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The Japanese Contributions to Martingales

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1 Probability theory in Japan before 1960: Itô's work on stochastic analysis

Modern probability theory, as founded and developed by distinguished pioneers such as A. N. Kolmogorov, P. Lévy, N. Wiener, and so on, attracted great interest and attention from Japanese mathematicians, including K. Yosida, S. Kakutani, K. Itô and G. Maruyama, and others. Around 1935, Kiyosi Itô (1915–2008), then a student at the University of Tokyo, found Kolmogorov's recently published book, "Grundbegriffe der Wahrscheinlichkeitsrechnung" one day in a bookstore. As he often recollected in later years,[2] this fortuitous discovery of Kolmogorov's book gave him one of his motivations for devoting his future life to the study of probability theory.

Although the study of modern probability theory in Japan certainly started before 1940, the war disrupted communications with other advanced countries. Under these circumstances, Itô completed two important contributions ([I 1], [I 2]) that are now considered the origin of *Itô's stochastic analysis* or *Itô's stochastic calculus*. In the first work, he gave a rigorous proof of what is now called the *Lévy-Itô theorem* for the structure of sample functions of Lévy processes, through which we have a complete understanding of the Lévy-Khinchin formula for canonical forms of infinitely divisible distributions. In the second work, he developed a complete theory of stochastic differential equations determining sample functions of diffusion processes whose laws are described by Kolmogorov's differential equations. In this work, he introduced the important notion of a *stochastic integral* and the basic formula now known as *Itô's formula* or *Itô's lemma* and thus founded

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[2]Cf. e.g. [I 9] in the page next to Preface.

a kind of Newton-Leibniz differential and integral calculus for a class of random functions now often called *Itô processes*. As we will see below, this work by Itô was further developed and refined in the martingale framework. Itô himself did not say anything about martingale theory in this work; the theory of martingales began to be noticed in Japan only after the publication of Doob's book ([Do]) in 1953.

Itô published two books in Japanese on modern probability theory, [I 3] in 1944 and [I 6] in 1953, in which he introduced the modern theory of probability to the mathematical community in Japan. He thus provided beginning Japanese researchers and students in this field with excellent textbooks. He incorporated his results in [I 1] and [I 2] into these books, especially into [I 6], the more advanced of the two. Although the term *martingale* cannot be found in the books, some fundamental ideas and techniques of martingale theory are implicit in them. In particular, von Mises's important ideas of "Stellenauswahl" (selection) from a random sequence and "Regellosigkeit" (irregularity) are explained in Sections 26 and 27 of [I 3] and in Section 17 of [I 6]. In modern martingale theory, selection is a typical example of a *martingale transform*, a discrete-time version of the stochastic integral (for martingale transforms, cf. e.g., [Wi], p. 97 and [Ik-W], p. 25). Such ideas and notions are used to obtain an extension of the famous *Kolmogorov maximal inequality* for sums of independent random variables, which plays an important role when we define stochastic integrals. As we know, Doob extended this inequality in his martingale theory, so that it is now well known as the Kolmogorov-Doob maximal inequality.

After the war ended and international communications began to recover, Itô sent an expanded and refined English version of [I 2] to Doob, asking him to help with its publication in United States. Doob was kind enough to arrange its publication in *Memoirs of AMS* in 1951 ([I 4]). When Doob's book ([Do]) appeared in 1953, Itô was impressed by its beautiful theory of martingales and was glad to see its treatment of his stochastic integrals in the martingale framework.[3] But Itô was given a chance to visit the Institute for Advanced Study at Princeton University from 1954 to 1956, and during that period and for some time thereafter, his main effort was devoted to the study of one-dimensional diffusion processes jointly with Henry P. McKean.[4]

At Princeton, W. Feller had already almost completed his theory of the analytical description of one-dimensional diffusion processes. At his suggestion, Itô began to work with McKean, who was then a student of Feller's, on a pathwise theoretic construction of one-dimensional diffusion processes.

Apparently this work was done independently of Doob's martingale theory,[5] but in fact it has much to do with martingale theories: the Itô-McKean

[3]Cf. [I-Sel], p. xv.

[4]Cf. [I-Sel], p. xv, [I 10], p. 2.

