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Edge-Coloring and f-Coloring for Various Classes of Graphs

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In an ordinary edge-coloring of a graph each color appears at each vertex at most once. An f-coloring is a generalized edge-coloring in which each color appears at each vertex v at most $f(v)$ times where $f(v)$ is a positive integer assigned to v. This paper gives efficient sequential and parallel algorithms to find ordinary edge-colorings and f-colorings for various classes of graphs such as bipartite graphs, planar graphs, and graphs having fixed degeneracy, tree-width, genus, arboricity, unicyclic index or thickness.

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1 Introduction

This paper deals with a *simple* graph G which has no multiple edges or self-loops. An *edge-coloring* of a graph G is to color all the edges of G so that no two adjacent edges are colored with the same color. The minimum number of colors needed for an edge-coloring is called the *chromatic index* of G and denoted by $\chi'(G)$. In this paper the *maximum degree* of a graph G is denoted by $\Delta(G)$ or simply by Δ. Vizing showed that $\chi'(G) = \Delta$ or $\Delta + 1$ for any simple graph G [10, 30]. The *edge-coloring problem* is to find an edge-coloring of G with $\chi'(G)$ colors. Let f be a function which assigns a positive integer $f(v)$ to each vertex $v \in V$. Then an *f-coloring* of G is to color all the edges of G so that, for each vertex $v \in V$, at most $f(v)$ edges incident to v are colored with the same color. Thus an f-coloring of G is a decomposition of G to edge-disjoint spanning subgraphs in each of which vertex degrees are bounded above by f. An ordinary edge-coloring is a special case of an f-coloring for which $f(v) = 1$ for every vertex $v \in V$. The minimum number of colors needed for an f-coloring is called the *f-chromatic index* of G and denoted by $\chi'_f(G)$. The *f-coloring problem* is to find an f-coloring of G with $\chi'_f(G)$ colors. Let $\Delta_f(G) = \max_{v \in V} \lceil d(v)/f(v) \rceil$ where $d(v)$ is the *degree* of vertex v, then $\chi'_f(G) = \Delta_f$ or $\Delta_f + 1$ for any simple graph G [13].

The edge-coloring and f-coloring have applications to scheduling problems like the file transfer problem in a computer network [5, 24, 25]. In the model a vertex of a graph G represents a computer, and an edge does a file which one wishes to transfer between the two computers corresponding to its ends. The integer $f(v)$ is the number of communication ports available at a computer v. The edges colored with the same color represent files that can be transferred in the network simultaneously. Thus an f-coloring of G with $\chi'_f(G)$ colors corresponds to a scheduling of file transfers with the minimum finishing time.

Since the ordinary edge-coloring problem is NP-complete [15], the f-coloring problem is also NP-complete in general. Therefore it is very unlikely that there exists an exact algorithm which solves the ordinary edge-coloring problem or the f-coloring problem in polynomial time. However, the following approximate algorithms are known. Any simple graph G can be edge-colored with $\Delta + 1$ colors in polynomial time [27, 29]; the best known algorithm takes time $O(\min\{n\Delta \log n, m\sqrt{n \log n}\})$ [12], where we denote by n the number of the vertices and m the number of the edges in G. Furthermore, the proof in [13] immediately yields an approximate algorithm to f-color any simple graph with $\Delta_f + 1$ colors in time $O(mn)$. On the other hand, exact algorithms to edge-color G with $\chi'(G)$ colors are known for restricted classes of graphs as follows:

(a) an $O(m \log n)$-time algorithm for bipartite graphs [6, 11];
(b) a linear-time algorithm for planar graphs of $\Delta \geq 19$ [4];
(c) an $O(n \log n)$-time algorithm for planar graphs of $\Delta \geq 9$ [3];
(d) an $O(n^2)$-time algorithm for planar graphs of $\Delta \geq 8$ [12, 27];
(e) a linear-time algorithm for series-parallel multigraphs [34]; and
(f) a linear-time algorithm for partial k-trees [32].

Concerning parallel edge-coloring algorithms, NC parallel exact algorithms have been obtained only for a few restricted classes of graphs such as bipartite graphs [20], series-parallel simple graphs [2], series-parallel multigraphs [35], partial k-trees [32] and planar graphs with maximum degree $\Delta \geq 9$ [3, 4]. However, NC parallel approximate algorithms to edge-color G with $\Delta + 1$ colors have not been known so far except for the case when Δ is small [18]. On the other hand, no efficient exact algorithms for the f-coloring problem have been obtained even for restricted classes of graphs.

In this paper we consider various classes of graphs specified by invariants like the degeneracy. The *degeneracy* $s(G)$ of a graph G is the minimum number s such that G can be reduced to an empty graph by the successive deletion of vertices with degree at most s [1]. Clearly the degeneracy has a favorable implication on the vertex-coloring: any graph G can be vertex-colored with at most $s(G)+1$ colors [9, 22, 23, 28]. On the other hand, Vizing [16, 31] showed that the degeneracy has a surprising implication on the edge-coloring: $\chi'(G) = \Delta(G)$ if $\Delta(G) \geq 2s(G)$. Thus Vizing gave a lower bound on $\Delta(G)$ for $\chi'(G) = \Delta(G)$ to hold true. In this paper we express such a lower bound in terms of various other graph-invariants like tree-width, arboricity, unicyclic index, thickness, and genus. It is rather straightforward to derive from Vizing's proof an $O(mn)$ algorithm for edge-coloring a graph G with $\Delta(G)$ colors if $\Delta(G) \geq 2s(G)$. We give more efficient sequential and NC parallel algorithms to edge-color a graph G whose maximum degree $\Delta(G)$ is roughly larger than twice the lower bounds, say $\Delta(G) \geq 4s(G)$. Our sequential algorithm takes time $O(n \log n)$ if $s(G)$ is bounded and $\Delta(G) \geq 4s(G)$. We next give a simple but useful transformation of a graph G to a new graph G_f such that an ordinary edge-coloring of G_f immediately induces an f-coloring of the original graph G with the same number of colors. Using the transformation, we finally give efficient sequential and NC parallel algorithms to f-color various classes of graphs with large $\Delta(G)$. In the paper the parallel computation model we use is a concurrent-read exclusive-write parallel random access machine (CREW PRAM). An early version of the paper was presented at [33].

