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# Bounds for Orthogonal 3-D Graph Drawing

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**Abstract**

This paper studies 3-D orthogonal grid drawings for graphs of arbitrary degree, in particular $K_n$, with vertices drawn as boxes. It establishes asymptotic lower bounds for the volume of the bounding box and the number of bends of such drawings and exhibits a construction that achieves these bounds. No edge route in this construction bends more than three times. For drawings constrained to have at most $k$ bends on any edge route, simple constructions are given for $k = 1$ and $k = 2$. The unconstrained construction handles the $k \geq 3$ cases.

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# 1 Introduction

This paper offers upper and lower bounds for the volume and the total number of bends in 3-D orthogonal grid drawings for graphs of arbitrary degree. In particular, we study how the volume depends on the maximum number of bends permitted per edge. All of our constructions have a total number of bends that is asymptotically optimal, and one construction also exhibits asymptotically optimal volume. To state the main results clearly, we first give some terminology and the drawing conventions and volume measure used.

A *grid point* is a point in $R^3$ whose coordinates are all integers. A *grid box* is the set of all points $(x, y, z)$ in $R^3$ satisfying $x_0 \leq x \leq x_1$, $y_0 \leq y \leq y_1$ and $z_0 \leq z \leq z_1$ for some integers $x_0, x_1, y_0, y_1, z_0, z_1$. A grid box is said to have *dimensions* $a \times b \times c$ whenever $x_1 = x_0 + a - 1$, $y_1 = y_0 + b - 1$, and $z_1 = z_0 + c - 1$. The *volume* of such a box is defined to be the number of grid points it contains, namely $abc$. For example, a single grid point is a $1 \times 1 \times 1$ box of volume 1. The *volume* of a drawing is the volume of its *bounding box*, which is the smallest volume grid box containing the drawing. Often we refer to the bounding box as an $X \times Y \times Z$-grid.

Throughout this paper, a *3-D orthogonal grid drawing* of a graph $G = (V, E)$ is a drawing that satisfies the following. Distinct vertices of $V$ are represented by disjoint grid boxes. While in general these boxes may be degenerate, i.e., they may have dimension 1 with respect to one or more coordinate directions, such degeneracies can be avoided, as we describe later. An edge $e = (v_1, v_2)$ of $E$ is drawn as a simple path that follows grid lines, possibly turning ("bending") at grid points; the endpoints of the path for $e$ are grid points that are extremal points for the boxes representing $v_1$ and $v_2$. The intermediate points along the path for an edge do not belong to any vertex box, nor do they belong to any other edge path. See Fig. 1. In what follows, graph theoretic terms such as *vertex* are typically used to refer both to the graph theoretic object and to its representation in a drawing.

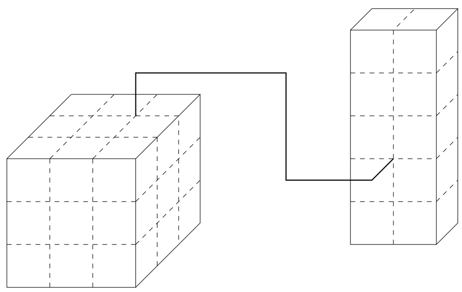

Figure 1: Two boxes joined by a 4-bend edge.

### 1.1 Focus of this paper

The focus of this paper is on establishing upper and lower bounds for the volume and the number of bends of 3-D orthogonal grid drawings. In particular, we give upper bounds on the volume that depend on the allowed maximum number of bends per edge. More useful than upper and lower bounds would be an algorithm that computes for an arbitrary input graph an embedding that minimizes volume or number of bends. However, the problem of bend minimization or volume minimization is apparently computationally intractable. See [8].

We exclusively study embeddings of the complete graph $K_n$ for the following reason. Any simple graph $G$ on $n$ vertices is a subgraph of the complete graph $K_n$. Thus, a drawing of $K_n$ immediately provides a drawing for $G$ by deleting irrelevant edges. Consequently, upper bounds for $K_n$ yield upper bounds for all other simple graphs on $n$ vertices. Furthermore, no simple graph on $n$ vertices can yield larger lower bounds than $K_n$.

Since our focus is on bounds, our constructions are not designed with the intention of giving attractive looking drawings. In particular, vertex boxes may be degenerate as previously described. Such degeneracies may be easily removed from a drawing by inserting extra axis-aligned planes of grid points. This increases the volume of a drawing by a multiplicative constant and does not affect the order of the upper bound.

The boxes produced by our upper bound constructions can be poorly proportioned in two respects. The surface area can be large relative to the degree of the vertex. Also, the vertex boxes can be far from cube-shaped. Algorithms that proportion vertices better have recently been presented in [20], [1].

### 1.2 Relation to other work

The problem of embedding a graph in a rectangular grid has been addressed in the context of VLSI design (see [16] for an overview). However, the objectives of graph drawing and those of VLSI, while similar, are often prioritized differently. For example, while bend (or "jog") minimization within layers is an issue in circuit layout design, apparently this cost measure does not have a high priority (see [16, p. 222]). By contrast, in graph drawing, the notion that bend minimization is important for diagram readability has been widely accepted ([21] provides some experimental evidence for this). Another difference between graph drawing and VLSI is that in 3-D VLSI design one of the dimensions is radically different from the other two: connections between layers ("vias") are undesirable, whereas bends within a layer ("jogs") are of minor importance. Also, one of the dimensions is usually restricted to a small number of layers. In 3-D graph drawing there are no such differences between directions. With advances in fabrication technology, it has become practical for VLSI design to use more than two or three layers; hence our results may nevertheless be of interest in that field.

In the field of graph drawing, for graphs drawn orthogonally in the 2-D grid, early research mainly considered graphs of maximum degree 4 and represented vertices as single grid points. See for example [23], [2], [19]. More recently, 2-D orthogonal grid drawings of higher degree graphs have been investigated, where vertices have been drawn as rectangular boxes. See for example [12], [18], [3].