[5]The situation at the time was similar in all work on Markov processes, perhaps with the exception of Doob's work. The same objects were given different names in each theory; for example, *stopping times* were called *Markov times* in Markov process theory. Since then,

construction of sample paths of one-dimensional diffusions makes a *random time change* in the paths of a one-dimensional Wiener process (Brownian motion), while, in Doob's theory, a time change is formulated as an *optional sampling*, and, indeed, Doob's optional sampling theorem plays a key role in his theory of martingales. Also, Itô-McKean's time change is based on Brownian local time. Later, a general notion of local time was established by the French school (cf. e.g., [RY], p. 206 and [RW], p. 96) in the martingale framework, and it plays an important role in the stochastic calculus of the random functions called semimartingales.

2 Japanese contributions to martingales from 1961 to 1970

Itô returned to Kyoto from Princeton in 1956. His and McKean's joint work continued at Kyoto University for several years; there McKean gave a series of lectures that stimulated much younger researchers (T. Hida, N. Ikeda, M. Motoo, M. Nisio, H. Tanaka, T. Ueno, T. Watanabe,...) as well as graduate students (M. Fukushima, H. Kunita, K. Sato, S. Watanabe, T. Yamada,...). Many were interested in the problem of extending the theory of Feller and Itô-McKean from one-dimensional diffusions to multidimensional cases, particularly the problem of diffusion processes with Wentzell's boundary conditions. World-wide, modern probability theory had been developing from the pioneering works by Kolmogorov, Lévy and Wiener and others. In the theory of Markov processes, the most advanced countries around 1960 were the United States and the Soviet Union. The main themes were Markov processes and related problems in analysis, potential theory in particular, and functionals of sample functions such as additive and multiplicative functionals, as studied by W. Feller, S. Kakutani, J. L. Doob, M. Kac, G. A. Hunt, and many others in United States, E. B. Dynkin and his group in Moscow, A. B. Skorohod and his group in Kiev, and so on.

The theory of martingales became known very gradually at the time, mostly from the work of Doob and the influence of his book [Do]. It was recognized as useful because of the (sub-,super-) martingale convergence theorems and the theorems on the existence of regular modifications of sample functions of (sub-,super-) martingales in continuous time. I personally came to know of it for the first time in Khinchin's paper [Kh] treating McMillan's theorem in information theory by Doob's martingale convergence theorem. In Markov process theory, the existence of a nice Markov process (Hunt process) has usually been based on the existence of regular modifications for sample functions of (sub-,super-) martingales in continuous time.

Doob applied his martingale theory to the study of Markov processes and potentials in an essential way. One of his typical ideas was the following: if $u(x)$ is a harmonic or a sub(super)harmonic function in a domain D of $\mathbf{R}^d$, and, if $B(t)$ is a d-dimensional Brownian motion (i.e., Wiener process) starting from a point in D, then the stochastic process $t \in [0, \tau_D) \mapsto u(B(t))$,

the two theories have gradually mixed together, bringing remarkable progress to each.

where τ_D is the first exit time from D of $B(t)$, is a continuous (local) martingale (resp. sub(super)martingale). So, for example, if $u(x)$ is a positive superharmonic function, then the process $t \mapsto u(B(t))$ has bounded and continuous sample functions with probability one, because of a well known result of Doob on sample paths of positive supermartingales. So we can say, for example, that Brownian motion behaves so as to avoid any discontinuity or any point at which a positive superharmonic function assumes an infinite value.

Gradually it began to be understood that there is a deep interplay between Markov process theory and the martingale theory: Many important results in Markov processes, formulated in a more abstract and general framework in the martingale theory, may be regarded as basic and abstract results and principles, so that the original results in Markov processes are just typical applications in a more concrete or special situation. The following subsections are all concerned, more or less, with this kind of progress in martingale theory.

