Commentary on the notes for Paul Lévy's 1919 lectures on the probability calculus at the Ecole Polytechnique

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INTRODUCTION

After the terrible experience of the First World War [12, 11], French academic institutions faced organizational difficulties. This was true in particular for the Ecole Polytechnique, whose students had all been mobilized and sent to the front. The Ecole Polytechnique was part of France's military (and still is today, though with a special status). Its students (the so-called *polytechnicians*) generally served in the artillery. The situation was worse for civilian superior schools, such the Ecole Normale Supérieure, for their students, sent into the infantry, had been subjected to outright slaughter from the very outset of the war (see [15] for the example of René Gateaux). But the polytechnicians who were still students in 1914 had been deprived of several years of graduate education, and no new students had been recruited in 1915 and 1916. So urgent measures were required to keep the school functioning in the post war years. The number of students admitted to the Polytechnique in 1919 and 1920 was doubled by special competitions, officially called *spéciales*, which were reserved to students who had gone to the front. Other academic competitions were similarly expanded; for example, there was a special recruitment at the *Agrégation*, the competition for the opportunity to teach at the secondary level.

Paul Lévy (1886–1971) had entered the Polytechnique as a student in 1904, and he had been appointed as *répétiteur* (tutor) there in 1913. A tutor's task was to help the students to understand the professors' lectures, and hence Lévy was well acquainted with the curriculum and the methods of instruction at the Polytechnique before his own appointment as professor.

Lévy was one of the most prominent mathematicians of the 20th century, and we can hardly give a full picture of his imposing personality in this brief introduction. Let us merely recall that his mathematical career can be divided naturally into two periods, with the division falling just at the time when he became a professor the Ecole Polytechnique.

In the first period of his career, Lévy was a specialist in what was then called *functional analysis* but is probably better described by as the *functional calculus*. This study of analytical properties of functions of functions, introduced by Volterra at the end of the 19th century in

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order to model problems in mechanics, was taken up by Hadamard, who used it successfully in potential theory. After defending a brilliant thesis in 1911 on themes in potential analysis proposed by Hadamard, Lévy was considered the leading young specialist in the field. He then had the good fortune, at the very end of the war, to be given by Hadamard the task of editing papers left incomplete by Gateaux when he was killed in 1914 [15]. This mission was a real springboard for Lévy, who not only edited but considerably developed Gateaux's papers, first for the cours Peccot he was asked to teach at the Collège de France in 1919, and above all for his important book *Leçons d'Analyse Fonctionnelle*, published in 1922. To Lévy's disappointment, the book did not revive the functional calculus in France, and he almost abandoned the field.

Lévy is honored today mainly for the second period of his mathematical career - that is, for his huge achievements in probabilistic studies. The document we present here may be considered the birth of this meteoric second career, as it seems to be the first evidence of Lévy's interest in probability. But it would be a mistake to miss the important connections between Lévy's studies in functional calculus and his second mathematical career. In fact, an amazing reorientation of Lévy's mind seems to have taken place in these years. It became clear to him that several aspects of his work, in particular about infinite dimensional integration, found their best expression in a probabilistic context. Wiener studied Lévy's book on the functional calculus carefully and found there the tools for constructing the measure for Brownian motion. Wiener testified to this in 1923, paying tribute to Lévy for having explained to him how the construction could be achieved. Later, in his autobiography, Lévy himself noted his contribution to Wiener's work, remarking bitterly about how he missed the chance to formalize Brownian motion himself. It is fair to say that at the end of his life Lévy was somewhat inclined to moan about missed opportunities. To conclude this quick presentation of Lévy's personality, we may refer the reader to Lévy's letters to Fréchet [1], which give a vivid picture of the man and his achievements.

In 1919, one of the professors of mathematics at the Ecole Polytechnique was the number theorist Georges Humbert (1859–1921), who had taught calculus there starting in 1895. During the academic year 1918–1919, already severely affected by the disease he died from two years later, Humbert decided to entrust the *répétiteur* Paul Lévy with some parts of his teaching. Lévy was a natural choice to substitute for Humbert, because already in 1914 he had obtained an important position as professor of mathematics at the Ecole des Mines de Paris. The following year, Humbert's poor health finally forced him to resign, and the 31 year old Lévy was appointed full professor at the Ecole Polytechnique, a position he kept until his retirement - except for the bitter period of 1942–1944 when he was dismissed from his position for being Jewish, in accordance with the Vichy racial laws [1]. Years later, when Lévy was an unsuccessful candidate to succeed Elie Cartan in the Paris Academy of Science, the issue of his appointment as professor at the Polytechnique in 1920 unexpectedly surfaced in a quarrel between Julia and Lévy (see the letters on this topic from 1950 onward in [1]).