At present, there are few results on 3-D orthogonal grid drawings. Rosenberg showed that any graph of maximum degree 6 can be embedded in a 3-D grid of volume $O(n^{3/2})$ and that this is asymptotically optimal [22]. No bounds on the number of bends were given. Recently, Eades, Symvonis and Whitesides gave a method for drawing graphs of maximum degree 6 in a grid of side-length $4\sqrt{n}$, with vertices represented by single grid points and each edge having at most 7 bends [9]. They also gave a simple method for drawing such graphs in a grid of side-length $3n$, creating at most 3 bends on each edge. This volume was subsequently improved to at most $4.66n^3$ [20] and then to volume at most $2.37n^3$ [27]. For graphs with maximum degree 5, volume $n^3$ and 2 bends per edge suffice [27].

As for drawings with higher degree, only two papers are known to the authors. Papakostas and Tollis showed how to draw graphs in a 3-D grid of volume $O(m^3) \leq O(n^6)$ [20]. Very recently, Biedl [1] extended the techniques presented here and showed how to draw graphs in a 3-D grid of volume $O(n^3)$. Neither paper matches our upper bound volume of $O(n^{2.5})$. However, both papers yield constructions where vertex boxes have a more cube-like appearance. This suggests a trade-off between cube-like appearance of of vertex boxes and volume.

## 1.3 Results of this paper

Our results concern volume and number of bends for 3-D orthogonal grid drawings. Since we give upper and lower bounds, we first explain what functions are being bounded, and then we state the results.

**convention:** From now on, the terms *drawing* and 3-D *orthogonal grid drawing* are used interchangeably.

Let $vol(n)$ denote the minimum, taken over all drawings of $K_n$, of the volumes of the drawings. Here, there are no restrictions on these drawings of $K_n$, other than that they are understood, by the above convention, to be 3-D orthogonal grid drawings.

Similarly, let $vol_k(n)$ denote the minimum, taken over all drawings of $K_n$ that have $k$ or fewer bends on any edge, of the volumes of the drawings. Let $bend(n)$ denote the minimum, taken over all drawings of $K_n$, of the total number of bends in the drawings.

**main results:** Our main results are that

- $vol(n) \in \Theta(n^{2.5})$;
- $vol_1(n)$ and $vol_2(n) \in O(n^3)$;
- for $k \geq 3$, $vol_k(n) \in \Theta(n^{2.5})$; and
- $bend(n) \in \Theta(n^2)$.

Note that for $k \geq 3$, the upper and lower bounds on the volume match (within a constant factor) when a maximum of $k$ bends per edge is allowed. The constructions of this paper have reasonably small constant factors for the volume. Only for the $k = 1$ and $k = 2$ cases do the bounds on the volume not match; in each of these cases we give an $O(n^3)$ volume drawing of $K_n$ and leave as an open problem whether this drawing indeed has asymptotically optimum volume.

# 2 Lower Bounds

## 2.1 A lower bound on the volume

Recall that $vol(n)$ is the minimum possible volume for a drawing of $K_n$. This definition is valid since, as later sections show, every $K_n$ has a drawing if edges are allowed to bend. The main result of this section is to show that $vol(n)$ is in $\Omega(n^{2.5})$.

A *z-line* is a line that is parallel to the $z$-axis; *y-lines* and *x-lines* are defined analogously. A $(z = z_0)$-plane is a plane that is orthogonal to the $z$-axis and intersects the $z$-axis at coordinate $z_0$; $(x = x_0)$-planes and $(y = y_0)$-planes are defined analogously.

**Theorem 1** *$vol(n) \in \Omega(n^{2.5})$. In fact, for any $0 < \varepsilon < \frac{1}{4}$, we have $vol(n) \geq \min\{(\frac{1}{4} - \varepsilon)n^{5/2}, f(n)\}$ where $f(n) \in \Theta(n^3)$.*

**Proof:** Consider a drawing of $K_n$ in a grid of dimensions $X \times Y \times Z$. Let $0 < \varepsilon < \frac{1}{4}$ be given and choose $0 < \delta < \frac{1}{4}$ so small that $\frac{1}{4}(1 - 4\delta)^{5/2} > \frac{1}{4} - \varepsilon$. We distinguish three cases.

**Case 1: A line intersects many vertices**

Assume that there exists a $z$-line intersecting at least $2\delta n$ vertices. Set $t = \lceil 2\delta n \rceil$, and let $v_1, \ldots, v_t$ be any $t$ of the vertices intersected by the $z$-line, listed in order of occurrence along the line. Let $z_0$ be a not necessarily integer $z$-coordinate such that the $(z=z_0)$-plane intersects none of these $t$ vertices and separates the first $\lfloor \frac{t}{2} \rfloor$ of them from the remaining $\lceil \frac{t}{2} \rceil$. See the top left picture in Fig. 2.

Since the $\lfloor \frac{t}{2} \rfloor \cdot \lceil \frac{t}{2} \rceil \geq \frac{1}{4}t^2 - 1$ edges connecting these two groups must cross the $(z=z_0)$-plane, this plane must contain at least $\frac{1}{4}t^2 - 1$ points having integer $x$- and $y$-coordinates. Hence $XY \geq \frac{1}{4}t^2 - 1$. Also, $Z \geq t$ since the $z$-line intersects at least $t$ vertices. Thus $XYZ \geq \frac{1}{4}t^3 - t \geq 2\delta^3 n^3 - 2\delta n \in \Theta(n^3)$.

**Case 2: A plane intersects many vertices**

Assume now that no $x$-line, $y$-line or $z$-line intersects as many as $2\delta n$ vertices, but that there exists a $(z=z_0)$-plane intersecting at least $(1 - 2\delta)n$ vertices.

