
$$\bar{r}' = f(\bar{r}),$$

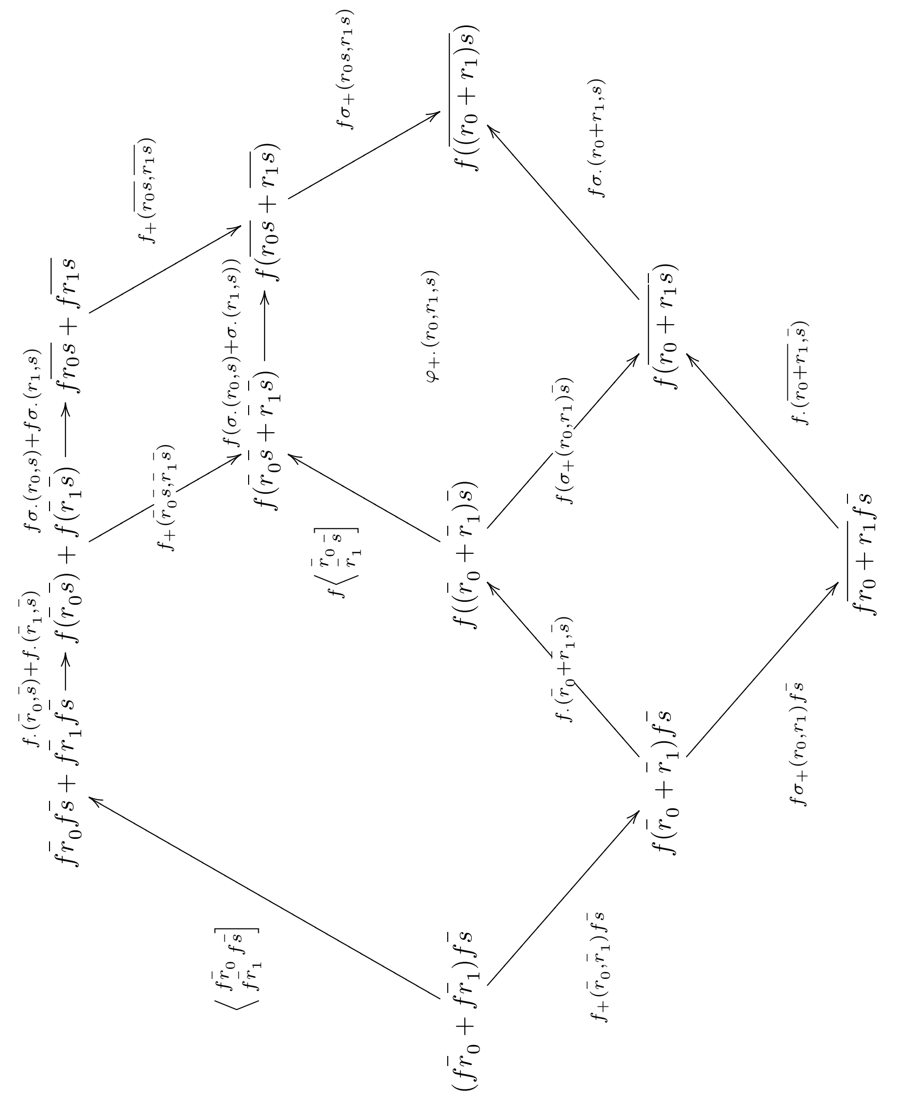
$$\sigma'_\cdot(r,s) = \left(f\bar{r} f\bar{s} \xrightarrow{f_\cdot(\bar{r},\bar{s})} f(\bar{r}\bar{s}) \xrightarrow{f\sigma_\cdot(r,s)} f\overline{rs} \right),$$