1. Lévy's 1919 lectures

As we have already said, Lévy's task at the Polytechnique in 1919 was only to give some of the lectures in the syllabus for the course on analysis. Among these were three lectures devoted to the probability calculus. It had long been usual for Polytechnique students, who were intended for professions close to the applied and experimental sciences, to learn some probability. The main point of interest was the law of errors, the major tool used to quantify the accuracy of experimental measures in sciences such as astronomy or ballistics. The war had put the emphasis

on ballistics, as the construction of error tables for artillery was an essential duty of the engineers involved in the trench warfare that had drug on after 1915 [7].

We may suppose, then, that Levy's lectures in 1919 were responding to a renewal of interest in lessons in probability at the school. In his autobiography [14], he mentions (p.71) that they were supposed to complete General Bourgeois' lectures.

This resulted in the document we present here, which was transcribed as Chapter VI of Georges Humbert's *Cours d'Analyse*. Lecture notes at the Ecole Polytechnique were traditionally reproduced by lithography from handwritten notes made by students, which had been transcribed by a scribe with good handwriting. This transcription was supposed to be reviewed by the professor in charge of the lectures. Lévy, when he was a student at the Polytechnique in 1906, had been asked by Poincaré to transcribe two lectures on probability, after another student had produced a transcription that Poincaré judged unsatisfactory. See Lévy's comments in [14], p. 69.

We do not know whether Lévy had reviewed the part of the text devoted to his lectures of 1918–19. It is separated from Humbert's part by a divider, with the title *Supplement to the lectures on calculus by M. Paul Lévy, substitute professor*. After the third page the handwriting changes, becoming less attractive. Perhaps the scribe has changed.

Chapter VI bears the title *Concepts of the probability calculus and the kinetic theory of gases*. As a result of Poincaré, Hadamard and Borel becoming interested in the theories of Maxwell, Boltzmann, and Gibbs, statistical mechanics had become the prototypical example of stochastic modeling, illustrating an acceptable practical application of probability theory. Borel in particular had devoted much energy to convincing mathematicians that it was possible to put statistical mechanics on a stable mathematical foundation: his epoch-making paper [3], completed and extended by his book [5], provide evidence of his effort. His celebrated popular book *Le Hasard* [6], also published in 1914, gives many insights into Borel's thoughts on these questions. Lévy's relationship with Borel with respect to probability theory became complicated in the 1920s. Lévy placed himself in Borel's footsteps, but by trying to extend Borel's results using more sophisticated mathematical methods, he put himself in opposition to Borel, who was oddly repelled by the idea of classing probability as a field of mathematics and did not think it should require deep mathematical notions such as characteristic functions. A rather bitter trace of this opposition between Lévy and Borel can be found in black and white in the introduction to the book on probability, mainly drawn from his lectures at the Polytechnique, that Lévy published in 1925 [13]. See [8] for more on this opposition between Borel and Lévy.

Anyone who has read Lévy knows that he had a very personal style, often difficult for anyone else to understand. His education at the Polytechnique, atypical at a time when most leading French mathematicians were educated at the Ecole Normale, gave him a taste for mathematics deeply tinted with physics and physical reasoning. This appears clearly in the correspondence with Fréchet [1], who was a typical product of the French mathematical school. It is also reflected by von Neumann's comment that *reading Lévy, one soon realizes that the author thinks mathematics in a different way from the rest of us*.[3] The notes for his lectures at the Polytechnique, though obviously at a rather elementary level, have several obscurities that confirm this rule. As often happened at the time (for instance in Poincaré's own probability textbook [16]), probability theory is presented as a mixture of general 'philosophical' ideas and particular methods for dealing with well defined situations (e.g. law of errors of measurement). This type of text helps us appreciate the unification and apparent simplification that Kolmogorov's axiomatization offered

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[3]Quoted by K.L.Chung in [1] (p. 9)

to the next generation. Cramér, in his 1937 treatise [10], took a position in favor of the use of Kolmogorov's axiomatization because of its simplicity. See also [18] and [17]. These thoughts raise, of course, the question of what Levy's students managed to understand from his lectures. The first author of this commentary remembers having met several people who had preserved vivid memories of Lévy's lectures on probability at the Ecole Polytechnique, but they may have been impressed more by the strength emanating from the passionate man than by the clarity of his explanations...

