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Drawing Graphs on Two and Three Lines

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Abstract

We give a linear-time algorithm to decide whether a graph has a planar LL-drawing, i.e., a planar drawing on two parallel lines. We utilize this result to obtain planar drawings on three lines for a generalization of bipartite graphs, also in linear time.

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1 Introduction

Let $G = (A \cup B, E)$ be a *partitioned graph*, i.e., a graph with a partition of its vertex set into two disjoint sets A and B. We will refer to the vertices of A as A-vertices and to the vertices of B as B-vertices, respectively. The question of drawing partitioned graphs on parallel lines arises from drawing bipartite graphs, i.e., partitioned graphs such that A and B are independent sets. A natural way to draw such graphs is to draw all vertices of A on one – say horizontal – line, all vertices of B on a parallel line, and all edges as straight-line segments between their end-vertices. Such a drawing will be denoted by *BA-drawing*.

If G is planar, a drawing without edge crossings would be desirable. Harary and Schwenk [7] and Eades et al. [4] showed that a bipartite graph G has a planar BA-drawing if and only if G is a caterpillar, i.e., a tree such that the set of all vertices of degree larger than one induces a path.

To obtain planar drawings of a larger class of bipartite graphs, Fößmeier and Kaufmann [6] proposed *BAB-drawings*. Again, every edge is a straight-line segment between its end-vertices and all vertices of A are drawn on one horizontal line, but the vertices of B may be drawn on two parallel lines – one above the A-vertices and one below. Fößmeier and Kaufmann [6] gave a linear-time algorithm to test whether a bipartite graph has a planar BAB-drawing. An example of a BAB-drawing of a general graph can be found in Fig. 1. Another generalization of planar BA-drawings of bipartite graphs are planar drawings of leveled graphs which are considered by Jünger et al. [8].

Planar drawings for non-bipartite partitioned graphs are considered by Biedl et al. [1, 2]. A complete characterization of graphs that have a planar BA-drawing is given in [1]. Felsner et al. [5] considered line-drawings of unpartitioned graphs. They gave a linear time algorithm that decides whether a tree has a planar straight line drawing on a fixed number of lines. A fixed-parameter approach for the problem whether an arbitrary graph has a straight-line drawing on a fixed number of lines with a fixed maximum number of crossings was given in [3].

We will show how to decide in linear time whether an arbitrary graph has a planar *LL-drawing*, i.e., a straight line drawing on two parallel lines without edge crossings. Examples for planar LL-drawings can be found, e.g., in Fig. 5 and 7. Our algorithm works even if the end-vertices of some edges are constrained to be on different lines. For planar BAB-drawings, we relax the condition of bipartiteness to partitioned graphs with the only constraint that the neighbor of each vertex of B with degree one is in A. Actually, even this restriction is only necessary in very special cases, which are discussed in Section 2.2. Note that LL-drawings of unpartitioned graphs are the special case of BAB-drawings with $A = \emptyset$.

This paper is organized as follows. In Section 2, we consider planar BAB-drawings. First, we decompose the input graph such that the A-vertices of each component induce a path. We then show that we can substitute each of these components by a graph that contains only B-vertices, but simulates the possible planar BAB-drawings of the component. Finally, in Section 3, we show how to decide whether an unpartitioned graph has a planar drawing on two parallel lines.

2 BAB-Drawings

Let $G = (A \cup B, E)$ be a partitioned graph such that every B-vertex of degree one is adjacent to an A-vertex. Since we are interested in straight-line drawings, we assume that G is simple. As an intermediate step for the construction, however, we also make use of *parallel edges*: We say that two edges are parallel, if they are incident to the same two vertices. Let $n := |A \cup B|$ be the number of vertices of G. Since we want to test planarity, we can reject every graph with more than $3n-6$ edges. So we can assume that the number of edges is linear in the number of vertices. For a subset $S \subseteq A \cup B$ we denote by $G(S)$ the graph that is induced by S. If $H = (S, E')$ is a subgraph of G, we denote by $G - H$ the subgraph of G that is induced by $(A \cup B) \setminus S$.

2.1 Decomposition

If G has a planar BAB-drawing, the connected components of $G(A)$ have to be paths. Therefore, the vertex set of a connected component of $G(A)$ will be called an *A-path* of G. By $\mathcal{P}(A)$, we denote the set of A-paths. $\mathcal{P}(A)$ can be determined in linear time. Next, we want to decompose G into components. A *subdivision B-path between two vertices* b_1 *and* b_k is a set $\{b_2, \dots, b_{k-1}\} \subseteq B$ such that

- $\{b_i, b_{i+1}\} \in E$ for $i = 1 \dots k-1$ and
- degree $b_i = 2$ for $i = 2 \dots k-1$.