2 Preliminary

In this section we define terminology and observe relationships between various graph invariants.

A graph with vertex set V and edge set E is denoted by $G = (V, E)$. The vertex set and the edge set of a graph G is often denoted by $V(G)$ and $E(G)$, respectively. We denote the number of vertices in G by $n(G)$ or simply n, and denote the number of edges in G by $m(G)$ or simply m. We say that a graph G is *trivial* if $m(G) = 0$. The degree of v in G is denoted by $d(v, G)$ or simply by $d(v)$. We denote by $\Delta(G)$ the *maximum degree* of vertices of G and by $\delta(G)$ the *minimum degree*. The graph obtained from G by deleting all vertices in $V' \subseteq V(G)$ is denoted by $G - V'$. The graph obtained from G by deleting all edges in $E' \subseteq E(G)$ is denoted by $G - E'$.

We then define various invariants of graphs. Let s be a positive integer. A graph G is *s-degenerate* if the vertices of G can be ordered $v_1, v_2, \cdots, v_n$ so that $d(v_i, G_i) \leq s$ for each i, $1 \leq i \leq n$, where $G_i = G - \{v_1, v_2, \cdots, v_{i-1}\}$ [1, 9, 22, 23]. Thus G is s-degenerate if and only if G can be reduced to a trivial graph by the successive removal of vertices having degree at most s. The *degeneracy* $s(G)$ of G is the minimum integer s for which G is s-degenerate. The degeneracy $s(G)$ is also called the Szekeres-Wilf number [28]. The degeneracy of a graph can be computed in linear time [23]. Every planar graph G has a vertex of degree at most five, that is, $\delta(G) \leq 5$ [1, 27], and hence

$$s(G) \leq 5. \tag{1}$$

Obviously any graph G can be vertex-colored with at most $s(G) + 1$ colors [9, 22, 23, 28]. Vizing showed that $\chi'(G) = \Delta(G)$ if $\Delta(G) \geq 2s(G)$ [16, 31].

A graph $G = (V, E)$ is a *k-tree* if either it is a complete graph on k vertices or it has a vertex $v \in V$ whose neighbors induce a clique of size k and $G - \{v\}$ is again a k-tree. A graph is a *partial k-tree* if it is a subgraph of a k-tree [32]. The *tree-width* $k(G)$ of graph G is the minimum integer k such that G is a partial k-tree. Clearly

$$s(G) \leq k(G). \tag{2}$$

The *arboricity* $a(G)$ of a graph G is the minimum number of edge-disjoint forests into which G can be decomposed. Nash-Williams [26] proved that $a(G) = \max_{H \subseteq G} \lceil m(H)/(n(H) - 1) \rceil$, where H runs over all nontrivial subgraphs of G. We have

$$a(G) \leq s(G), \tag{3}$$

because any subgraph H of G is $s(G)$-degenerate and hence $m(H) \leq s(G)(n(H) -1)$ and $m(H)/(n(H) - 1) \leq s(G)$. Furthermore, if G is planar, then

$$a(G) \leq 3, \tag{4}$$

because $m(H) \leq 3n(H) - 3$ for any nontrivial subgraph H of G.

We now introduce a rather unfamiliar invariant $a'(G)$ which we call the *unicyclic index* of a graph G: $a'(G)$ is the minimum number of edge-disjoint unicyclic graphs, that is, graphs with at most one cycle, into which G can be decomposed. Since a forest is a unicyclic graph and a unicyclic graph can be decomposed to one or two forests, we have

$$a'(G) \leq a(G) \leq 2a'(G). \tag{5}$$

The *thickness* $\theta(G)$ of a graph G is the minimum number of edge-disjoint planar subgraphs into which G can be decomposed. Clearly

$$\theta(G) \leq a'(G) \leq a(G) \leq 3\theta(G) \tag{6}$$

since every unicyclic graph is planar and every planar graph can be decomposed into at most three edge-disjoint forests [8].

The *genus* $g(G)$ of a graph G is the minimum number of handles which must be added to a sphere so that G can be embedded on the resulting surface. Of course, $g(G) = 0$ if and only if G is planar. It is known [14, 17] that if $g(G) \geq 1$ then

$$\delta(G) \leq \left\lfloor \left(5 + \sqrt{48g(G)+1}\right)/2 \right\rfloor. \tag{7}$$

Furthermore any subgraph H of G satisfies $g(H) \leq g(G)$. Therefore, if $g(G) \geq 1$ then

$$s(G) \leq \left\lfloor \left(5 + \sqrt{48g(G)+1}\right)/2 \right\rfloor. \tag{8}$$

One can observe that the following upper bound holds on the minimum degree.

Lemma 1 *The following* (a)–(c) *hold for any nontrivial graph G:*

(a) $\delta(G) \leq 2a(G) - 1$ [8]*;*

(b) $\delta(G) \leq 2a'(G)$*; and*

(c) *if $a'(G)$ is bounded and $U = \{u \in V \mid d(u,G) \leq 2a'(G)\}$, then $|U| \geq n/(2a'(G)+1)$ and hence $|U| = \Theta(n)$.*

Proof: (a) One may assume that G has no isolated vertices. Let n' be the number of vertices v of G such that $1 \leq d(v) \leq 2a(G) - 1$. Then clearly $n' + 2a(G)(n - n') \leq 2m$. On the other hand, G can be decomposed into $a(G)$ edge-disjoint forests, and any forest has at most $n - 1$ edges. Therefore $m \leq a(G)(n-1)$. Thus $n' \geq 2a(G)/(2a(G)-1) > 1$, and hence $\delta(G) \leq 2a(G)-1$.