At the end of the text, Lévy gives only a few references for further reading. For probability theory, he merely refers to the treatises by Borel (he probably means Borel's 1909 book on probability [4], but he might mean *Le Hasard*), and Carvallo [9]. Emmanuel Carvallo (1856– 1945), director of studies at the Ecole Polytechnique in 1919, was a strong supporter of increasing the amount of probability in the curriculum. Lévy may not have read Carvallo's book, for as he himself admitted, he read little of the work of others. But he recalls in his autobiography ([14], p.71) that it was Carvallo, most probably on Humbert's recommendation, who had asked him to give the lectures at the Polytechnique.

2. CONTENTS

Chapter VI, the chapter devoted to probability, extends for 29 pages, from page 495 to page 523 of Humbert's *Cours d'Analyse*. It is divided into 24 sections introduced by numbers between parentheses. Stochastic modeling of the kinetic theory of gases occupies an important place, extending for 11 pages, from page 513 to page 523. As we have already noted, Lévy criticized the model proposed by Borel and wanted to replace it by one that he found more compatible with the physical hypotheses.

Let us now briefly describe the different sections of Chapter VI.

Section (1) presents the usual uniform probability situation, where the probability is defined as the quotient of the number of favorable cases over the total number of cases, which is correct when the different cases are equally probable. Lévy emphasizes in section (2) that the previous 'definition' is not a mathematical definition. On the contrary, the mathematical definition of probability attributes an arbitrary number to each event (obviously, the situation implies that there are a finite number of possible outcomes for the experiment). So from a mathematical viewpoint, if we attribute the value $\frac{1}{10}$ to the drawing of one ball among 10 balls, this is an arbitrary choice among all the possible ones. The task of the mathematical theory of probability is to compute the probability of events which depend on events for which the probability is given.

Section (3) reflects on the practical value of the probability calculus. For Lévy, this value can be only measured through statistics: it is when direct access to a detailed description is not easily possible that probabilistic information is unavoidable. Such is the case with social phenomena, provided the range of interpretation is well delimited. Lévy takes as an example a country's mortality rate, which is rather stable from one year to the next, though it may be changed by unusual events such as wars or epidemics. The examples resonated strongly in 1919, just following the end of the war and the terrible epidemic of Spanish flu, which had caused many more deaths than the war. In the short section (4), Lévy adopts Borel's subjective point of view on probability: the most important case is that of an event with a minute probability. From the practical point of view, one should act as if such an event is impossible. Section (5) gives a short description of what can be done in the continuous case. Lévy only considers probabilities defined by densities. He briefly exposits Bertrand's paradoxes concerning the choice of a chord

on a circle (without quoting Bertrand) in order to emphasize the need to specify a probability distribution precisely.

The long sequence of sections (6) to (16) is devoted to limit theorems - more precisely, to what we now call the central limit theorem. The formulation of the notion of convergence is generally informal: the results are said to be valid when n equals infinity.

Beginning with the simplest case, the game of heads and tails, Lévy proves de Moivre's theorem on the convergence of normalized means to the Gaussian distribution. De Moivre is not quoted, and the convergence is described, in the subsequent section (7) as *Bernoulli's law*. An elementary majoration of the Gaussian distribution function allows to conclude that almost certainly the gain will increase less rapidly than $\sqrt{n}$.

In section (8), Lévy generalizes the previous sections to the case of an asymetric game where at each step the gambler may obtain values a and b with probabilities λ and μ. He says that, with the same kind of calculations, one finds that the deviation of the gain from nE, where $E = a\lambda + b\mu$, follows a Gaussian distribution with standard deviation $\sqrt{\lambda\mu} \mid b - a \mid$.

Now comes in (9) the case when there are more than 2 possibilities, namely p alternatives $a_1, \ldots, a_p$ with probabilities $\lambda_1, \ldots, \lambda_p$. Lévy's idea is that it is possible to proceed by induction in order to reduce the game to a unique gain $E = a_1\lambda_1 + \cdots + a_p\lambda_p$, making at each step a Gaussian error by virtue of the previous section.