For an A-path $P \in \mathcal{P}(A)$, the *A-path component* G_P is the graph induced by the union of the following sets of vertices

- set P,
- set B_P of all B-vertices that are incident to P,
- all *subdivision B-paths* between two vertices of B_P that are not *subdivision B-paths* between two vertices of $B_{P'}$ for any $P' \in \mathcal{P}(A) \setminus \{P\}$.

Edges that would be contained in several A-path components are omitted in any of these components. Figure 1 shows a graph with three A-path components. A vertex of an A-path component G_P that is adjacent to a vertex of $G - G_P$ is called a *connection vertex* of G_P. Given a BAB-drawing of G, we call the first and last vertex of G_P on each of the three lines a *terminal* of G_P. By the restriction on B-vertices of degree one, the following lemma is immediate.

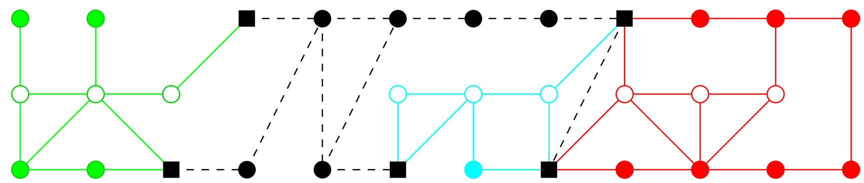

Figure 1: Decomposition of a graph into three A-path components. White vertices are A-vertices, all other vertices are B-vertices. Rectangularly shaped vertices are connection vertices. Dashed edges are not contained in any of the three A-path components.

Lemma 1 *Let P be an A-path of G.*

1. *All connection vertices of G_P are in B_P.*
2. *In any planar BAB-drawing of G, every connection vertex of G_P is a terminal of G_P.*
3. *If G has a planar BAB-drawing, G_P has at most four connection vertices.*

Let $P, P' \in \mathcal{P}(A)$ be distinct and $b_1, b_2 \in B_P \cap B_{P'}$ be two connection vertices of both G_P and $G_{P'}$. Then b_1 and b_2 are drawn on different lines in any planar BAB-drawing. In this case, we add edge $\{b_1, b_2\}$ to G. We will refer to such an edge as a *reminder edge*. Note that by adding the reminder edges, we might create parallel edges.

Lemma 2 1. *The sum of the sizes of all A-path components is in $\mathcal{O}(n)$.*

2. *There is a linear-time algorithm that either computes all A-path components and reminder edges or returns an A-path component that has more than four connection vertices.*

Proof:

1. By definition, each edge is contained in at most one A-path component. The number of A-path components is at most $|A|$ and each A-path component is connected. Thus the sum of the number of vertices in all A-path components is at most $|E| + |A| \in \mathcal{O}(n)$.
2. First, each subdivision B-path is substituted by a single edge between its two end-vertices and all sets B_P are computed. This can be done in linear time, e.g. by depth first search. Then for each $P \in \mathcal{P}(A)$, we examine the incident edges e of all $b \in B_P$. If both end vertices of e are contained in $B_P \cup P$, we add e to G_P, if not, we classify b as a connection vertex and stop the examination of this vertex b. This guarantees that for each vertex, at most one edge that is not in G_P is touched. If the number of connection vertices of G_P is greater than 4, we can return G_P, else we

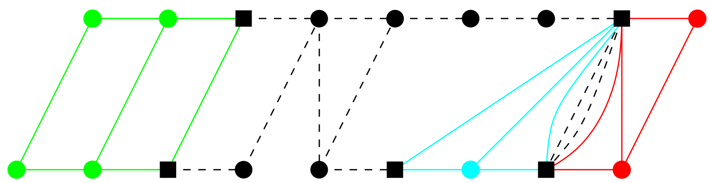

Figure 2: The graph G' constructed form the graph G in Fig. 1 by inserting the reminder edges and substituting all A-path components

add a 'pre-reminder' edge labeled G_P between all six pairs of connection vertices of G_P to G.

In a final walk through the adjacency list of each connection vertex, we can use the pre-reminder edges to determine the reminder edges and those edges of G between connection vertices that have to be added to an A-path component. Finally, subdivision vertices are reinserted into the replacement edges. □

2.2 Substitution

In this section, we show how to substitute an A-path component G_P by a graph that contains only B-vertices. There will be a one-to-one correspondence between the connection vertices of G_P and a subset of the vertex set of its substitute. The vertices of the substitute that correspond to connection vertices of G_P will also be called connection vertices. The substitute of G_P will be constructed such that it simulates all possible planar BAB-drawings of G_P in which the connection vertices are terminals. In the remainder of this section, a BA-drawing, a BAB-drawing, or an LL-drawing of an A-path component or its substitute requires always that the connection vertices are terminals.