A vertex is *left* of an $(x=x_0)$-plane if all the points in its grid box have $x$-coordinates less than $x_0$. The notion of *right* of an $(x=x_0)$-plane is analogous. As an $(x = x_0)$-plane is swept from smaller to larger values of $x_0$, the $y$-line determined by the intersection of this $(x = x_0)$-plane with the $(z = z_0)$-plane sweeps the $(z = z_0)$-plane. At any time, this $y$-line intersects fewer than $2\delta n$ vertices by assumption.

During the sweep by the $(x=x_0)$-plane, an integer $x^*$ is encountered where, for the last time, there are fewer than $(\frac{1}{2}-2\delta)n$ vertices left of the $(x=x^*)$-plane and intersecting the $(z=z_0)$-plane. See the top right picture in Fig. 2. Since the $y$-line determined by the $(x=x_0)$-plane intersects fewer than $2\delta n$ vertices, and the $(z=z_0)$-plane intersects at least $(1-2\delta)n$ vertices by assumption, at least $(1-2\delta)n-(\frac{1}{2}-2\delta)n-2\delta n = (\frac{1}{2}-2\delta)n$ vertices intersect the $(z=z_0)$-plane and lie right of the $(x=x^*)$-plane. All these vertices also lie to the right of $(x=x^*+\frac{1}{2})$-plane.

By definition of $x^*$, the number of vertices that intersect the $(z=z_0)$-plane and that lie left of the $(x=x^*+1)$-plane is at least $(\frac{1}{2}-2\delta)n$. All these vertices also lie to the left of $(x=x^*+\frac{1}{2})$-plane.

There are at least $(\frac{1}{2}-2\delta)^2n^2$ edges between the vertices on the left and the vertices on the right of the $(x=x^*+\frac{1}{2})$-plane, so $YZ \geq (\frac{1}{2}-2\delta)^2n^2 = \frac{1}{4}(1-4\delta)^2n^2$. Apply exactly the same argument in the $y$-direction to obtain $XZ \geq \frac{1}{4}(1-4\delta)^2n^2$. Finally, note that $XY \geq (1-2\delta)n \geq (1-4\delta)n$, since the $(z=z_0)$-plane intersects $(1-2\delta)n$ vertices. Consequently, $XYZ = \sqrt{YZ\cdot XZ\cdot XY} \geq \sqrt{\frac{1}{16}(1-4\delta)^5n^5} = \frac{1}{4}(1-4\delta)^{5/2}n^{5/2} > (\frac{1}{4}-\varepsilon)n^{5/2}$ by the choice of $\delta$.

**Case 3: No plane intersects many vertices**

Assume now that no plane intersects as many as $(1-2\delta)n$ vertices. As an $(x=x_0)$-plane is swept from smaller to larger values of $x_0$, by an argument analogous to the one in Case 2 a value $x^*$ (not necessarily integral) will be encountered for which at least $\delta n$ vertices lie left of the $(x=x^*)$-plane and at least $\delta n$ vertices lie right of the $(x=x^*)$-plane. See the bottom picture in Fig. 2. Consequently, the $(x=x^*)$-plane contains at least $\delta^2n^2$ points with integer $y$- and $z$-coordinates, and $YZ \geq \delta^2n^2$. Since the same argument holds for the other two directions, $XYZ \geq (\delta^2n^2)^{3/2} = \delta^3n^3 \in \Theta(n^3)$.

For all sufficiently large $n$, the bound given by Case 2 is the smallest of the three; hence $vol(n) \in \Omega(n^{5/2})$. □

## 2.2 A lower bound on the bends

Recall that $bend(n)$ is the minimum possible number of bends for a drawing of $K_n$. This definition is valid since, as later sections show, every $K_n$ has a drawing if edges are allowed to bend. The main result of this section is that $bend(n)$ is in $\Omega(n^2)$.

To prove this result, we use the fact that there exist graphs that have no 0-bend 3-D orthogonal drawing [11]. We present here a simple proof of this fact.

If no bends are permitted in the drawing, then the edges correspond to axis-parallel visibility lines between pairs of boxes. Such visibility representations have been studied in 2-D by Wismath [26], [15] and by Tamassia and Tollis [24], and in 3-D with 2-D objects in [4], [10], [11]. A 3-D orthogonal drawing of a graph with no bends splits the edges into three classes, depending on the