2.1 The Doob-Meyer decomposition theorem for supermartingales

The theory of Markov processes and potentials has advanced a great deal. In particular, G. A. Hunt developed a very general theory of *excessive functions* for a given Markov process in a restricted but reasonably general and convenient class (these Markov processes are now known as *Hunt processes*). In Moscow, Dynkin emphasized[6] the importance of the study of additive and multiplicative functionals of Markov processes in connection with potential theory. (Actually, some of its importance had already been demonstrated in Itô and McKean's work.) P. A. Meyer, who originally studied potential theory in the famous French school guided by Brelot, Choquet and Deny, made a deep study of additive functionals (AFs) of a Hunt process ([Me 1]) from a potential theoretic point of view. His results, viewed in a martingale framework, could be understood as giving an abstract and general principle in stochastic processes. It is indeed a realization of Doob's idea that a submartingale should be a sum of a martingale and a process with increasing sample paths, so that a supermartingale should have a representation as a martingale minus an increasing process. Meyer rewrote his results on AFs of a Hunt process in a framework of martingale theory ([Me 2], [Me 3]) and thus obtained a general result concerning the representation of a supermartingale as a difference of a martingale and an increasing process. This is now called the *Doob-Meyer decomposition of supermartingales*, which is certainly one of the most basic and important results in martingale theory.

K. Itô and S. Watanabe ([It-W]) studied, in contrast with the *additive* decomposition of Doob-Meyer, the *multiplicative* decomposition of a positive supermartingale into a product of a positive martingale and a positive

[6]Cf. Introduction of [Dy], which was originally his plenary lecture at ICM 1962, Stockholm.

decreasing process. This problem originated in the study of multiplicative functionals (MFs) in connection with transformations of Markov processes, a problem much studied at that time. This paper introduced the notion of *local martingales*, which is now a basic tool in localization arguments in martingale theory.

We would mention here some relevant important developments, around that time, in the study on *positive martingales*. C. Doléans-Dade ([D-D], 1970) obtained a general expression of so-called *exponential martingales*. I. V. Girsanov ([G], 1960) (with later generalization and refinement by P. A. Meyer and others of the French school) established the *Girsanov theorem* which is concerned with the transformation of the martingale character under an equivalent change of the underlying probability.[7] These cannot be considered Japanese contributions, but G. Maruyama ([Ma], 1954) and M. Motoo ([Mo], 1960–61) had already studied, in their works on diffusion processes associated with Kolmogorov differential equations, important examples of exponential martingales and the Girsanov transformations they define, even though they did not state their results in terms of martingale theory.

2.2 Stochastic integrals for square-integrable martingales and semimartingales

Stochastic integrals were first introduced by K. Itô ([I 2]) in 1942. J. L. Doob ([Do]) pointed out the martingale character of stochastic integrals and suggested that a unified theory of stochastic integrals should be established in a framework of martingale theory. His program was accomplished by H. Kunita and S. Watanabe ([Ku-W]) and P. A. Meyer ([Me 5]), among others.

I would like to comment on these works in more detail. Here again, they have their origin in the theory of Markov processes, particularly in the work of M. Motoo and S. Watanbe ([MW] and [Wa]) on square-integrable additive functionals (AFs) of a Hunt process having zero expectations.[8] A main aim of the work in [MW] and [Wa] was to study the structure of the space $\mathbf{M}$ formed by the square-integrable AFs having zero expectations, particularly to understand and generalize a result of A. D. Wentzell ([We]) in the case of Brownian motion; if $X(t) = (X_1(t), \ldots, X_d(t))$ is a d-dimensional Brownian motion, the space $\mathbf{M}$ consists of AF $A(t)$ represented in the form $A(t) = \sum_{i=1}^{d} \int_0^t f_i(X(s))dX_i(t)$ as a sum of Itô's stochastic integrals.[9] In this study, a fundamental role is played by a *random inner product* $\langle M, N \rangle$, $M, N \in \mathbf{M}$, which is defined to be a continuous AF with almost all sample paths locally of bounded variation. Using this random inner product, important and useful notions such as *stochastic integrals*, *stable subspaces*, *orthogonality* and *projection* of subspaces in $\mathbf{M}$, *basis* of a subspace, and so on, can be introduced and studied. The orthogonal complement $\mathbf{M}_d$ of the subspace

[7] Cf. e.g., [P], p. 109.

[8] For an AF, it is equivalent that it have zero expectation and that it be a martingale with respect to the natural filtration of the process.

[9] f_i are Borel functions on $\mathbf{R}^d$ with certain integrability conditions.