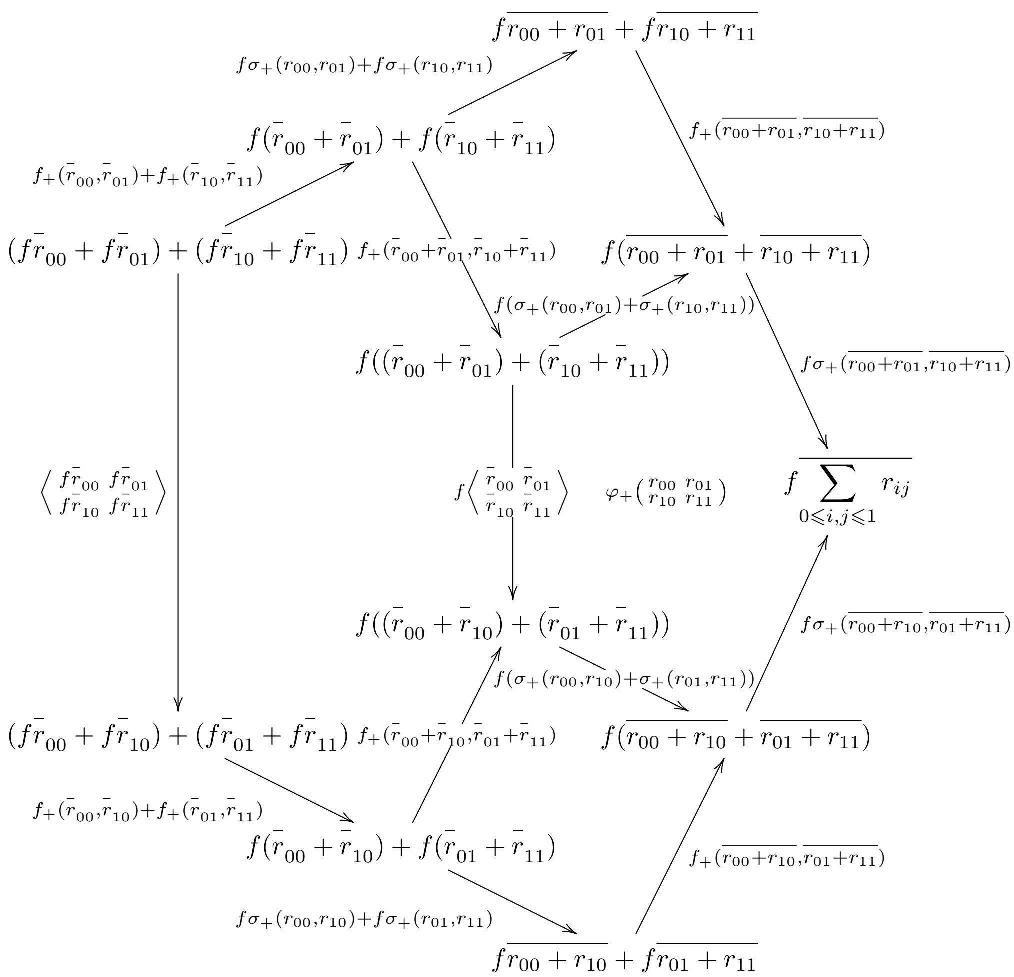
$$\sigma'_+(r_0,r_1) = \left(f\bar{r}_0 + f\bar{r}_1 \xrightarrow{f_+(\bar{r}_0,\bar{r}_1)} f(\bar{r}_0 + \bar{r}_1) \xrightarrow{f\sigma_+(r_0,r_1)} f\overline{r_0 + r_1} \right).$$

In view of (1.3), (1.4), since $\boldsymbol{f}$ induces identity on π_1, i. e. $f_\#$ is the identity map, one has the diagrams









where unlabeled polygons strictly commute by coherence of $\boldsymbol{f}$ and naturality. These diagrams show that the cocycles φ and φ' representing characteristic classes of, respectively, $\mathscr{R}$ and $\mathscr{R}'$, are cohomologous. □

We have thus defined a map

$$\langle - \rangle : \mathrm{Crext}(R; B) \to H^3(R; B).$$

Let us now construct a map in the opposite direction.

For a 3-cocycle $\varphi = (\varphi_{\cdot}, \varphi_{\cdot+}, \varphi_{+\cdot}, \varphi_{+})$ of R with coefficients in B let $\mathscr{R}_\varphi$ be the following categorical ring. The set of objects of $\mathscr{R}_\varphi$ is R. The set of morphisms is $B \times R$, where (b, r) is a morphism from r to r. Identities are morphisms of the form $(0, r)$, and composition is given by $(b, r) \circ (b', r) = (b + b', r)$. Categorical group structure is as follows. Addition of objects and morphisms is given by

$$(b_0, r_0) + (b_1, r_1) = (b_0 + b_1, r_0 + r_1);$$

the neutral object is $(0, 0)$, with the neutrality constraints given by $\lambda(r) = \rho(r) = (0, r)$, the associativity constraint is given by

$$\langle r, s, t\rangle = \left(\varphi_+ \left(\begin{smallmatrix} r & s \\ 0 & t \end{smallmatrix}\right), r + s + t\right),$$

and the symmetry by

$$\{r, s\} = \left(\varphi_+ \left(\begin{smallmatrix} 0 & r \\ s & 0 \end{smallmatrix}\right), r + s\right).$$