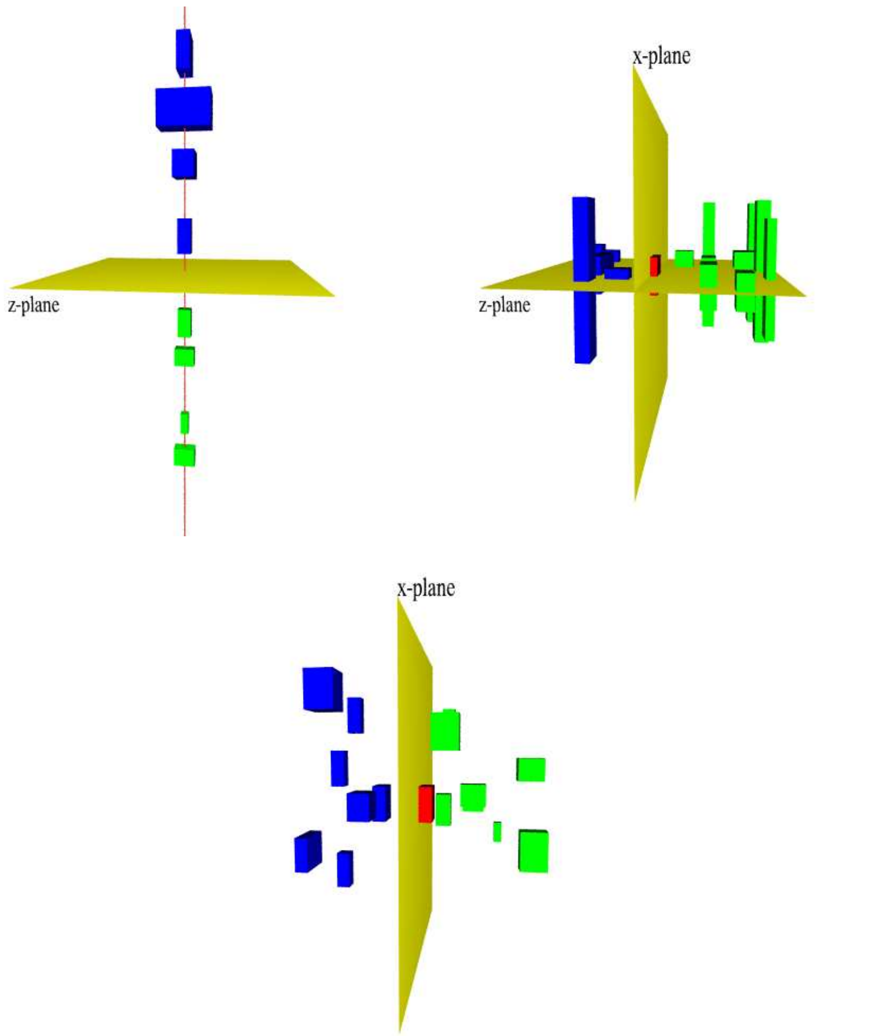


Figure 2: Cases 1,2,3 for the lower bound.

direction of visibility. Each class of edges forms a graph that has a visibility representation using only one direction of visibility. Our lower bound result depends on the fact that $K_{56}$ has no such visibility representation, as shown in [10].

**Lemma 1** *For all sufficiently large* $n$*,* $K_n$ *has no bend-free 3-D orthogonal grid drawing.*

**Proof:** The *3-Ramsey number* $R(r, b, g)$ is the smallest number such that any arbitrary coloring of the edges of $K_{R(r,b,g)}$ with colors red, blue and green induces either a red $K_r$, or a blue $K_b$, or a green $K_g$ as a subgraph. This number exists and is finite; see for example [13].

Assume $K_n$, $n > R(56, 56, 56)$, is drawn without a bend. Color an edge red if it is parallel to an $x$-line, green if it is parallel to a $y$-line, and blue if it is parallel to a $z$-line. By the choice of $n$, we must have a monochromatic $K_{56}$, which contradicts the fact that $K_{56}$ has no visibility representation using only one direction of visibility. Therefore $K_n$ must have a bend in any 3-D orthogonal grid drawing. □

Fekete and Meijer [11] independently proved this lemma. They were interested in obtaining good bounds for the minimum such $n$, and therefore gave a longer proof to show that $K_{184}$ requires a bend in any 3-D orthogonal drawing.

One consequence of this lemma is that $bend(n) \in \Omega(n^2)$.

**Theorem 2** *$bend(n) \in \Omega(n^2)$.*

**Proof:** Let $c$ be an integer (e.g., 184) such that any 3-D orthogonal grid drawing of $K_c$ has a bend. For $n > c$, the graph $K_n$ contains $\binom{n}{c}$ copies of a $K_c$. Each of these copies must have a bend. Any edge of $K_n$ belongs to exactly $\binom{n-2}{c-2}$ of these copies of $K_c$. Consequently, the number of edges with a bend must be at least

$$\frac{\binom{n}{c}}{\binom{n-2}{c-2}} = \frac{n!}{c!(n-c)!}\frac{(c-2)!(n-c)!}{(n-2)!} = \frac{n(n-1)}{c(c-1)} \geq \frac{n^2}{c^2}$$

for $n \geq c$. □

# 3 Constructions

The lower bound of Section 2.1 provides a volumetric goal for layout strategies for drawings with at most $k$ bends per edge. This section presents a construction that achieves this lower bound with a small constant factor. For the $k = 1$ case, two strategies are described and then modified to give a drawing for the $k = 2$ case. A simple construction that realizes the $\Omega(n^{2.5})$ lower bound for volume is described in Subsection 3.3. The construction generates at most 3 bends on any edge and hence is valid for each $k \geq 3$. Whether the lower bound is attainable when $k = 1$ or 2 remains an open problem.

In each of the constructions, vertices are first placed as points in a 2-D $xy$-plane. Next, all the edges are routed in the same $xy$-plane, with overlap and crossings of edges temporarily permitted. Then a number $Z$ of $z$-planes is introduced, and edges are assigned to these planes so that no edges overlap or cross. The vertices are stretched into segments of $z$-lines.

While the VLSI and MCM literature proposes many layout constructions of similar flavor (see e.g. [14]), our work differs from those results in several aspects. Our constructions provide proof techniques for obtaining upper bounds for $K_n$; by contrast, the VLSI literature aims to provide usable layout heuristics and algorithms for arbitrary input graphs. Another important difference is that the constraint on the maximum number of bends per edge that we study in this paper is apparently not an issue for the VLSI and MCM technologies.

## 3.1 Drawings of $O(n^3)$ volume for $k = 1$

In this section, we describe two strategies to draw $K_n$ with at most $k = 1$ bend on any edge. For simplicity, we assume in the description of our constructions that $n$ is divisible by 4. When this is not the case, slightly modified constructions yield the same asymptotic bounds.

The first layout scheme draws $K_n$ in an $n \times n \times n$-grid. The second scheme then makes two drawings of $K_{n/2}$ (without recursion) using the first scheme; then it positions these drawings in an $\frac{n}{2} \times n \times \frac{n}{2}$-grid and supplies the edges between the two parts.