Suppose indeed that the gains may take the values $a_1, \ldots, a_p$ with probabilities $\lambda_1, \ldots, \lambda_p$. Applying the result from section (8), one may replace the original game with a game with gains

$$\frac{\lambda_1 a_1 + \lambda_2 a_2}{\lambda_1 + \lambda_2}, a_3, \ldots, a_p$$

obtained with probabilities $\lambda_1 + \lambda_2, \lambda_3, \ldots, \lambda_p$. From this results a Gaussian distributed error E_1. Then one may replace this last game with a game with gains

$$\frac{\lambda_1 a_1 + \lambda_2 a_2 + \lambda_3 a_3}{\lambda_1 + \lambda_2 + \lambda_3}, a_4, \ldots, a_p$$

obtained with probabilities $\lambda_1 + \lambda_2 + \lambda_3, \lambda_4 \ldots, \lambda_p$, committing a Gaussian distributed error E_2. And so on. At the end, one obtains a (deterministic) game with a single possible gain $\lambda_1 a_1 + \cdots + \lambda_p a_p$. The resultant error is $E_1 + \cdots + E_{p-1}$. Lévy asserts that these errors are independent, an assertion underlined in the text with the comment *point easy to check*. The Gaussian form of the resultant error, the desired extension of section (8), would follow from this independence. However, this assertion is clearly false: the independence is only asymptotic. Lévy repeats the same error the following year; the underlined sentence appears again in the notes for his 1920–21 lectures. No one knows whether someone asked Lévy for further explanations, but when he wrote his treatise [13], probably with the written versions of his lectures in hand, he saw the problem. On page 231 of [13], he writes *one may immediately conclude that the distribution of the sum [of errors] is Gaussian if the two terms of the sum are independent. But they are not*. And he gives a proof *à la Lévy* (but this time unquestionable!) of the asymptotic independence of the two terms.

In section (10), Lévy comments briefly on how the result (the central limit theorem) can be extended to the continuous case when the distribution of the gain has a density f with respect to Lebesgue measure (our formulation in modern terms, obviously). The idea is to approximate f by a step function to come back to the case with a finite number of possible gains, and that can

be done with an error as small as desired. The text emphasizes that this way of reasoning is only heuristic.

Sections (11) to (16) are devoted to the problem of convergence to the normal distribution when different distributions describe the different steps of the game. After the technical observation that subtracting the expectation makes the probability distributions of the games centered (section (11)), Lévy introduces the sequence of moments E_k of the different steps and the moments F_k of the result of the first k steps (section (12)). In section (13), Lévy introduces reasonable necessary conditions for a convergence of the normalized sequence. These conditions are the classical ones for the method of moments (see for example [2], pp. 407–410), namely that the size of the step-gains can be controlled by the variance of the resulting gain. Then he observes that the sequence of moments characterizes the distribution, by the Fourier inversion theorem, which he briefly quotes. According to the transcriber, Lévy said that he could not prove the theorem during his lectures (section (14)). Lévy does not use the expression 'characteristic function' but the moments are called 'characteristic coefficients'. In the two subsequent sections (15) and (16), he shows that the moments F_k tend to those of the Gaussian distribution. In the preface to his book [13], Lévy emphasized that the introduction of characteristic functions (*starting in my 1920 lectures at the Ecole Polytechnique*, he writes, p. VI) was his fundamental contribution - and, as we mentioned earlier, it was the source of Borel's skepticism about the importance of Lévy's work on probability. Reading the preface to [13], one cannot help thinking that Lévy wrote the book only to advertise the use of characteristic function in probability - and to oppose Borel.

Section (17) bears the title *Diverse applications*. It is an introduction to the final sections of the chapter, all devoted to the kinetic theory of gases. Lévy divides the application of the preceding theory in two categories. The first concerns the statistical description of deviations from the mean, the task of the law of errors. The second category arises when the number of events at stake is huge (microscopic scale), so that the application leads to a situation which can be considered deterministic (macroscopic scale). Kinetic theory is a perfect paradigm for such a situation: a macroscopic reversal is best thought of as an extremely improbable event, which we consider impossible in accordance with our earlier discussion of the practical value of probability.