Having found a suitable substitute H for G_P the substitution process works as follows. Delete all vertices of G_P but the connection vertices of G_P from G. Insert H into G, identifying the connection vertices of G_P with the corresponding vertices of H. An example of the graph resulting from the graph in Fig. 1 by inserting the reminder edges and substituting all A-path components can be found in Fig. 2.

We say that two terminals of G_P are on the *same side* of G_P in a BAB-drawing if they are both first or both last vertices on their lines. They are on *different sides* if one is a first and the other one a last vertex. Essentially, there are six questions of interest:

#: How many connection vertices are contained in G_P?

η: Does the number of connection-vertices equal the number of B-vertices of G_P?

τ: Does G_P have a planar BA-drawing?

τ_v: For each connection vertex v of G_P, is there a planar BAB-drawing of G_P that induces a BA-drawing of $G_P - v$?

σ_{vw}: For each pair v, w of connection-vertices of G_P, is there a planar BAB-drawing of G_P, such that v and w are on the same side of G_P?

δ_{vw}: For each pair v, w of connection-vertices of G_P, is there a planar BAB-drawing of G_P, such that v and w are on different sides of G_P?

Note that τ implies τ_v and that τ_v implies σ_{vw} and δ_{vw} for any pair v, w of connection vertices of G_P. Thus, provided that there exists some planar BAB-drawing of G_P, these six questions lead to the cases listed in Table 1. Note that there is one case with two parallel edges.

We say that an edge in an LL-drawing is *vertical* if the end vertices of e are drawn on different lines. Recall that an LL-drawing is the special case of a BAB-drawing in which the set of A-vertices is empty. Note also that in case $A = \emptyset$ a BA-drawing is an LL-drawing in which all vertices are drawn on a single line. Finally, for the next lemma, we allow parallel edges in an LL-drawing. We require, however, that they have to be vertical. A close look to Table 1 immediately shows that the planar LL-drawings of the substitutes simulate the possible BAB-drawings of the corresponding A-path components in the following sense.

Lemma 3 *Let Q be one of the decision questions above. Then the answer to Q is yes for an A-path component if and only if the answer is yes for its substitute.*

Let G' be the graph constructed from G by substituting each A-path component in the way described above. Let G_L be the graph obtained from G' by deleting all parallel edges. Each remaining edge in G_L that had a parallel edge in G' will also be called a reminder edge. Further, for each substitute that is not biconnected[1] and for each pair v,w of its connection vertices that is not yet connected by an edge, insert a reminder edge $\{v, w\}$ into G_L, if there is a subdivision path of length at least 2 between v and w in G_L.

Lemma 4 *1. The size of G' is linear in the size of G.*

2. G_L has a planar LL-drawing with every reminder edge drawn vertically if and only if G has a planar BAB-drawing.

Proof: Item 1 follows immediately from the fact that every A-path component contains at least one A-vertex and is replaced by at most 9 vertices. It remains to show Item 2.

"$\Leftarrow$": Suppose we have a planar BAB-drawing. By our general assumption on B-vertices of degree one, for any two vertices v and w of G_P all vertices that are drawn between v and w do also belong to G_P. Thus, by Lemma 3, we can construct a planar LL-drawing of G_L by replacing each A-path component G_P by its substitute.

[1] A graph is biconnected, if deleting any vertex does not disconnect the graph.

<table>
<tr><th colspan="4">1 connection vertex</th></tr>
<tr><th>η</th><th>τ</th><th>τ_v</th><th></th></tr>
<tr><td>+</td><td>⊕</td><td>⊕</td><td>v</td></tr>
<tr><td></td><td>+</td><td>⊕</td><td>v – •</td></tr>
<tr><td></td><td></td><td>+</td><td>v
| \
• – •</td></tr>
<tr><td></td><td></td><td></td><td>v – •
| / |
• – •</td></tr>
</table>

<table>
<tr><th colspan="4">4 connection vertices</th></tr>
<tr><th>σ_{vx}</th><th>σ_{vy}</th><th>σ_{vz}</th><th></th></tr>
<tr><td>+</td><td>+</td><td>+</td><td>v – •   • – y
\ /
w – • – • – • – x</td></tr>
<tr><td>+</td><td>+</td><td></td><td>v – • – • – x
| |
y – • – • – z</td></tr>
<tr><td></td><td>+</td><td></td><td>v – • – x
| | |
y – • – z</td></tr>
</table>