(b) and (c) Since every vertex in $V - U$ has degree $\geq 2a'(G) + 1$, we have $(2a'(G)+1)(n - |U|) \leq 2m$. Since any unicyclic graph has at most n edges, we have $m \leq a'(G)n$. Thus we have $|U| \geq n/(2a'(G)+1)$. Hence $U \neq \phi$ and $\delta(G) \leq 2a'(G)$. If $a'(G)$ is bounded, then $|U| = \Theta(n)$. □

By Lemma 1 and Eqs. (1), (2), (4)–(6) and (8) we can immediately derive the following upper bounds on $s(G)$ in terms of $k(G)$, $a(G)$, $a'(G)$, $\theta(G)$ and $g(G)$. Note that $a(H) \leq a(G)$, $a'(H) \leq a'(G)$, $\theta(H) \leq \theta(G)$ and $g(H) \leq g(G)$ for any subgraph H of G.

Lemma 2 *The following* (a) – (f) *hold:*

(a) $s(G) \leq k(G)$*;*
(b) $s(G) \leq 2a(G) - 1$*;*
(c) $s(G) \leq 2a'(G)$*;*
(d) $s(G) \leq 6\theta(G) - 1$*;*
(e) $s(G) \leq \left\lfloor \left(5 + \sqrt{48g(G)+1}\right)/2 \right\rfloor$ *if $g(G) \geq 1$; and*
(f) $s(G) \leq 5$ *if G is planar.*

The relationships among these graph-invariants are illustrated in Figure 1.

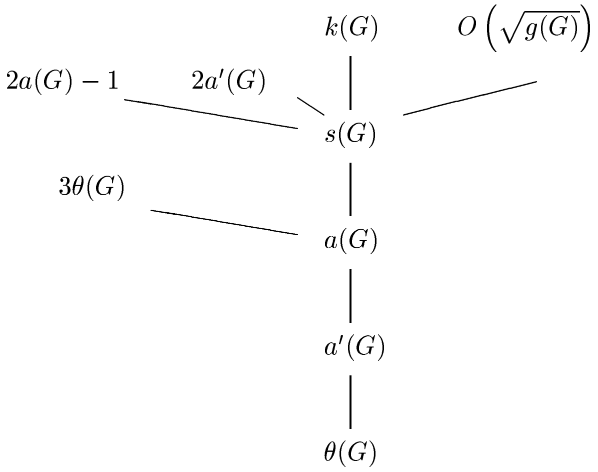

Figure 1: Relationships among graph-invariants.

3 Chromatic Index

By the classical Vizing's theorem, $\chi'(G) = \Delta$ or $\Delta + 1$ for any simple graph G [10, 30]. Vizing also showed that $\chi'(G) = \Delta$ if $\Delta \geq 2s(G)$. In this section we give various lower bounds on $\Delta(G)$ for $\chi'(G) = \Delta(G)$ to hold true, expressed in terms of various invariants such as $k(G)$, $a(G)$, $a'(G)$, $\theta(G)$ and $g(G)$.

For vertices u and v, we denote by $d_u^*(v)$ the number of v's neighbors, other than u, having degree $\Delta(G)$. An edge $(u, v) \in E$ is *eliminable* if either $d(u) + d_u^*(v) \leq \Delta(G)$ or $d(v) + d_v^*(u) \leq \Delta(G)$ [27, 29]. The following lemma is an expression of a classical result on "critical graphs," called "Vizing's adjacency lemma" (see, for example, [10, 27, 29]). In other words, the edges that are excluded in a critical graph by the adacency lemma are eliminable. Note that the definition is not symmetric with u and v.

Lemma 3 *If (u, v) is an eliminable edge of a simple graph G and $\chi'(G - (u, v)) \leq \Delta(G)$, then $\chi'(G) = \Delta(G)$.*

Thus, if we remove an eliminable edge (u, v) and can color the remaining graph $G - (u, v)$ with $\Delta(G)$ colors, then the obtained coloring can be extended to the edge (u, v) without using more colors.

Vizing [31] obtained the following two theorems. We give proofs for them, which yield an $O(mn)$ algorithm to edge-color G with Δ colors if $\Delta(G) \geq 2s(G)$, as we will show in the succeeding section.

Theorem 1 [31] *Any nontrivial graph G has an eliminable edge if $\Delta(G) \geq 2s(G)$.*

Proof: Let $U = \{u \in V(G) \mid d(u, G) \leq s(G)\}$. Then $U \neq \phi$ because the definition of the degeneracy implies that G has at least one vertex of degree

$\leq s(G)$. Furthermore $V - U \neq \phi$ since $\Delta \geq 2s(G) > s(G)$ and hence the vertices of degree Δ are not contained in U. Thus $H = G - U$ is not empty and $s(H) \leq s(G)$. Therefore H has a vertex v of degree $\leq s(G)$. Since $s(G) + 1 \leq d(v, G)$ and $d(v, H) \leq s(G)$, G has an edge (u, v) joining v and a vertex $u \in U$. Since $u \in U$, $d(u) \leq s(G) < 2s(G) \leq \Delta$. Thus none of v's neighbors in U has degree Δ, and hence $d^*_u(v) \leq d(v, H) \leq s(G)$. Therefore $d(u) + d^*_u(v) \leq 2s(G) \leq \Delta$, and hence edge (u, v) is eliminable. □

Theorem 2 [31] *$\chi'(G) = \Delta(G)$ if $\Delta(G) \geq 2s(G)$.*

Proof: Assume that G is a nontrivial graph with $\Delta(G) \geq 2s(G)$. Then by Theorem 1 G has an eliminable edge e_1. Let $G_1 = G - \{e_1\}$, then $s(G_1) \leq s(G)$. If $\Delta(G_1) = \Delta(G)$, then G_1 has an eliminable edge e_2. Thus there exists a sequence of edges $e_1, e_2, \cdots, e_j$ such that

(i) $\Delta(G_j) = \Delta(G) - 1$ where $G_j = G - \{e_1, e_2, \cdots, e_j\}$; and

(ii) every edge e_i, $1 \leq i \leq j$, is eliminable in $G_{i-1} = G - \{e_1, e_2, \cdots, e_{i-1}\}$.