$\mathbf{M}_c$ formed of all continuous elements in $\mathbf{M}$ was studied in [Wa]. There, a random point process was defined by jumps of sample paths of the Hunt process and its *compensator*, called the *Lévy measure* of the Hunt process, was introduced and studied.

During a period around 1963, H. Kunita and I conceived the idea of extending results in [MW] and [Wa] to a more general and abstract situation in which the natural filtration associated with the Hunt process is replaced by a general filtration and an AF is replaced by a general càdlàg adapted process. By the Doob-Meyer decomposition theorem, we can still define the random inner product $\langle M, N\rangle$ for square-integrable martingales M and N.[10] Stochastic integrals with respect to a square-integrable martingale can be characterized by this random inner product and can be constructed along the lines of Itô and Doob.

In this period, I was visiting Paris as a scholarship student (*boursier*) of the French Government. Very fortunately, I had an opportunity to attend a lecture on the decomposition of supermartingales by Meyer at the Collège de France, just before he moved from Paris to Strasbourg. After the lecture, he kindly invited me to his home and we exchanged information on our current work.

Thus, the works [Ku-W] and [Me 5], which finally appeared in the same year, are very much related; indeed, as Meyer kindly stated in [Me 5], his work was motivated by the work [Ku-W]. If we review the work in [Ku-W] now from the standpoint of martingale theory, it should be said that, as far as discontinuous stochastic processes are concerned, it is rather restricted and incomplete in many points. As we know, a mathematically complete and satisfactory theory was established by Meyer and his French (Strasbourg) school (cf. e.g., [DM], [JS], [P] as important texts treating the theory), and [Me 5] was a starting point for this French contribution.

The class of stochastic processes introduced in Itô's original paper [I 2] (now often called Itô processes) can be naturally extended to a class of stochastic processes called *semimartingales*.[11] *Itô's formula* or *Itô's lemma* leads to a kind of Newton-Leibniz differential and integral calculus for semimartingales.

For a semimartingale, we have a decomposition of a sample function as the sum of a continuous semimartingale and a discontinuous semimartingale. Roughly speaking, a process is a discontinuous semimartingale if its sample function can be obtained as a *compensated sum* of jumps. The continuous part is a sum of a (locally) square-integrable continuous martingale and a

[10] A standard terminology now is *predictable quadratic co-variation* of M and N. Meyer [Me 5] introduced another random inner product $[M, N]$, called the *quadratic co-variation* of M and N, which plays important role in the study of discontinuous semimartingales.

[11] The term *semimartingale* and its notion were introduced by Meyer ([Me 5]). Note that this terminology is used differently in Doob's book ([Do]); there, the term semimartingale is used to mean submartingale, and the term *lower semi-martingale* to mean supermartingale.

continuous process with sample functions (locally) of bounded variation. So a semimartingale has sample functions similar to those of a Lévy process. We can associate with a semimartingale a system of quantities which correspond, in the case of a Lévy process, to its Lévy-Khinchin characteristic: What corresponds to the covariance of the Gaussian component in the Lévy process is the predictable quadratic co-variation of the continuous martingale part of the semimartingale. What corresponds to the Lévy measure of the Lévy process is the compensator of a point process defined by the size of jumps of sample paths of the semimartingale.[12]

Actually, Lévy processes are a particular case of semimartingales. Indeed, it is the most fundamental case, in which the associated characteristic quantities are deterministic (i.e., non-random). In particular, a d-dimensional Wiener process $X(t)$ is characterized as a d-dimensional continuous martingale $X(t) = (X_1(t), \ldots, X_d(t))$ with the predictable quadratic co-variation satisfying the condition $\langle X_i, X_j \rangle(t) = \delta_{i,j} t,\ i,j = 1, \ldots, d$. In [Do], this characterization of the Wiener process in the frame of martingale theory is attributed to P. Lévy. Also, there is a similar martingale characterization theorem for a Poisson process ([Wa]) and Poisson point processes (cf. e.g. [Ik-W], [JS]).

Thus, we can see that Itô's works on Lévy processes in [I 1] and on stochastic integrals and Itô processes in [I 2] have grown into a unified general theory of semimartingales. In this framework, many important stochastic models can be defined and constructed by appealing to the theory of stochastic differential equations or the method of martingale problems.