<table>
<tr><th colspan="7">2 connection vertices</th></tr>
<tr><th>η</th><th>τ</th><th>τ_v</th><th>τ_w</th><th>σ_{vw}</th><th>δ_{vw}</th><th></th></tr>
<tr><td>+</td><td>+</td><td>⊕</td><td>⊕</td><td>⊕</td><td>⊕</td><td>v – w</td></tr>
<tr><td>+</td><td></td><td>⊕</td><td>⊕</td><td>⊕</td><td>⊕</td><td>v
‖
w</td></tr>
<tr><td></td><td>+</td><td>⊕</td><td>⊕</td><td>⊕</td><td>⊕</td><td>v – • – w</td></tr>
<tr><td></td><td></td><td>+</td><td>+</td><td>⊕</td><td>⊕</td><td>v – w
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<tr><td></td><td></td><td>+</td><td></td><td>⊕</td><td>⊕</td><td>v – •
| \ |
w – •</td></tr>
<tr><td></td><td></td><td></td><td></td><td>+</td><td>+</td><td>v – • – •
| |
w – • – •</td></tr>
<tr><td></td><td></td><td></td><td></td><td>+</td><td></td><td>v – • – •
| | |
w – • – •</td></tr>
<tr><td></td><td></td><td></td><td></td><td></td><td>+</td><td>v – • – •
| | |
• – • – w</td></tr>
</table>

<table>
<tr><th colspan="7">3 connection vertices</th></tr>
<tr><th>τ_v</th><th>τ_w</th><th>τ_x</th><th>σ_{vw}</th><th>σ_{vx}</th><th>σ_{wx}</th><th></th></tr>
<tr><td>+</td><td>+</td><td>+</td><td>⊕</td><td>⊕</td><td>⊕</td><td>v – w
| /
x</td></tr>
<tr><td>+</td><td>+</td><td></td><td>⊕</td><td>⊕</td><td>⊕</td><td>v – • – x
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w</td></tr>
<tr><td></td><td></td><td>+</td><td>+</td><td>⊕</td><td>⊕</td><td>v – • – • – w
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x</td></tr>
<tr><td></td><td></td><td>+</td><td></td><td>⊕</td><td>⊕</td><td>x – v
| \ |
w – •</td></tr>
<tr><td></td><td></td><td></td><td>+</td><td>+</td><td>+</td><td>v – •   • – •
\ /
w – • – • – • – x</td></tr>
<tr><td></td><td></td><td></td><td></td><td>+</td><td>+</td><td>v – • – • – •
| |
x – • – • – w</td></tr>
<tr><td></td><td></td><td></td><td>+</td><td></td><td></td><td>v – • – •
| | |
w – • – x</td></tr>
</table>

Table 1: Substitutes for the A-path components. Cases that correspond up to the names of the connection vertices are omitted. A + means that the corresponding question is answered by yes. A ⊕ means that the answer to this property is implied. No entry means that the according property is not fulfilled.

As mentioned before, pairs of connection vertices, that are contained in several A-path components are drawn on different lines in a planar BAB-drawing of G. Parallel edges occur in two cases: they are edges from different substitutes and hence, between a pair of connection vertices that are contained in at least two A-path components. Or they are edges of the substitute with two parallel edges. In either case, the end vertices of these edges are on different lines.

Suppose now that H is a non-biconnected substitute and that there is a subdivision path Q in $G_L - H$ between two connection vertices v and w of H. If H is not drawn on one line, then the whole path Q has to be drawn on the same side of H. Else $H = v{-}\bullet{-}w$ and the LL-drawing can be changed such that v and w are on different lines.

Hence, there is a planar LL-drawing of G_L in which all reminder edges are vertical.

"⇒": Suppose now that there exists a planar LL-drawing with the indicated properties. We first show that we may assume that the drawing of each substitute H of an A-path component G_P fulfills the following two properties.

1. The connection-vertices of H are all terminals of H.
2. The convex hull of H contains no edge that does not belong to H and no edge that was parallel to an edge not belonging to H – except if it is incident to two connection vertices that are drawn on the same side of H.

Let H be a substitute. The above properties are immediately clear for H if H is biconnected – we enforce Condition 2 by requiring that reminder edges have to be vertical.

Now, let H be one of the substitutes that is not biconnected. We show that it is possible to change the LL-drawing of G_L such that it remains planar but fulfills Condition 1+2. Let v and w be connection vertices of H that are either not both terminals of H or that are not drawn on the same side of H. We show that there is no path in $G_L - H$ between v and w. This is immediately clear if v and w are on different lines in the planar LL-drawing.

So suppose v and w are on the same line and that there is a path P between v and w in $G_L - H$. Since there is a path in H between v and w that does not contain any other connection vertex of H, it follows from the existence of a planar LL-drawing that P is a subdivision path. Hence, by construction $\{v, w\}$ is an edge of G_L. Hence there cannot be another path between v and w.