By the classical Vizing's theorem [10], $\chi'(G_j) \leq \Delta(G_j) + 1 = \Delta(G)$. Therefore, applying Lemma 3 repeatedly, we have $\chi'(G) = \Delta(G)$. □

A *minor* of a graph G is a graph obtained from G by repeated deletions and contractions of edges. We say that a class $\mathcal{G}$ of graphs is *minor closed* if any minor of G belongs to $\mathcal{G}$ for every graph $G \in \mathcal{G}$. A classical result of Mader [7, 21] implies that every graph G in any minor closed class $\mathcal{G}$ has a degeneracy bounded by a constant $h(\mathcal{G})$, that is, $s(G) \leq h(\mathcal{G})$, where $h(\mathcal{G})$ is a constant depending on the class $\mathcal{G}$. For example, $h(\mathcal{G}) = 5$ for the class $\mathcal{G}$ of planar graphs.

Thus we have the following corollary from Theorem 2 and Lemma 2.

Corollary 1 *$\chi'(G) = \Delta(G)$ if one of the following* (a) – (g) *holds:*

(a) *G belongs to a minor closed class $\mathcal{G}$ and $\Delta(G) \geq 2h(\mathcal{G})$;*

(b) $\Delta(G) \geq 2k(G)$ [32]*;*

(c) *$\Delta(G) \geq 4a(G) - 2$;*

(d) *$\Delta(G) \geq 4a'(G)$;*

(e) *$\Delta(G) \geq 12\theta(G) - 2$;*

(f) *$g(G) \geq 1$ and $\Delta(G) \geq 2\left\lfloor \left(5 + \sqrt{48g(G) + 1}\right)/2 \right\rfloor$; and*

(g) *G is planar and $\Delta(G) \geq 10$.*

A result better than Corollary 1(g) is known [10, 27]: $\chi'(G) = \Delta(G)$ if G is planar and $\Delta(G) \geq 8$.

4 Finding Edge-Colorings

The proofs of Theorems 1 and 2 yield an exact algorithm to edge-color a graph G with Δ colors if $\Delta(G) \geq 2s(G)$. However, the algorithm takes $O(mn)$ time, since it repeats operations of "shifting a fan sequence" and "switching an alternating path" $O(m)$ times and each operation takes $O(n)$ time [27]. In this section we give a more efficient exact algorithm of complexity $O(n \log n)$ for the case where $a'(G)$ is bounded and $\Delta(G)$ is large: $\Delta(G) \geq 4a'(G)$. Remember that $a'(G) \leq s(G)$. Furthermore we give an NC parallel exact algorithm for this case. Our algorithms first decompose a given graph G of large maximum degree to several edge-disjoint subgraphs of small maximum degrees by using Zhou, Nakano and Nishizeki's algorithm [32], and then find edge-colorings of the subgraphs by using Chrobak and Nishizeki's algorithm (for planar graphs) [3], and finally superimpose the edge-colorings of subgraphs to obtain an edge-coloring of G.

The main result of this section is the following.

Theorem 3 *If the unicyclic index $a'(G)$ is bounded and $\Delta(G) \geq 4a'(G)$, then graph G can be edge-colored by $\Delta(G)$ colors in $O(n \log n)$ sequential time or in $O(\log^3 n)$ parallel time with $O(n \log^3 n)$ operations.*

By Lemma 2 and Eqs.(2)–(6) we have the following corollary.

Corollary 2 *Graph G can be edge-colored by $\Delta(G)$ colors in $O(n \log n)$ sequential time or in $O(\log^3 n)$ parallel time with $O(n \log^3 n)$ operations if one of the following* (a) – (g) *holds:*

(a) *G belongs to a minor closed class $\mathcal{G}$ and $\Delta(G) \geq 4h(\mathcal{G})$;*

(b) *$a(G)$ is bounded and $\Delta(G) \geq 4a(G)$;*

(c) *$s(G)$ is bounded and $\Delta(G) \geq 4s(G)$;*

(d) *$k(G)$ is bounded and $\Delta(G) \geq 4k(G)$;*

(e) *$\theta(G)$ is bounded and $\Delta(G) \geq 12\theta(G)$;*

(f) *$g(G) \geq 1$ is bounded and $\Delta(G) \geq 4 \left\lfloor \left(5 + \sqrt{48g(G)+1}\right)/2 \right\rfloor$; and*

(g) *G is planar and $\Delta(G) \geq 12$.*

Zhou *et al.* [32] obtained a result stronger than Corollary 2(d): a linear-time sequential and an optimal parallel edge-coloring algorithm for any graphs with bounded $k(G)$, i.e., partial k-trees.

In the remaining of this section we prove Theorem 3. We use Chrobak and Nishizeki's algorithm [3] which edge-colors a planar graph G of $\Delta(G) \geq 9$ by Δ colors in $O(n \log n)$ sequential time or in $O(\log^3 n)$ parallel time with $O(n \log^3 n)$ operations and hence is stronger than Corollary 2(g). Their algorithm relies on the following fact: any planar connected graph G has $\Theta(n)$ eliminable edges if $\Delta(G) \geq 9$ [3]. We have the following lemma on graphs which are not always planar.