### 3.1.1 Drawing $K_n$ in an $n \times n \times n$-grid for $k = 1$

Enumerate the vertices as $v_1, \ldots, v_n$. Place vertex $v_i$ at $(i, i)$. Route edge $e = (v_i, v_j)$, where $i < j$, with one bend via $(i, i), (i, j), (j, j)$. Note that no vertex or part of an edge is placed at a point $(x, y)$ with $y < x$.

Now partition the edges of $K_n$ into $n$ edge sets $E_i^a, E_i^b$, $i = 1, \ldots, \frac{n}{2}$, defined as $E_i^a = \{(v_{i-l+1}, v_{i+l}) | l = 1, \ldots, \frac{n}{2}\}$ and $E_i^b = \{(v_{i-l}, v_{i+l}) | l = 1, \ldots, \frac{n}{2} - 1\}$ (all additions are modulo $n$). It is easy to check that these sets indeed partition the edges of $K_n$, and that neither crossings nor overlaps occur either among edges in $E_i^a$ or among edges in $E_i^b$. Hence only $n$ $z$-planes are needed. See Fig. 3.

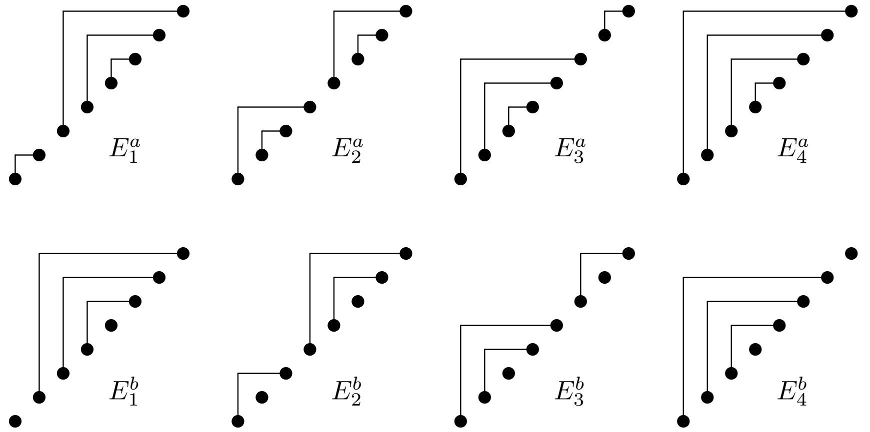


Figure 3: The sets $E_1^a, \ldots, E_4^a$ and $E_1^b, \ldots, E_4^b$ for $K_8$.

This gives the following lemma.

**Lemma 2** *If $n$ is even, there exists a drawing of $K_n$ in an $n \times n \times n$-grid with one bend per edge such that the points $\{(x, y, z) | y < x\}$ are unused.*

**Proof:** Represent each vertex $v_j$, $i \leq j \leq n$, as a line segment (hence a grid box) with endpoints $(j, j, 1)$ and $(j, j, n)$. Route the edges in $E_i^a$, $1 \leq i \leq n/2$, in the $(z = i)$-plane as described above. Similarly, route the edges in $E_i^b$, $1 \leq i \leq n/2$, in the $(z = n/2 + i)$-plane. This gives a crossing-free drawing with the desired properties. $\square$

**Remark:** Note that $E_i^a$ and $E_i^b$ can be drawn in the same plane by reflecting the edges of $E_i^a$ with respect to the diagonal line through the vertices. This yields a drawing of $K_n$ in an $n \times n \times \frac{n}{2}$-grid. This strategy is closely related to the *pagenumber* of a graph (see for example [6]), and in fact, may prove a useful idea for drawing sparse graphs.

### 3.1.2 Drawing $K_n$ in an $\frac{n}{2} \times n \times \frac{n}{2}$-grid for $k = 1$

Let $K^1$ and $K^2$ denote two drawings of $K_{n/2}$ with coordinates as described in the proof of the previous lemma. Thus each drawing has an $\frac{n}{2} \times \frac{n}{2} \times \frac{n}{2}$ bounding box and initially, $K^1$ and $K^2$ are superimposed. Rotate $K^2$ and its bounding box about the $y$-axis clockwise by 90 degrees (looking towards $+\infty$). Then rotate it about the $x$-axis by 180 degrees. In this rotated $K^2$, vertex $v_j$ contains the points $\{(x, -j, j) | 1 \leq x \leq \frac{n}{2}\}$. See Fig. 4.

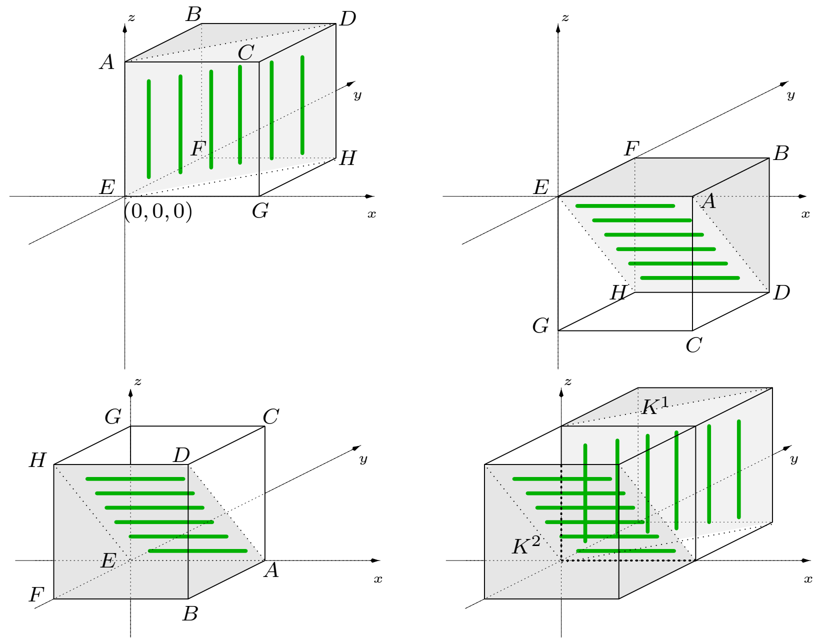


Figure 4: Rotate $K^2$ twice: first by 90 degrees about the $y$-axis, and then by 180 degrees about the $x$-axis. Finally, we show the combination of $K^1$ and the rotated $K^2$. The gray area is the area that contains edges of $K^2$.