Hence, the convex hull of a substitute H can only contain edges of H and subdivision paths between a connection vertex of H and a vertex of degree one. It is illustrated in Fig. 3 how to achieve Condition 1+2 for

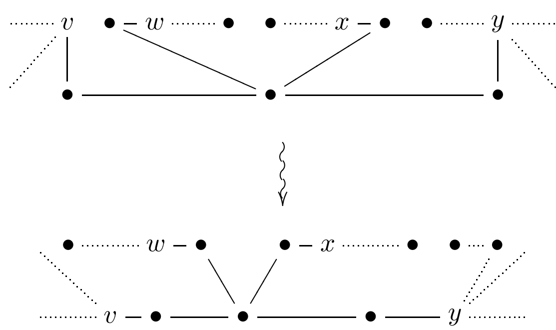

Figure 3: How to move the rest of the graph out of the convex hull of a substitute in a planar LL-drawing of G_L. Parts of G_L that are not contained in the substitute are indicated by dotted lines.

the substitute of an A-path component in which each pair among four connection vertices can be on the same side in a planar BAB-drawing. Other cases can be handled similarly.

Now, by Lemma 3, G_P can be reinserted into the drawing. □

Note that the precondition on B-vertices of degree one was only needed for "$\Leftarrow$". Suppose it was not guaranteed. Then there might be also a subdivision B-path S between a B-vertex of degree one and a vertex $b \in B_P$ for an A-path P, which is drawn between b and another vertex of G_P. If b is only contained in G_P and subdivision B-paths between b and a B-vertex of degree one, we can add S to G_P, but if b was also contained in other components, b would be a connection vertex that is not necessarily a terminal and there need not exist a suitable substitute any more. There is no problem, if there are at least three subdivision paths between b and vertices of degree one, since then at least one of them has to be drawn on the other line then b and then any of them can be drawn there, thus neither has to be added to G_P.

Lemma 5 *For each component, the six questions that determine the substitute, can be answered in time linear in the size of the component.*

Proof: It is obvious how to decide Properties # and η. To decide Properties σ_{vw} and δ_{vw}, suppose, we have an A-path component G_P with an A-path P : $a_1, \ldots, a_k$. For technical reasons, we add vertices a_0 and a_{k+1}, one at each end, to P.

Let W be the set of vertices of a connected component of $G_P(B)$. Suppose first that we wouldn't allow edges between B-vertices on different lines. In this case $G(W)$ has to be a path and the vertices in W have to occur in the same order as their adjacent vertices in P. Furthermore, let a_ℓ (a_r) be the A-vertex with lowest (highest) index that is adjacent to W. Then we say that the edges $\{a_i, a_{i+1}\}, i = \ell, \ldots, r-1$ are *occupied* by $G(W)$. Finally, if a vertex b in $G(W)$

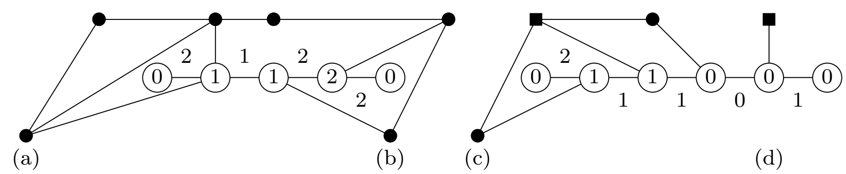

Figure 4: Three reasons for changing a line: a) a chord, b) a change in the order of adjacent A-vertices, and c) a connection vertex. d) A connection vertex occupies a part of the A-path. The numbers indicate how many times an edge or a vertex in the A-path is occupied.

is a connection vertex of G_P then b has to be a terminal. Thus, if a_ℓ is an A-vertex with the lowest (highest) index that is adjacent to b, then the edges $\{a_i, a_{i+1}\}, i = 0, \ldots, \ell$ (or $i = \ell, \ldots, k+1$) are also occupied by $G(W)$.

In general, if G_P has a planar BAB-drawing, $G(W)$ might be a path with at most two chords and the A-vertices adjacent to this path may change at most twice between increasing and decreasing order. Thus, there are three reasons for changing the line in W: a chord, a change in the order of the adjacent A-vertices, or a connection vertex. Note also that such a change of lines might occur in at most two connected components of $G_P(B)$. If there is a line change between b_1 and b_2 in W, then similar to the case of connection vertices above, $G(W)$ also occupies the edges between the adjacent A-vertices of b_1 and b_2 and one end of the A-path P. Note that in case of a line change some edges in $G(P)$ might be occupied twice.