Lemma 4 *If G is a connected graph, $\Delta(G)$ is bounded and $\Delta(G) \geq 4a'(G)$, then G has $\Theta(n)$ eliminable edges.*

Proof: Let $U = \{u \in V(G) \mid d(u,G) \leq 2a'(G)\}$, then $U \neq \phi$ because by Lemma 1(b) $\delta(G) \leq 2a'(G)$. Furthermore $V - U \neq \phi$ since $\Delta(G) \geq 4a'(G) > 2a'(G)$. Therefore the graph H obtained from G by deleting all the vertices in U is not empty. Let $W = \{w \in V(H) \mid d(w,H) \leq 2a'(G)\}$, and let E' be the set of edges $(u,v) \in E(G)$ such that $u \in U$ and $v \in U \cup W$. Then it suffices to prove the following (i) and (ii):

(i) each edge $(u,v) \in E'$ is eliminable; and

(ii) the number of edges in E' is $\Theta(n)$.

We first prove (i). Let (u,v) be an arbitrary edge in E'. Since $u \in U$, $d(u) \leq 2a'(G) < \Delta$. On the other hand $d^*_u(v) \leq 2a'(G)$: if $v \in U$ then $d^*_u(v) \leq d(v,G) \leq 2a'(G)$; and if $v \in W$ then $d^*_u(v) \leq d(v,H) \leq 2a'(G)$ since none of v's neighbors in U has degree Δ. Therefore $d(u) + d^*_u(v) \leq 4a'(G) \leq \Delta$, and hence edge (u,v) is eliminable.

We next prove (ii). Since at least one edge in E' is incident to each vertex in W, we have

$$|E'| \geq |W|. \tag{9}$$

By applying Lemma 1(c) to graph H we have

$$|W| \geq \frac{n(H)}{2a'(H)+1}. \tag{10}$$

Since $a'(H) \leq a'(G)$ and $n(H) = n - |U|$, we have

$$|W| \geq \frac{n-|U|}{2a'(G)+1}. \tag{11}$$

If $|U|$ is small, say $|U| \leq \frac{2\Delta-1}{2\Delta+1}n$, then by Eqs. (9) and (11) we have

$$\begin{aligned} |E'| &\geq \frac{1}{2a'(G)+1}(n-|U|) \\ &\geq \frac{2}{(2a'(G)+1)(2\Delta+1)}n, \end{aligned}$$

and hence $|E'| = \Theta(n)$ since both $\Delta(G)$ and $a'(G)$ are bounded. Thus it suffices to verify $|E'| = \Theta(n)$ for the case when $|U| > \frac{2\Delta-1}{2\Delta+1}n$, that is,

$$n - |U| < \frac{2}{2\Delta+1}n. \tag{12}$$

Edges in $E(G) - E'$ either join two vertices in W or are incident to vertices in $V-U-W$. The number of former edges is at most $a'(G)|W|$, and the number of latter edges is at most $\Delta(n-|U|-|W|)$. Therefore we have

$$\begin{aligned} |E'| &\geq m - a'(G)|W| - \Delta(n-|U|-|W|) \\ &= m - \Delta(n-|U|) + (\Delta - a'(G))|W|. \end{aligned} \tag{13}$$

Since G is connected, $m \geq n-1$. Therefore by Eqs. (11), (12) and (13) we have

$$\begin{aligned} |E'| &\geq n - 1 - \Delta(n - |U|) + \frac{\Delta - a'(G)}{2a'(G)+1}(n - |U|) \\ &= n - 1 - \frac{a'(G)(2\Delta+1)}{2a'(G)+1}(n - |U|) \\ &> n - 1 - \frac{2a'(G)}{2a'(G)+1}n \\ &= \frac{1}{2a'(G)+1}n - 1. \end{aligned}$$

Thus $|E'| = \Theta(n)$ since $a'(G)$ is bounded. □

Lemma 4 implies that if G is a connected planar graph, $\Delta(G)$ is bounded and $\Delta(G) \geq 12$ then G has $\Theta(n)$ eliminable edges. Thus Lemma 4 does not implies the fact proved by Chrobak and Nishizeki [3], but is a kind of generalization of the fact for (not always planar) graphs.

Chrobak and Nishizeki's algorithm [3] correctly edge-colors any (not always planar) graph G with Δ colors if Δ is bounded and G has $\Theta(n)$ eliminable edges. Therefore by Lemma 4 we have the following lemma.

Lemma 5 *If $\Delta(G)$ is bounded and $\Delta(G) \geq 4a'(G)$, then G can be edge-colored by $\Delta(G)$ colors in $O(n \log n)$ sequential time or in $O(\log^3 n)$ parallel time with $O(n \log^3 n)$ operations.*

By Lemma 5, in order to prove Theorem 3, it suffices to give an algorithm to edge-color G with Δ colors only for the case in which Δ is not bounded, say $\Delta \geq 8s(G)(> 4a'(G))$. Chrobak and Nishizeki's algorithm [3] uses Chrobak and Yung's algorithm [4] for the case in which Δ is large, say $\Delta \geq 19$. However, the algorithm in [4] works only for *planar* graphs with $\Delta \geq 19$. Our idea is to decompose (not always planar) graph G of large maximum degree into several edge-disjoint subgraphs $G_1, G_2, \cdots, G_j$ of small maximum degrees $\Delta(G_i)$ such that $\Delta(G) = \sum_{i=1}^{j} \Delta(G_i)$ and $4s(G) \leq \chi'(G_i) = \Delta(G_i) < 8s(G)$ for each i, and hence an edge-coloring of G with $\Delta(G)$ colors can be obtained simply by superimposing edge-colorings of G_i with $\Delta(G_i)$ colors. Note that the edge-coloring of G_i can be found within the required time bounds as shown in Lemma 5 since $\Delta(G_i)$ is bounded.

Let c be a bounded positive integer, and let $E_1, E_2, \cdots, E_j$ be a partition of E. Denote by $G_i = G[E_i]$ the subgraph of G induced by the edge set E_i. We say that $E_1, E_2, \cdots, E_j$ is a (Δ, c)-*partition* of E if $G_i = G[E_i]$, $1 \leq i \leq j$, satisfies

(i) $\Delta(G) = \sum_{i=1}^{j} \Delta(G_i)$; and

(ii) $\Delta(G_i) = c$ for each i, $1 \leq i \leq j-1$, and $c \leq \Delta(G_j) < 2c$.