2.3 Martingale representation theorems

In the case of a Wiener process, the martingale representation theorem[13] states that *every local martingale with respect to the natural filtration of a Wiener process can be expressed as the sum of a constant and a stochastic integral of a predictable integrand $f(s)$ with $\int_0^t f(s)^2 ds < \infty$ for every t, a.s.* As mentioned above, this kind of representation theorem first appeared in Wentzell's study of AFs, and its extension to general Hunt processes has been a main motivation of our work in [MW]. Its further extension to the case of general square-integrable martingales motivated our work in [Ku-W]. In [Ku-W], we presented several useful results for the representation of martingales. However, the notion of a *basis* in the sense of [MW] could not be stated explicitly. Later, this notion was completely established by M. H. A. Davis

[12]The notion of the compensator for a point process is key in the martingale theoretic approach to point processes. Indeed, it has much to do with semimartingale theory; the discontinuities of a semimartingale define a point process on the real line and, conversely, a point process on a general state space defines a discontinuous semimartingale by a projection of the state space to the real line. Cf. e.g. [Ik-W], [JS], [Ka-W], for the martingale-theoretic approach to point processes and applications.

[13]We state it in the one-dimensional case; its multi-dimensional extension is straightforward.

and P. Varaiya ([DV]).

The martingale representation theorem for a Wiener process as stated above has played an important role in financial mathematics. In this field, this theorem is very well known as *Itô's representation theorem.*[14] Indeed, this is because an essential part of the proof of this theorem is to prove that every square-integrable functional F of Wiener process paths $\{w(t); 0 \leq t \leq T\}$ can be represented as $F = E(F) + \int_0^T f(s)dw(s)$ by Itô's stochastic integral. Such a representation can be obtained, as Itô remarked on page 168 (Th. 5.1) of [I 5], by expanding F into an orthogonal sum of multiple Wiener integrals and then rewriting the multiple Wiener integrals as iterated Itô stochastic integrals.

3 Japanese contributions to martingales after 1971

During this period, stochastic analysis based on semimartingales was developed and used around the world. It became one of the most important and useful methods in probability theory and its applications. Many standard textbooks, including [Ik-W], [KS], [RW], [RY], and [JS], treated stochastic analysis based on semimartingales and martingale methods. Here, I review some work in this period in which we can find some Japanese contributions.

3.1 Fisk-Stratonovich symmetric stochastic integrals. Itô's circle operation

K. Itô ([I 7]), based on the general results in [Ku-W] and [Me 5], reformulated the stochastic calculus in terms of stochastic differentials. This put Itô's formula in a form convenient for applications. The fact that Itô's formula needs extra terms as compared with the standard Newton-Leibniz rule is most interesting and mysterious in the stochastic calculus; it might be a surprise for beginners. This causes a difficulty when we want to apply the stochastic calculus for stochastic processes moving on a differentiable manifold. The process given in each local coordinate is a semimartingale but the rule of the calculus is not a usual one, so that some difficulty always arises when we want to obtain coordinate-free notions and results. For a typical example, see a very troublesome construction of a solution of stochastic differential equations on a manifold in [Mc].

On the other hand, Stratonovich ([Stra]) and Fisk ([F]) introduced a type of stochastic integral (sometimes called a *symmetric stochastic integral*) different from Itô's. Itô noticed that this kind of stochastic integral can be immediately defined by modifying Itô integrals; for two continuous semimartingales X and Y, the symmetric stochastic integral of X by Y, denoted as $\int X \circ dY$, is, by definition, a continuous semimartingale Z given by

$$Z(t) = (\int_0^t X(s) \circ dY(s)) := \int_0^t X(s)dY(s) + \frac{1}{2}\langle M^X, M^Y\rangle(t),$$

[14]Cf. D. W. Stroock's interesting remark on page 180 of [Stro].

where the first term on the right-hand side is Itô's stochastic integral and the second term is the predictable quadratic co-variation of the martingale parts M^X and M^Y of X and Y, respectively. Itô wrote this in stochastic-differential form as $X \circ dY = XdY + \frac{1}{2}dXdY$. This operation on stochastic differentials is often called *Itô's circle operation*. Under this new operation, we have the same rule of transformations as that of Newton-Leibniz in the ordinary differential calculus. In other words, under Itô's circle operation Itô's formula has the same form as in ordinary differential calculus. As it has turned out in many later works, this circle operation is an indispensable tool in the study of random motions on manifolds, producing many fruitful results (cf. e.g., [Ik-W], [RW]).