Each vertex $v_i$ in $K^1$ sees each vertex $v_j$ in the rotated $K^2$ along the $y$-line segment $[(i,i,j),(i,-j,j)]$. Therefore, these edges can be drawn as straight line segments, thus producing a drawing of $K_n$. The unused $(y=0)$-plane can be deleted to give a drawing with dimensions $X = Z = \frac{n}{2}$ and $Y = n$.

**Theorem 3** *For a given $n$, let $N \geq n$ be the smallest number that is divisible by 4. $K_n$ can be drawn in an $\frac{N}{2} \times N \times \frac{N}{2}$-grid with at most one bend per edge and total number of bends at most $N^2/4 - N/2$.*

**Proof:** Draw $K_N$ as described above, ignoring the $N - n$ vertices not belonging to $K_n$, and their incident edges. The volume bounds follow directly from the construction. There are $N^2/4$ edges drawn without a bend, and all other edges have one bend, so the total number of bends is at most $N^2/4 - N/2$. □

**Remark:** Since $N \leq n+3$, our construction has a volume of $\frac{1}{4}n^3 + O(n^2)$.

## 3.2 A smaller $O(n^3)$ volume drawing for $k = 2$

A similar strategy can be applied when a maximum of $k = 2$ bends on an edge is allowed. For simplicity, we assume in the description of our constructions that $n$ is divisible by 4. When this is not the case, slightly modified constructions yield the same asymptotic bounds.

We draw $K_n$ with at most two bends per edge by first making two copies of a drawing for $K_{\frac{n}{2}}$ (without recursion) and then placing them in a grid of side-length $\frac{n}{2}$ and supplying the edges connecting the two parts.

### 3.2.1 Drawing in an $n \times \frac{n}{2} \times n$-grid

Enumerate the vertices as $\{v_1, \ldots, v_n\}$ and place $v_i$ at $(x,y) = (i,1)$ in a 2-D $xy$-plane. To route edge $e = (v_i, v_j)$, where $i < j$, let $y = \lceil \frac{j-i}{2} \rceil$ and route $e$ via the points $(i,1),(i,y),(j,y),(j,1)$, creating two bends if $y > 1$ and no bends if $y = 1$.

Define the edge sets $E_i^a$ and $E_i^b$ as above. Again there are neither crossings nor overlaps among edges in the same set and so $n$ $z$-planes suffice. Since the largest $y$-coordinate is $\lceil \frac{n-1}{2} \rceil$, the bounding box has dimensions $n \times \frac{n}{2} \times n$.

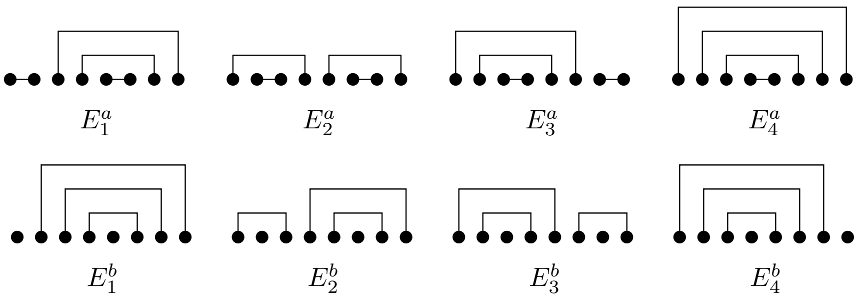


Figure 5: The edge sets of $K_8$ drawn with at most two bends per edge.

**Lemma 3** *If $n$ is even, there exists a drawing of $K_n$ in an $n \times \frac{n}{2} \times n$-grid, with a total of $n^2 - 3n + 2$ bends and at most two bends per edge, such that the line segment (grid box) for vertex $v_i$ contains the points $\{(i, 1, z) | 1 \leq z \leq n\}$.*

**Proof:** Represent each vertex $v_j$, $i \leq j \leq n$, as a line segment (hence a grid box) with endpoints $(j, 1, 1)$ and $(j, 1, n)$. Route the edges in $E_i^a$, $1 \leq i \leq n/2$, in the $(z = i)$-plane as described above. Similarly, route the edges in $E_i^b$, $1 \leq i \leq n/2$, in the $(z = n/2 + i)$-plane. This gives a crossing-free drawing with the desired volume bounds. The edges $(v_i, v_{i+1})$ for $i = 1, \ldots, n-1$ are drawn straight; all other edges have two bends, so the total number of bends is $2(n(n-1)/2 - (n-1)) = n^2 - 3n + 2$. □

#### 3.2.2 Drawing in an $\frac{n}{2} \times \frac{n}{2} \times \frac{n}{2}$-grid

Let $K^1$ and $K^2$ denote two drawings of $K_{n/2}$ with coordinates as described in the proof of the previous lemma. Thus each drawing has an $\frac{n}{2} \times \frac{n}{4} \times \frac{n}{2}$ bounding box and initially, $K^1$ and $K^2$ are superimposed. Rotate the bounding box of $K^2$ as described in Section 3.1.2 and Fig. 4. Then vertex $v_j$ of the rotated $K^2$ contains the points $\{(x, -1, j) | 1 \leq x \leq \frac{n}{2}\}$. See Fig. 6.