Clearly $s(G_i) \leq s(G)$ for each i, $1 \leq i \leq j$. Theorem 2 implies that $\chi'(G) = \Delta(G)$ since $\Delta(G) \geq 8s(G) > 2s(G)$. Choose $c = 4s(G)$, then $\Delta(G_i) \geq c =$

$4s(G) > 2s(G_i)$ and hence $\chi'(G_i) = \Delta(G_i)$ for each i, $1 \le i \le j$. Since $\Delta(G_i) < 2c = 8s(G) \le 16a'(G) = O(1)$ by Lemma 2, $\Delta(G_i)$ is bounded for $1 \le i \le j$. Furthermore $\Delta(G_i) \ge 4s(G) \ge 4s(G_i) \ge 4a'(G_i)$ by Eqs. (3) and (5). Therefore by Lemma 5 one can find an edge-coloring of G_i with $\Delta(G_i)$ colors in the claimed time. Since

$$\Delta(G) = \sum_{i=1}^{j} \Delta(G_i),$$

edge-colorings of G_i with $\Delta(G_i)$ colors, $1 \le i \le j$, can be immediately superimposed to an edge-coloring of G with $\Delta(G)$ colors. Zhou *et al.* [32] obtained the following result on the (Δ, c)-partition.

Lemma 6 *If $\Delta(G) \ge 2c \ge 8s(G)$, then a (Δ, c)-partition of E can be found in linear sequential time or in $O(\log n)$ parallel time with $O(n)$ operations.*

Thus we have the following algorithm to edge-color a graph G such that $a'(G)$ is bounded and $\Delta(G) \ge 4a'(G)$.

EDGE-COLOR(G);

{ assume that $a'(G)$ is bounded and $\Delta(G) \ge 4a'(G)$ }

begin

  if $\Delta(G) < 8s(G)$ **then** {$\Delta(G)$ is bounded }

1. edge-color G with $\Delta(G)$ colors by Lemma 5;

  else {$\Delta(G) \ge 8s(G)$}

  begin

2. find a $(\Delta, 4s(G))$-partition $E_1, E_2, \cdots E_j$ of $E(G)$;

3. **for** $i := 1$ to j **do**

   edge-color of G_i with $\Delta(G_i)$ colors where $G_i = G[E_i]$;

4. extend these optimal edge-colorings of $G_1, G_2, \cdots, G_j$

   to an optimal edge-coloring of G with $\Delta(G)$ colors

  end

end;

We are now ready to prove Theorem 3.

Proof of Theorem 3: By Lemmas 5 and 6 clearly the algorithm above correctly finds an edge-coloring of a graph G with Δ colors. Therefore it suffices to prove the complexities. By Lemma 6 line 2 can be done in linear time or optimally in parallel. By Lemma 5 line 1 can be done in $O(n \log n)$ sequential time or in $O(\log^3 n)$ parallel time with $O(n \log^3 n)$ operations, since $\Delta(G) < 8s(G) \le 16a'(G) = O(1)$. At line 3, for each i, $1 \le i \le j$, by Lemma 5 one can find an edge-coloring of G_i with $\Delta(G_i)$ colors in $O(n(G_i) \log n(G_i))$ sequential time or in $O(\log^3 n(G_i))$ parallel time with $O(n(G_i) \log^3 n(G_i))$ operations. Since $G_i = G[E_i]$, $n(G_i) \le 2|E_i|$. Therefore

$$\sum_{i=1}^{j} n(G_i) \le 2 \sum_{i=1}^{j} |E_i| = 2|E|.$$

Since the unicyclic index $a'(G)$ is bounded, $|E| = O(n)$. Thus line 3 can be totally done in $O(n \log n)$ sequential time or in $O(\log^3 n)$ parallel time with $O(n \log^3 n)$ operations. At line 4, since $\Delta(G) = \sum_{i=1}^{j} \Delta(G_i)$, one can immediately superimpose these edge-colorings of $G_1, G_2, \cdots, G_j$ to an edge-coloring of G with $\Delta(G)$ colors. Thus the algorithm spends $O(n \log n)$ sequential time in total or in $O(\log^3 n)$ parallel time with $O(n \log^3 n)$ operations. $\Box$

It should be noted that the algorithm EDGE-COLOR does not need to know an actual decomposition of G into $a(G)$ unicyclic subgraphs.

5 f-Coloring

In this section we give efficient sequential and NC parallel algorithms for the f-coloring problem on various classes of graphs.

We first show that the f-coloring problem on a graph G can be reduced to the edge-coloring problem on a new graph G_f defined below. We may assume without loss of generality that $f(v) \le d(v)$ for each $v \in V$. For each vertex $v \in V$, replace v with $f(v)$ copies $v_1, v_2, \cdots, v_{f(v)}$, and attach the $d(v)$ edges incident with v to the copies; attach $\lceil d(v)/f(v) \rceil$ or $\lfloor d(v)/f(v) \rfloor$ edges to each copy v_i, $1 \le i \le f(v)$. Let G_f be the resulting graph. It should be noted that the construction of G_f is not unique. Figure 2 illustrates G and an example of G_f, where the number next to vertex v is $f(v)$. Since an edge-coloring of G_f immediately induces an f-coloring of G with the same number of colors, we have

$$\chi'_f(G) \le \chi'(G_f). \tag{14}$$

However, Eq. (14) does not always hold in equality. For example, $\chi'_f(G) = 2$ for a graph G in Figure 2(a) as indicated by solid and dotted lines, but $\chi'(G_f) = 3$ for the graph G_f in Figure 2(b) as indicated by thin, thick and dotted lines. Clearly $\Delta(G_f) = \Delta_f(G) = \max_{v \in V} \lceil d(v)/f(v) \rceil$. If G is a simple graph, then G_f is also a simple graph and hence $\chi'(G_f) \le \Delta(G_f) + 1 = \Delta_f(G) + 1$. Thus an edge-coloring of G_f with $\chi'(G_f)$ colors does not always induce an f-coloring of G with $\chi'_f(G)$ colors, but induces a near-optimal f-coloring of G with at most $\Delta_f(G) + 1$ colors.