We label every edge and every vertex in $G(P)$ with the number of times it is occupied as indicated in Fig. 4, where the number of times a vertex $v \in P$ is *occupied* by $G(W)$ is defined as follows. v is occupied k times by $G(W)$, if both adjacent edges in the A-path are occupied k-times by W and v is not adjacent to an end vertex of W and $k-1$ times if it is adjacent to such an end vertex. It is also occupied $k-1$-times if one adjacent edge is occupied k-times and the other one $k-1$ times. There exists a planar BAB-drawing with the corresponding choice of the direction of the connection vertices, if and only if

1. in total, every edge in $G(P)$ is occupied at most twice and
2. each vertex $v \in P$ that is adjacent to an end vertex of a connected component of $G_P(B)$ is occupied at most once.

This test works in linear time, since we can reject the graph, as soon as an edge gets a label higher than 2 and because there are at most four connection vertices and at most two components that change lines for which different orientations have to be checked. A similar approach can be used to answer questions τ and τ_v. □

The results in the next section show how to test the conditions of Lemma 4(2), which completes the algorithm for deciding whether a planar BAB-drawing exists.

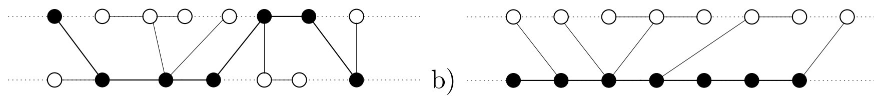

Figure 5: LL-drawings of a tree. The spine is drawn black.

3 LL-Drawings

Given a graph $G = (V, E)$ and a set of edges $U \subseteq E$, we show how to decide in linear time whether there exists a planar LL-drawing of G with the property that each edge in U is drawn vertically. See also [9] for the construction of planar LL-drawings. We will first discuss trees and biconnected graphs. Then we will decompose the graph to solve the general case. For an easier discussion, we assume that an LL-drawing is a drawing on two horizontal lines.

3.1 LL-Drawings of Trees and Biconnected Graphs

First, we consider trees. Felsner et al. [5] characterized trees that have a planar LL-drawing and called them strip-drawable. We give a slightly different characterization.

Lemma 6 *A tree T has a planar LL-drawing, if and only if it contains a* spine*, i.e., a path S such that $T - S$ is a collection of paths.*

Proof:

"⇒": The unique path between a leftmost and a rightmost vertex of T in a planar LL-drawing is a spine of T. See Fig. 5a for an illustration.

"⇐": A planar LL-drawing can be achieved by placing the spine on one of the lines and the path components in the order of their adjacency to the spine on the other line. See Fig. 5b for an illustration. □

As indicated in [5], the inclusion minimal spine $S_{\min}$ can be computed in linear time. If a vertex v is required to be an end vertex of a spine, we only have to check if there is a path between v and an end vertex of $S_{\min}$ in $T - S_{\min}$. We will use this fact when we examine the general case. Finally, the following lemma characterizes whether the edges in a set $U \subseteq E$ can be drawn vertically.

Lemma 7 *A set U of edges of a tree T can all be drawn vertically in a planar LL-drawing of T if and only if there exists a spine S that contains at least one end vertex of each edge in U.*

Recall that a graph is biconnected if deleting any vertex does not disconnect the graph. Before we characterize biconnected graphs that have a planar LL-drawing, we make the following observation.

Lemma 8 *Every graph that has a planar LL-drawing is outer planar.*

Proof: Since all vertices are placed on two – say horizontal – lines, every vertex is either a top most or bottom most vertex and is thus incident to the outer face. □

For general graphs, the existence of a spine is still necessary for the existence of a planar LL-drawing, but it is not sufficient. For example, see the graph of Fig. 6, which is the smallest outer planar graph that has no planar LL-drawing. For a biconnected graph, a planar LL-drawing can be constructed if and only if the inner faces induce a path in the dual graph.

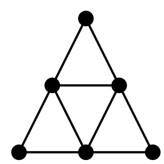

Figure 6: The smallest outer planar graph that has no planar LL-drawing.

Theorem 1 *A biconnected graph G has a planar LL-drawing if and only if*

1. *G is outer planar and*
2. *the set of inner faces induces a path in the dual graph*[2] *of an outer planar embedding of G.*

Proof: Recall that only outer planar graphs can have a planar LL-drawing. Thus, let G be an outer planar biconnected graph. We assume without loss of generality that G has at least two inner faces. Let G^* be the dual of the outer planar embedding of G, and let $G^* - f_o$ be the subgraph of G^* that results from deleting the outer face f_o. In general, $G^* - f_o$ is a tree. We have to show that G has a planar LL-drawing if and only if $G^* - f_o$ is a path.

"⇒": Consider the faces of G according to a planar LL-drawing of G. Then the boundary of each face contains exactly two vertical edges. Hence every vertex in $G^* - f_o$ has at most degree 2. Thus, the tree $G^* - f_o$ is a path.