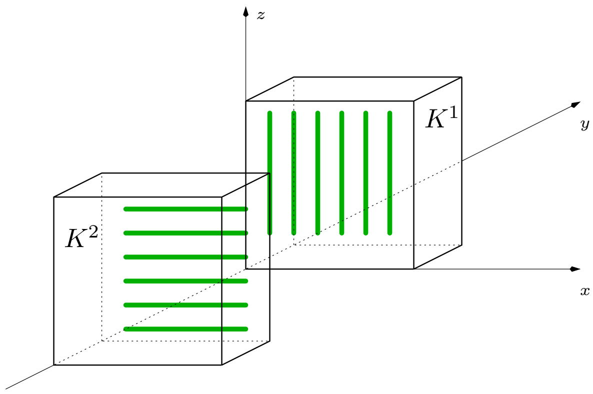


Figure 6: The combination of $K^1$, and the rotated $K^2$ (we moved $K^2$ farther away for clarity).

Each vertex $v_i$ in $K^1$ sees each vertex $v_j$ in the rotated $K^2$ along the $y$-line segment $[(i, 1, j), (i, -1, j)]$. Therefore, these edges can be drawn as straight lines, thus producing a drawing of $K_n$. The unused $(y=0)$-plane can be deleted to give a drawing with dimensions $X = Y = Z = \frac{n}{2}$.

**Theorem 4** *For a given $n$, let $N \geq n$ be the smallest number that is divisible by 4. $K_n$ can be drawn in a $\frac{N}{2} \times \frac{N}{2} \times \frac{N}{2}$-grid with at most two bends per edge and total number of bends at most $N^2/2 - 3N + 4$.*

**Proof:** Draw $K_N$ as described above, ignoring the $N - n$ vertices not belonging to $K_n$, and their incident edges. The volume bounds follow directly from the construction, and the bound on the number of bends follows from Lemma 3, since we have at most $2((\frac{N}{2})^2 - 3\frac{N}{2} + 2)$ bends. $\square$

**Remark:** Since $N \leq n + 3$, our construction has a volume of $\frac{1}{8}n^3 + O(n^2)$.

### 3.3 An $O(n^{2.5})$ volume drawing for $k = 3$

In this section, we draw $K_n$ with at most $k = 3$ bends on any edge and with volume $O(n^{2.5})$. Case 2 of the lower bound proof suggests what general form such a drawing might take. For simplicity, we assume in the description of our constructions that $n = r^2$ for some integer $r$. When this is not the case, slightly modified constructions yield the same asymptotic bounds.

Enumerate the vertices as ordered pairs $(i, j)$, where $1 \leq i \leq r$, $1 \leq j \leq r$, and place vertex $(i, j)$ at $(2i, 2j)$ in the 2-D $xy$-plane. Suppose edge $e$ joins vertex $(i_1, j_1)$ and vertex $(i_2, j_2)$. After possible renaming, we may assume that $i_1 \leq i_2$, and that if $i_1 = i_2$, then $j_1 > j_2$. Call $e$ an L-*edge* if $j_1 > j_2$ and a $\Gamma$-*edge* otherwise. Fig. 7 shows some L-edges.

Initially route each L-edge via the points $(2i_1, 2j_1), (2i_1{+}1, 2j_1), (2i_1{+}1, 2j_2{+}1), (2i_2, 2j_2{+}1), (2i_2, 2j_2)$, thus with three bends. Route each $\Gamma$-edge via points $(2i_1, 2j_1), (2i_1 + 1, 2j_1), (2i_1 + 1, 2j_2 - 1), (2i_2, 2j_2 - 1), (2i_2, 2j_2)$.

Split the L-edges into $r(r - 1)$ groups $E_{d_x,d_y}$, with $0 \leq d_x \leq r - 1$ and $1 \leq d_y \leq r - 1$. Each group $E_{d_x,d_y}$ consists of those edges $((i_1, j_1), (i_2, j_2))$ for which $i_2 = i_1 + d_x$ and $j_2 = j_1 - d_y$. These groups cover all L-edges since $i_1 \leq i_2$ and $j_1 > j_2$ for any L-edge.

Now split each group $E_{d_x,d_y}$ into at most $d_x + d_y$ sets of edges as follows. For $p = 0, \ldots, d_x + d_y - 1$, let $E^p_{d_x,d_y}$ be the edges in $E_{d_x,d_y}$ for which $j_2 - i_1 = p$ modulo $(d_x{+}d_y)$. In other words, the lower left "corners" of the L-edges in $E^p_{d_x,d_y}$ lie on diagonals that intersect the $y$-axis at the value $2p$ modulo $(2d_x + 2d_y)$. See Fig. 7. It is easy to check that no two edges in $E^p_{d_x,d_y}$ overlap or intersect (except at endpoints), since the corners of the L's are placed on a sequence of diagonals; these diagonals have a vertical spacing of $2d_x + 2d_y$ between adjacent diagonals. Also, note that $E^p_{d_x,d_y}$ is non-empty only if $p \leq 2r - d_x - d_y$.[1]

Assign a $z$-plane to each set $E^p_{d_x,d_y}$ to obtain a legal drawing of the L-edges. Route the $\Gamma$-edges in an analogous fashion. This doubles the number of $z$-planes, yielding a drawing of $K_n$ in a grid with $X = Y = 2r = 2\sqrt{n}$. The $Z$ dimension is given by

$$2 \sum_{d_x=0}^{r-1} \sum_{d_y=1}^{r-1} \min\{d_x + d_y, 2r - d_x - d_y\},$$

which is shown in the following technical lemma to be no greater than $\frac{4}{3}r^3$.

**Lemma 4** $\sum_{d_x=0}^{r-1} \sum_{d_y=1}^{r-1} \min\{d_x + d_y, 2r - d_x - d_y\} < \frac{2}{3}r^3$.

[1] A java applet demonstrating the sets and their routings for $K_{100}$ can be found at `http://www.cs.uleth.ca/~wismath/ortho.html`. VRML constructions of any graph can be created with the OrthoPak software package available from the above Web site.