This part on kinetic theory extending from section (18) to the end of the notes is probably the first indication of Lévy's mathematical proof of the Maxwell-Boltzmann laws, which later constituted Chapter VIII of his treatise [13]. Here also, in a way, Lévy settles a score with Borel, whom he never quotes in the lecture notes but duly quotes and contests with respect to the proof of Maxwell's laws in [13], p. 309.

In Borel's original proof of 1906, reproduced in [5], Borel assumes that shocks are small and independent. In section (20), Lévy contests this hypothesis as physically absurd. The speed of a particle with maximum (resp. minimum) speed after a shock cannot increase (resp. decrease), so that these shocks cannot be the same for all the particles. In the present lecture notes, Lévy only illustrates how it is possible to obtain the value of some physical coefficients, such as the pressure on the surface of the container or van der Waals's formula linking mean speed, pressure and volume, without Borel's hypothesis. It is only in subsequent years that Lévy would formulate precisely his ideas on the non-independence of the particles with regard to mutual shocks. The final presentation constitutes section 70 of [13] (p. 309), entitled *Justification of Maxwell's law*. It must be said that Lévy's book went unnoticed by the French mathematical community. Evidence for this is provided by Hadamard's amazement in 1928 when he discovered Lévy's exposition of

the ergodic convergence for card shuffling ([1], p. 141). In his autobiography ([14] p. 81), Lévy mentions his surprise when physicists told him that they understood Maxwell's law thanks to his proof.

3. Conclusion

Lévy's lecture notes at the Ecole Polytchnique in 1919 nicely document his beginnings as a probabilist. Though they are evidently not developed in the detail needed (we must not forget that Lévy took the charge of the teaching at the drop of a hat), they indicate how Lévy perceived probability on the eve of his exceptional second mathematical career. They contain the seeds of the studies Lévy would pursue with virtuosity for 40 years: convergence in distribution, sums of random variables, the connection between mathematical probability and physics. They testify to what Lévy said about himself 50 years later in an interview to the French radio *France Culture*, accessible online in the June 2006 issue of JEHPS (https:// www.jehps.net issue: 2/1): *I have the feeling of being a mathematician, neither superior nor inferior to others, but different from them.*

References

[1] M. Barbut, B. Locker and L. Mazliak: Paul Lévy–Maurice Fréchet, 50 ans de correspondance mathématique, Hermann, 2004
[2] P. Billingsley: Probability and Measure (3rd Ed) Wiley and sons, 1995
[3] E. Borel: Sur les principes de la théorie cinétique des gaz, Ann. Sci. ENS, 23, 1906, 9–32
[4] E. Borel: Eléments de la théorie des probabilités, Hermann, 1909
[5] E. Borel: Introduction géométrique à quelques théories physiques, Gauthier-Villars, 1914
[6] E. Borel: Le Hasard, Alcan, 1914
[7] B. Bru: Problème de l'efficacité du tir à l'école d'artillerie de Metz. Apects théoriques et expérimentaux, Math. Sci. Hum., 136, 1996, 29–42
[8] B. Bru: Borel, Lévy, Neyman, Pearson et les autres. Matapli, 60, 1999, 51–60
[9] E. Carvallo: *Le calcul des probabilités et ses applications*, Gauthier-Villars, Paris, 1912
[10] H. Cramér: *Random Variables and Probability Distributions*. University Press, Cambridge, 1937
[11] E. Crawford and J. Olff-Nathan (ed.): La science sous influence: l'Université de Strasbourg enjeu des conflits franco-allemands, Strasbourg, La Nuée bleue, 2005
[12] C. Goldstein and L. Mazliak (eds): Mathématique et Mathématiciens en France autour de la Première Guerre Mondiale, to appear in 2009
[13] P. Lévy: Calcul des Probabilités, Gauthier-Villars, 1925
[14] P. Lévy: Quelques éléments de la pensée d'un mathématicien, Blanchard, 1970
[15] L. Mazliak: les fantômes de l'Ecole Normale, 2008. to appear
[16] H. Poincaré: Calcul des Probabilités, Gauthier-Villars, 2ème Edition, 1912
[17] G. Shafer and V. Vovk (2005): The origins and legacy of Kolmogorov's *Grundbegriffe*. *Working paper on* www.probabilityandfinance.com. *Shorter version* Statistical Science, 2006
[18] J. Von Plato: Creating Modern Probability, Cambridge University Press, 1994