"⇐": Let $G^* - f_o$ be a path and let f_1, f_2 be the vertices of $G^* - f_o$ with degree one. Chose two edges e_1, e_2 of G such that $e_i, i = 1, 2$ is incident to f_i and f_o. Deleting e_1 and e_2 from the boundary cycle of f_o results into two paths P_1 and P_2. A planar LL-drawing of G can be obtained as follows. Draw the boundary cycle of f_o such that P_1 is drawn on one line and P_2 is drawn on the other line. Then all vertices of G are drawn. Since $G^* - f_o$

[2]The dual graph G^* of a planar graph G with a fixed planar embedding is defined as follows. The vertices of G^* are the faces of G. For each edge e of G the dual graph G^* contains an edge that is incident to the same faces as e.

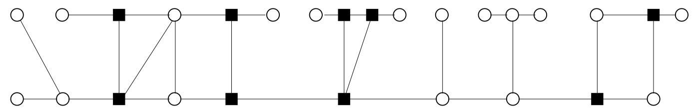

Figure 7: A graph with a planar LL-drawing. Connection vertices are solid.

is a path and by the choice of e_1 and e_2, non of the edges that is only incident to inner faces is incident to two vertices on the same line. Hence all remaining edges are vertical edges. Since G is outer planar, no two vertical edges cross. Hence the construction yields a planar LL-drawing of G. □

Corollary 1 *A set U of edges of a biconnected graph G can all be drawn vertically in a planar LL-drawing of G if and only if*

1. *G is outer planar,*
2. *the set of inner faces induces a path P in the dual graph of an outer planar embedding of G, and*
3. *the dual edges of U are all on a simple cycle including P.*

3.2 LL-Drawings of General Graphs

To test whether a general graph has a planar LL-drawing, we first split the graph into components like paths, trees and biconnected components. We give necessary conditions and show how to test them in linear time. Finally, by constructing a drawing, we show that these necessary conditions are also sufficient.

Suppose now without loss of generality that G is a connected graph. In this section, a vertex v is called a *connection vertex*, if it is contained in a cycle and its removal disconnects G. Figure 7 shows the connection vertices in an example graph. We denote the set of all connection vertices by V_c. The connection vertices can be determined with depth first search in linear time.

A subgraph L of G is called a *single line component*, if it is maximal with the property that L is an induced path and there exists a vertex $v \in V_c$ such that L is a connected component of $G - v$. By $\bar{L}$, we denote a single line component L including its incident connection vertex. If $\bar{L}$ is a path we call it a *strict single line component*, otherwise we call it a *fan.* Figure 8 illustrates different single line components.

Simply testing for all $v \in V_c$, whether the connected components of $G(V \setminus \{v\})$ are paths leads to a quadratic time algorithm for finding the single line components. But we can use outer-planarity to do it in linear time. Note that any single line component contains at most two connection vertices (see Fig. 8 for illustration). Thus, we can find them, by storing the three last visited connection vertices on a walk around the outer face and testing, whether the

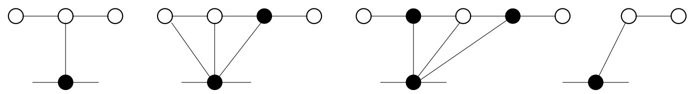

Figure 8: Fans and a strict single line component. Connection vertices are solid.

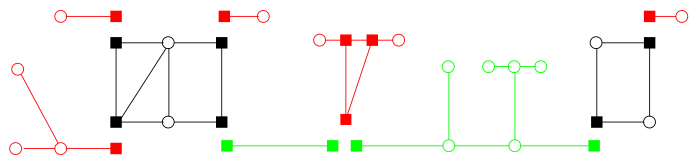

Figure 9: Components of the graph in Fig. 7. There are two trees, two two-lined biconnected components, two fans, and three strictly single lined components.

vertices between the first and second occurrence of the same connection vertex induce a path.

Let G' be the subgraph that results from G by deleting all single line components. The *two-lined components* of G are the components that we get by splitting G' at the connection vertices of G. Two-lined components that do not contain a cycle will be called *tree components.* All other two-lined components will be denoted by *two-lined biconnected components.* Figure 9 illustrates the different components of the graph in Fig. 7.

We have now defined all components that we need to characterize those graphs that have a planar LL-drawing. Before we formulate the characterization for the general case, note that G' is a tree if the graph does not contain two-lined biconnected components. Indeed, the characterization in this case is similar to the situation in which the graph is a tree.