Note that the latter two equalities are equivalent to the equality

$$\left\langle \begin{smallmatrix} r_{00} & r_{01} \\ r_{10} & r_{11} \end{smallmatrix} \right\rangle = \left(\varphi_+ \left(\begin{smallmatrix} r_{00} & r_{01} \\ r_{10} & r_{11} \end{smallmatrix}\right), \sum_{0\leqslant i,j\leqslant 1} r_{ij}\right).$$

Next, we define multiplication of objects and morphisms by

$$(b, r)(b', r') = (br' + rb', rr'),$$

the unit by 1, the associativity constraint for the multiplication by

$$[r, s, t] = (\varphi_\cdot(r, s, t), rst)$$

and the unitality constraints by

$$\lambda_\cdot(r) = (-\varphi_\cdot(1, 1, r), r),$$
$$\rho_\cdot(r) = (\varphi_\cdot(r, 1, 1), r).$$

Moreover, we define the distributivity constraints by the equalities

$$\left[r \begin{smallmatrix} s_0 \\ s_1 \end{smallmatrix}\right\rangle = (\varphi_{\cdot+}(r, s_0, s_1), r(s_0 + s_1))$$

and

$$\left\langle \begin{smallmatrix} r_0 \\ r_1 \end{smallmatrix} s\right] = (\varphi_{+\cdot}(r_0, r_1, s), (r_0 + r_1)s).$$

It then turns out that commutativity of the coherence diagrams necessary for $\mathscr{R}_\varphi$ to be a categorical ring correspond precisely to the equations expressing the cocycle condition for φ.

Now suppose we are given two cohomologous 3-cocycles φ, φ', i. e. there is a 2-cochain $\gamma = (\gamma_\cdot, \gamma_+)$ satisfying the required equalities. We then define a 2-homomorphism $\boldsymbol{f} = (f, f_+, f_\cdot, f_1) : \mathscr{R}_\varphi \to \mathscr{R}_{\varphi'}$ as follows. Since the underlying categories of $\mathscr{R}_\varphi$ and $\mathscr{R}_{\varphi'}$ are identical, we can define f to be the identity functor. Moreover we define

$$f_+(r_0, r_1) = (\gamma_+(r_0, r_1), r_0 + r_1),$$
$$f_\cdot(r, s) = (\gamma_\cdot(r, s), rs)$$

and

$$f_1 = (\gamma_\cdot(1, 1), 1).$$

Then again it is straightforward to verify that the coherence conditions for $\boldsymbol{f}$ to be a 2-homomorphism precisely amount to the equalities expressing the fact that φ' differs from φ by the coboundary of γ.

We have thus obtained a well-defined map

$$\mathscr{R}_{-} : H^3(R; B) \to \mathrm{Crext}(R; B)$$

in the opposite direction.

Now it is obvious that constructing the characteristic class $\langle \mathscr{R}_\varphi \rangle$ we can choose the maps $\sigma_{\cdot}$, σ_+ in the beginning of section 4 to be identities, which will produce the cocycle φ back. Thus $\langle \mathscr{R}_\varphi \rangle$ is equal to the cohomology class of φ. So one composite of our maps (from H^3 to itself) is in fact identity. For the other composite to be also the identity, it thus remains to construct, for any categorical ring $\mathscr{R}$, a 2-homomorphism between $\mathscr{R}$ and $\mathscr{R}_\varphi$ for some cocycle φ, inducing identity on π_0 and π_1. For this, let us return to the construction of the characteristic class of $\mathscr{R}$; for that construction, we have chosen an object $\bar{r}$ of $\mathscr{R}$ in each isomorphism class $r \in \pi_0(\mathscr{R}) = R$ and morphisms $\sigma_{\cdot}$, σ_+, which then produced the cocycle φ representing $\langle \mathscr{R} \rangle$. Obviously these choices can be made in such a way that $\bar{0} = 0$, $\bar{1} = 1$, $\sigma_+(0, r) = \lambda(\bar{r})$, $\sigma_+(r, 0) = \rho(\bar{r})$, $\sigma_{\cdot}(1, r) = \lambda_{\cdot}(\bar{r})$, and $\sigma_{\cdot}(r, 1) = \rho_{\cdot}(\bar{r})$. Let us then use these data to define a functor $f : \mathscr{R}_\varphi \to \mathscr{R}$. On objects this functor is given by $f(r) = \bar{r}$ and on morphisms by