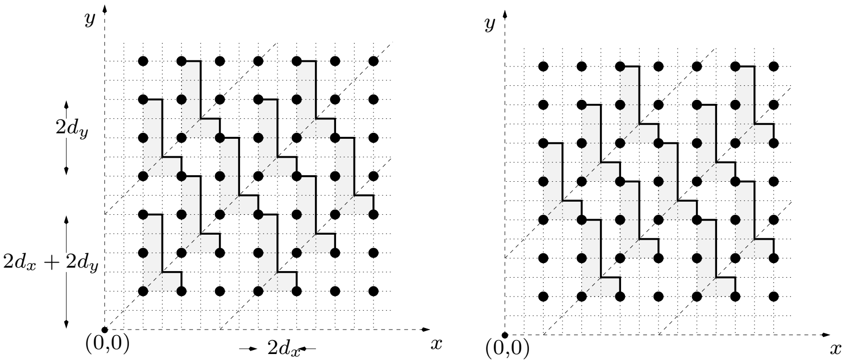


Figure 7: The edge sets $E^0_{1,2}$ and $E^2_{1,2}$.

**Proof:** We write the values of $\min\{d_x + d_y, 2r - d_x - d_y\}$ in the specified range as the following $r \times (r-1)$-rectangle:

$$
\begin{array}{r|ccccccccc}
d_y & & & & & & \multicolumn{4}{r}{\min\{d_x+d_y, 2r-d_x-d_y\}} \\
\text{r-1} & \text{r-1} & \text{r} & \text{r-1} & \text{r-2} & \cdots & \cdots & \cdots & 3 & 2 \\
\text{r-2} & \text{r-2} & \text{r-1} & \text{r} & \text{r-1} & \text{r-2} & & \cdot & & 3 \\
\cdot & \vdots & \text{r-2} & \text{r-1} & \text{r} & \text{r-1} & \text{r-2} & \cdot & & \vdots \\
\cdot & \vdots & & \text{r-2} & \text{r-1} & \text{r} & \text{r-1} & \text{r-2} & & \vdots \\
4 & \vdots & & \cdot & \text{r-2} & \text{r-1} & \text{r} & \text{r-1} & \text{r-2} & \cdot \\
3 & 3 & \cdot & & & \text{r-2} & \text{r-1} & \text{r} & \text{r-1} & \text{r-2} \\
2 & 2 & 3 & & & & \text{r-2} & \text{r-1} & \text{r} & \text{r-1} \\
1 & 1 & 2 & 3 & \cdots & \cdots & \cdots & \text{r-2} & \text{r-1} & \text{r} \\
\hline
 & 0 & 1 & 2 & \cdots & \cdots & \cdots & \text{r-3} & \text{r-2} & \text{r-1} \quad d_x
\end{array}
$$

The sum of the $r-1$ lower diagonals is $\sum_{k=1}^{r-1} k^2$. The sum of the $r-1$ upper diagonals is $\sum_{k=1}^{r-1} k(k+1)$. Hence the total sum is

$$
\begin{aligned}
\sum_{k=1}^{r-1} k^2 + \sum_{k=1}^{r-1} k(k+1) &= 2\sum_{k=1}^{r-1} k^2 + \sum_{k=1}^{r-1} k = \frac{(r-1)r(2r-1)}{3} + \frac{r(r-1)}{2} \\
&= \frac{(r-1)r(4r-2+3)}{6} = \frac{4r^3 - 3r^2 - r}{6} < \frac{2}{3}r^3.
\end{aligned}
$$

□

**Theorem 5** *For a given $n$, let $r = \lceil\sqrt{n}\,\rceil$ and let $N = r^2$. Then $K_n$ can be drawn in a $2r \times 2r \times \frac{4}{3}r^3$-grid with $\frac{3}{2}N^2 - \frac{15}{2}N + 6\sqrt{N}$ bends and at most three bends per edge.*

**Proof:** Draw $K_N$ as described above, ignoring the $N-n$ vertices not belonging to $K_n$, and their incident edges. The volume bounds follow directly from the construction.

Every edge has three bends, except the $2r(r-1) = 2N - 2\sqrt{N}$ edges where $d_x = 0$ and $d_y = 1$, or $d_x = 1$ and $d_y = 0$, which can be drawn without a bend. So the total number of bends is $3(N^2/2 - N/2) - 3(2N - 2\sqrt{N}) = \frac{3}{2}N^2 - \frac{15}{2}N + 6\sqrt{N}$. □

**Remark:** Since $r = \lceil \sqrt{n}\, \rceil < \sqrt{n}+1$, we have $N \leq n+2\sqrt{n}$, so our construction has a volume of $\frac{16}{3}N^{2.5} = \frac{16}{3}n^{2.5} + O(n^{2.25})$.

# 4 Conclusions

This paper is one of the first to address volume and bend considerations for 3-D orthogonal grid drawings of graphs. The focus has been on $K_n$, since it is the most difficult graph on $n$ vertices to draw in small volume or with restrictions on bends. In particular, we have

- provided a method for drawing $K_n$ with volume that is provably within a constant factor (same constant for all $n$) of best possible in the case that at most $k$ bends per edge are allowed, where $k \geq 3$;
- proved a lower bound of $\Omega(n^{2.5})$ and an upper bound of $O(n^3)$ on the volume of drawings of $K_n$ when $k = 1$ and $k = 2$;
- proved a lower bound of $\Omega(n^2)$ on the number of bends, which is matched by our constructions.

An open problem is to close the gap between the upper and lower bounds in the $k = 1$ and $k = 2$ cases, where at most 1 and at most 2 bends on each edge are permitted, respectively. Another interesting problem is to find upper and lower bounds that depend not only on the number of vertices $n$ but also on the number of edges $m$.

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