Remark 1 *A graph G that does not contain any two-lined biconnected component has a planar LL-drawing if and only if it contains a spine.*

Now, consider the set $\mathcal{L}$ of all two-lined components. Let $\mathcal{P}$ be the graph with vertex set $\mathcal{L}$ in which two two-lined components are adjacent if and only if they share a connection vertex. Suppose a planar LL-drawing for G is given. Then all two-lined biconnected components require at least two lines. All tree components also require at least two lines or are connected to two components that require two lines. Thus $\mathcal{P}$ has to be a path. We will refer to this property by saying that *the two-lined components induce a path.* For the same reason three two-lined components cannot share a vertex.

Theorem 2 *A graph G has a planar LL-drawing if and only if*

1. *G is outer planar,*
2. *the two-lined components induce a path,*

3. *each tree component T has a spine S such that the connection vertices of T are end vertices of S, and*

4. *for each two-lined biconnected component B there is a drawing with the following properties.*

   (a) *Connection vertices are leftmost or rightmost vertices of B on their line.*

   (b) *At most one two-lined component is connected to each vertex of B.*

   (c) *If a two-lined component or a fan is connected to a vertex of B, then a vertex on the same side is at most connected to a strict single line component.*

In the rest of this section, we show that the conditions in the above theorem are necessary. Sufficiency will be shown in the next section by demonstrating how to find a planar LL-drawing of G in linear time. We will also discuss the conditions for vertical edges in the next section.

So, suppose that the graph G has a planar LL-drawing. The necessity of Conditions 1-3 and 4b follows from Section 3.1 and the observations mentioned above. Clearly, no component can be connected to a two-lined biconnected component, if not to a leftmost or rightmost vertex. Hence, it follows that also Condition 4a is necessary.

To prove the necessity of Condition 4c, let B be a two-lined biconnected component. Let C_1, C_2 be two components that are connected to different vertices on the same side of B. Let v be the common vertex of B and C_1. First we observe that C_2 cannot be connected to C_1 nor to any component C_3 that contains a vertex $w \neq v$ of C_1. Else there would be a cycle in G that contains edges of C_1 and C_2 – contradicting the fact that they are different components. Hence, if C_1 cannot be drawn on a single line, it follows immediately that C_2 can only be a strict single line component. Suppose now that C_1 is a tree component that can be drawn on a single line. Then there is a component C_3 that has no drawing on a single line, such that C_1 and C_3 share a vertex $w \neq v$. Hence, it follows again that C_2 can only be a strict single line component.

3.2.1 Drawing and Sufficient Conditions.

In this subsection we sketch how to construct in linear time a planar LL-drawing if the conditions in Theorem 2 are fulfilled. We also mention which edges can be drawn vertically. Since a linear order on the two-lined components is given, we only have to show how to draw each component separately.

It was already shown in Section 3.1 how to find a spine and that a tree with a spine has a planar LL-drawing. For drawing two-lined biconnected components, first note that a biconnected graph has a unique outer planar embedding. Starting with a connection vertex on any line, we add the other vertices in the order of their appearance around the outer face and switch lines only if necessary, i.e., at a connection vertex or a chord. We do this procedure for each direction around the outer face. Those cases in which lines are switched at most twice, are the possible outer planar drawings of a two-lined biconnected component, with respect to Theorem 2. Hence, if a drawing exists, it can be found in linear time. Vertical edges in two-lined biconnected components only yield another condition for switching lines.

Let L be a single line component and let L be incident to the connection vertex v. Let v be contained in the two-lined component B. If a two-lined component or a fan is connected to another connection vertex on the same side of B then $\overline{L}$ is a strict single line component and no edge of $\overline{L}$ can be vertical. Else all edges between v and L can be vertical edges.

If no two-lined component is connected to the side of B that contains v, then among all single line components that are connected to v there is one that may have additional vertical edges. If it is a fan, also the edges indicated by dotted lines in Fig. 10 can be vertical. Note, however, that only one of the edges e_1 and e_2 can be vertical. If the single line component is strict, all edges can be vertical.

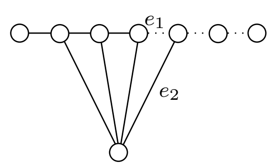

Figure 10: Edges that can be vertical.

4 Conclusion

We showed how to decide in linear time, whether an arbitrary graph has a planar drawing on two parallel lines, even if the input contains edges that have to be drawn vertically. We applied this result to decide in linear time whether a partitioned graph $G = (A \cup B, E)$ with the property that every B-vertex of degree one is adjacent to an A-vertex has a planar BAB-drawing. The algorithm worked in three steps. First, the graph was decomposed into A-path components. Then, each of these components was substituted by a graph that contains only B-vertices, but simulates the possible positions of the connection vertices. Finally, we test whether the resulting graph has a planar LL-drawing. We discussed that the restriction on the vertices of degree one is only needed in the following sense: If b is a connection vertex of an A-path component, the number of subdivision B-paths between b and vertices of degree one may not be one or two.

Acknowledgments

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