3.2 Itô-Tanaka's formula and local times

In the Itô-McKean theory, the local time of Brownian motion (i.e., the Brownian local time) plays a fundamental role. The notion of Brownian local time was first introduced by Lévy, and a rigorous and precise result on its existence as a sojourn time density and its continuity on the space variable was obtained by H. F. Trotter ([T], [IM]). However, Trotter's paper was rather hard to follow, at least for beginners.

Around 1962, H. Tanaka was visiting McKean at MIT and he sent a letter to his friends in Japan communicating a nice and much simpler proof of Trotter's theorem. His idea is to use Itô's calculus, particularly Itô's stochastic integral, in an essential way. Tanaka's proof was reproduced in McKean's book ([Mc]) and then spread widely.

An essential point in Tanaka's proof was an extension of Itô's formula. Itô's formula is concerned with a transformation of a semimartingale by $\mathcal{C}^2$-functions: If $f(x)$ is a $\mathcal{C}^2$-function and $X(t)$ is a continuous semimartingale, then $f(X(t))$ is also a continuous semimartingale and Itô's formula describes its semimartingale decomposition precisely. If $f(x)$ now is only a convex function, or a difference of two convex functions, it is still true that $f(X(t))$ is a continuous semimartingale. In its semimartingale decomposition, the continuous martingale part has the same form as in the case of $f(x)$ being a $\mathcal{C}^2$-function; it is given by the stochastic integral $\int_0^t f'(X(s))dM^X(s)$ with respect to the continuous martingale part M^X of X. Then all terms in this semimartingale decomposition except the part of the process of bounded variation can be known explicitly, so that this part has a representation as a difference of other terms which are known explicitly. Brownian local time at $a \in \mathbf{R}$ is obtained in this way when $X(t)$ is a one-dimensional Wiener process and $f(x)$ is the convex function given by $f(x) = \max\{x - a, 0\}$. This formula representing Brownian local time is called *Tanaka's formula*.

In a similar way, we can define the local time for every continuous semimartingale. This notion was established around the last half of the seventies by members of the French school, including Meyer, Azema, and Yor (cf. e.g. [RY], [RW]). Le Gall ([L]) obtained a nice application of this theory to the pathwise uniqueness problem for one-dimensional stochastic differential equations. Itô's formula for continuous semimartingales on $\mathcal{C}^2$-functions can be extended to functions that are differences of two convex functions in which the part of process of bounded variation can be expressed by an integral of local times. Such a formula is often called an *Itô-Tanaka formula*. Thus, we may say that the theory of local times for semimartingales is an important French contribution motivated by a Japanese contribution.

An important idea for extending Itô's formula beyond the Itô-Tanaka formula was given by M. Fukushima in his theory of Dirichlet forms and symmetric Markov processes associated with them ([FOT]). He introduced a class of stochastic processes *with zero energy* and extended Itô's formula using this notion. The notions of semimartigales and semimartingale decomposition, in this case, are thereby extended; the decomposition is now known as the *Fukushima decomposition* and is playing an important role in path-theoretic studies in symmetric Markov processes. In the case of a one-dimensional Wiener process, as such an important process as Brownian local time is defined by the Itô-Tanaka formula, many new important processes can be obtained through the Fukushima decomposition: a typical example is the *Cauchy principal value of Brownian local time*, introduced and studied by T. Yamada ([T]) and M. Yor ([Yo]), among others.

3.3 Problems concerning filtrations

The martingale theory is usually developed by fixing a filtration to which the martingale property is referred. So it is very important to see how changing the filtration affects the theory. Among many important problems of this kind, Th. Jeulin and M. Yor, among others, established a theory concerning an enlargement of filtrations ([JY]). This is certainly a French contribution, but as Yor has often pointed out, his original motivation for this study was work by K. Itô ([I 8]). In this paper, Itô discussed how to give meaning to a class of stochastic integrals by a Wiener process in which the integrands are not adapted to the natural filtration of the Wiener process.

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