$$f(b, r) = \bar{r} \xrightarrow{\lambda(\bar{r})^{-1}} 0 + \bar{r} \xrightarrow{b + \bar{r}} 0 + \bar{r} \xrightarrow{\lambda(\bar{r})} \bar{r}.$$

This f then extends to a 2-homomorphism $\boldsymbol{f} = (f, f_+, f_{\cdot}, f_1) : \mathscr{R}_\varphi \to \mathscr{R}$ with $f_{\cdot} = \sigma_{\cdot}$, $f_+ = \sigma_+$ and $f_1 =$ identity of 1.

Summarizing all of the above, we have thus proved

(4.4) Theorem. *For any ring R and any R-bimodule B there is a bijection*

$$H^3(R; B) \approx \mathrm{Crext}(R; B)$$

between the third Mac Lane cohomology of R with coefficients in B and equivalence classes of categorical rings $\mathscr{R}$ with $\pi_0(\mathscr{R}) = R$, $\pi_1(\mathscr{R}) = B$ and the resulting bimodule structure coinciding with the original one.

□

References

[1] Hans-Joachim Baues, Mamuka Jibladze, and Teimuraz Pirashvili, *Third Mac Lane cohomology*, Math. Proc. Cambridge Phil. Soc. (to appear).

[2] Hans-Joachim Baues and Teimuraz Pirashvili, *Shukla cohomology and additive track theories*, arxiv:math.KT/0401158.

[3] Samuel Eilenberg and Saunders Mac Lane, *Homology theories for multiplicative systems*, Trans. Amer. Math. Soc. **71** (1951), 294–330.

[4] Albrecht Fröhlich and C. T. C. Wall, *Graded monoidal categories*, Compos. Math. **28** (1974), 229–285.

[5] Mamuka Jibladze and Teimuraz Pirashvili, *Some linear extensions of a category of finitely generated free modules*, Bull. Acad. Sci. Georgia **123** (1986), 481–484.

[6] André Joyal and Ross Street, *Braided tensor categories*, Adv. Math. **102** (1993), 20–78.

[7] Saunders Mac Lane, *Homologie des anneaux et des modules*, Colloque de topologie algébrique, Louvain, 1956, Georges Thone, Liège, 1957, pp. 55–80. (French)

[8] ———, *Extensions and obstructions for rings*, Illinois J. Math. **2** (1958), 316–345.

[9] Teimuraz Pirashvili and Friedhelm Waldhausen, *Mac Lane homology and topological Hochschild homology*, J. Pure Appl. Algebra **82** (1992), 81–98.

[10] Nguyen Tien Quang, *Structure of Ann-categories and Mac Lane-Shukla cohomology*, East-West J. of Math. **5** (2003), 51–66.

[11] Umeshachandra Shukla, *Cohomologie des algèbres associatives*, Ann. Sci. École Norm. Sup. (3) **78** (1961), 163–209.
[12] Hoang Xuan Sinh, *Gr-categories*, Université Paris VII doctoral thesis, 1975.
[13] Enrico Vitale, *A Picard-Brauer exact sequence of categorical groups*, J. Pure Appl. Algebra **175** (2002), 383–408.

http://www.emis.de/ZMATH/
http://www.ams.org/mathscinet

This article may be accessed via WWW at http://jhrs.rmi.acnet.ge

M. Jibladze
jib@rmi.acnet.ge

A. Razmadze Mathematical Institute,
1 M. Alexidze Street, 0193 Tbilisi,
Georgia

T. Pirashvili
tp59@le.ac.uk

Department of Mathematics,
University of Leicester,
Leicester, UK