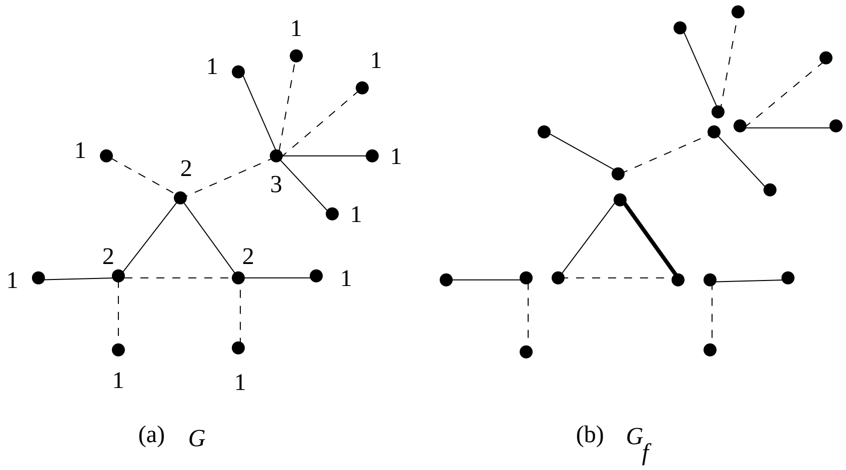

Figure 2: Transformation from G to G_f.

The number of edges in G_f is equal to that of G, but the number of vertices of G_f increases to $\sum_{v \in V} f(v)$ ($\leq 2m$). Furthermore one can easily observe that the following lemmas hold.

Lemma 7 *For a graph G there exists G_f such that*

(a) *G_f is bipartite if G is bipartite;*
(b) *G_f is planar if G is planar;*
(c) $g(G_f) \leq g(G)$;
(d) $s(G_f) \leq s(G)$;
(e) $a(G_f) \leq a(G)$;
(f) $a'(G_f) \leq a'(G)$; *and*
(g) $\theta(G_f) \leq \theta(G)$.

Lemma 8 *Let $\mathcal{G}$ be a class of graphs which are closed under the transformation above, that is, any G_f is contained in $\mathcal{G}$ for every $G \in \mathcal{G}$, and let α and β be real numbers. Then the following* (a) *and* (b) *hold.*

(a) *If there exists a sequential algorithm to edge-color any graph $G' \in \mathcal{G}$ by $\alpha\Delta(G') + \beta$ colors in polynomial time $T(m(G') + n(G'))$, then there exists a sequential algorithm to f-color any graph $G \in \mathcal{G}$ by $\alpha\Delta_f(G) + \beta$ colors in $O(T(m(G) + n(G)))$ time.*

(b) *If there exists a parallel algorithm to edge-color any graph $G' \in \mathcal{G}$ by $\alpha\Delta(G') + \beta$ colors in polylogarithmic parallel time $T(m(G') + n(G'))$ with polynomial operations $P(m(G') + n(G'))$, then there exists a parallel algorithm to f-color any graph $G \in \mathcal{G}$ by $\alpha\Delta_f(G) + \beta$ colors in $O(T(m(G) + n(G)))$ parallel time with $O(P(m(G) + n(G)))$ operations.*

Proof: (a) Let G be a graph in $\mathcal{G}$. One can construct G_f from G in linear time. Using the assumed algorithm, one can find an ordinary edge-coloring of G_f with $\alpha\Delta(G_f)+\beta$ colors in $T(m(G_f)+n(G_f))$ time. The edge-coloring of G_f immediately induces an f-coloring of G with $\alpha\Delta(G_f)+\beta = \alpha\Delta_f(G)+\beta$ colors. By the construction of G_f we have $m(G_f) = m(G)$ and $n(G_f) \le 2m(G)+n(G)$ and hence $m(G_f)+n(G_f) \le 3m(G)+n(G)$. Since the function T is polynomial, $T(m(G_f)+n(G_f)) = O(T(m(G)+n(G)))$. Thus an f-coloring of G can be found in $O(T(m(G)+n(G)))$ time in total.

(b) Similarly, G_f can be easily constructed from G in $O(\log(m(G)+n(G)))$ parallel time with $O(m(G)+n(G))$ operations. □

It is known that $\chi'(G) = \Delta(G)$ if G is a bipartite graph [19] and that $\chi'(G) = \Delta(G)$ if G is a planar graph with $\Delta(G) \ge 8$ [10, 27]. Therefore, by Theorem 2, Corollary 1 and Lemmas 7, 8, we have the following theorem.

Theorem 4 $\chi'_f(G) = \Delta_f(G)$ *if one of the following* (a)–(i) *holds*:

(a) *G belongs to a minor closed class $\mathcal{G}$ and $\Delta_f(G) \ge 2h(\mathcal{G})$;*
(b) *G is bipartite* [13]*;*
(c) *$\Delta_f(G) \ge 2s(G)$;*
(d) *G is a partial k-tree and $\Delta_f(G) \ge 2k$;*
(e) *$\Delta_f(G) \ge 4a(G) - 2$;*
(f) *$\Delta_f(G) \ge 4a'(G)$;*
(g) *$\Delta_f(G) \ge 12\theta(G) - 2$;*
(h) *$g(G) \ge 1$ and $\Delta_f(G) \ge 2\left\lfloor \left(5+\sqrt{48g(G)+1}\right)/2 \right\rfloor$; and*
(i) *G is planar and $\Delta_f(G) \ge 8$.*

Proof: Proofs of (a), (b), (c), (e), (f), (g) and (i) are immediate. If G is a partial k-tree, then G_f is not always a partial k-tree, but $s(G) \le k$. Therefore (d) above is an immediate consequence of (c). If $g(G) \ge 1$ and

$$\Delta_f(G) \ge 2\left\lfloor \left(5+\sqrt{48g(G)+1}\right)/2 \right\rfloor,$$

then $\Delta_f(G) = \Delta(G_f) \ge 12$ and hence $\chi'_f(G) \le \chi'(G_f) = \Delta(G_f) = \Delta_f(G)$ even if $g(G_f) = 0$. Thus (h) follows. □

By Theorem 3, Corollary 2, Lemmas 7, 8 and the algorithms in [3, 6, 12], we have the following results.

Theorem 5

(a) *Any graph G can be f-colored by at most $\Delta_f(G)+1$ colors in $O(\min\{m\Delta_f \log n, m\sqrt{m\log n}\})$ time.*

(b) *Any bipartite graph G can be f-colored by $\Delta_f(G)$ colors in $O(m\log n)$ time.*

(c) *Graph G can be f-colored by $\Delta_f(G)$ colors in $O(n \log n)$ time if one of the following* (i) – (viii) *holds:*

(i) *G belongs to a minor closed class $\mathcal{G}$ and $\Delta_f(G) \geq 4h(\mathcal{G})$;*

(ii) *$a'(G)$ is bounded and $\Delta_f(G) \geq 4a'(G)$;*

(iii) *$a(G)$ is bounded and $\Delta_f(G) \geq 4a(G)$;*

(iv) *$s(G)$ is bounded and $\Delta_f(G) \geq 4s(G)$;*

(v) *$k(G)$ is bounded and $\Delta_f(G) \geq 4k(G)$;*

(vi) *$\theta(G)$ is bounded and $\Delta_f(G) \geq 12\theta(G)$;*

(vii) *$g(G) \geq 1$ is bounded and $\Delta_f(G) \geq 4\left\lfloor \left(5+\sqrt{48g(G)+1}\right)/2 \right\rfloor$; and*

(viii) *G is planar and $\Delta_f(G) \geq 9$.*

Proof: (a) The algorithm in [12] edge-colors G_f with $\Delta(G_f)+1$ colors in time

$$O(\min\{m_f \Delta_f \log n_f, m_f \sqrt{m_f \log n_f}\}),$$

where m_f is the number of edges and n_f the number of vertices in G_f. Since $m_f = O(m)$, $n_f = O(m+n)$ and $\Delta(G_f) = \Delta_f(G)$, the claim holds.

(b) G_f is also bipartite. The algorithm in [6] edge-colors a bipartite graph G_f with $\Delta(G_f)$ colors in time $O(m_f \log n_f)$. Thus the claim holds similarly as (a).

(c) Similar to (b). Note that $s(G) \leq h(\mathcal{G})$, $a'(G) \leq a(G) \leq s(G) \leq k(G)$, $a(G) \leq 3\theta(G)$, and $a(G) \leq s(G) \leq \left\lfloor \left(5+\sqrt{48g(G)+1}\right)/2 \right\rfloor$. □

Theorem 6

(a) *Any bipartite graph G can be f-colored by $\Delta_f(G)$ colors in $O(\log^3 n)$ parallel time with $O(m)$ operations.*

(b) *Graph G can be f-colored by $\Delta_f(G)$ colors in $O(\log^3 n)$ parallel time with $O(n \log^3 n)$ operations if one of the following* (i) – (viii) *holds:*

(i) *G belongs to a minor closed class $\mathcal{G}$ and $\Delta_f(G) \geq 4h(\mathcal{G})$;*

(ii) *$a'(G)$ is bounded and $\Delta_f(G) \geq 4a'(G)$;*

(iii) *$a(G)$ is bounded and $\Delta_f(G) \geq 4a(G)$;*

(iv) *$s(G)$ is bounded and $\Delta_f(G) \geq 4s(G)$;*

(v) *$k(G)$ is bounded and $\Delta_f(G) \geq 4k(G)$;*

(vi) *$\theta(G)$ is bounded and $\Delta_f(G) \geq 12\theta(G)$;*

(vii) *$g(G) \geq 1$ is bounded and $\Delta_f(G) \geq 4\left\lfloor \left(5+\sqrt{48g(G)+1}\right)/2 \right\rfloor$; and*

(viii) *G is planar and $\Delta_f(G) \geq 9$.*

Proof: (a) G_f is also bipartite. The algorithm in [20] edge-colors G_f with $\Delta(G_f)$ colors in $O(\log^3 n_f)$ parallel time with $O(m_f)$ operations. Since $m_f = O(m)$, $n_f = O(m+n)$ and $\Delta(G_f) = \Delta_f(G)$, the claim holds.

(b) Similar to (a). □

It should be noted that the algorithms in Theorems 5.4 and 5.5 do not need to know an actual embedding or a decomposition related to an invariant.

6 Conclusion

In this paper we first gave efficient sequential and NC parallel algorithms to edge-color graph G with $\Delta(G)$ colors if $a'(G)$ is bounded and $\Delta(G) \geq 4a'(G)$, where $a'(G)$ is the unicyclic index of G. Our algorithms are based on the following two algorithms: the edge-coloring algorithm (for planar graphs) by Chrobak and Nishizeki [3], and the algorithm for decomposing a graph of large maximum degree to edge-disjoint subgraphs of small maximum degrees by Zhou, Nakano and Nishizeki [32]. We next introduced a simple but useful reduction of an f-coloring to an ordinary edge-coloring, and derived various sufficient conditions for $\chi'_f(G) = \Delta_f(G)$ to hold true. Using the reduction, we finally gave efficient sequential and NC parallel f-coloring algorithms.

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