Journal of Convex Analysis
Volume 7 (2000), No. 2, 395–412

Bounded Linear Regularity, Strong CHIP, and CHIP are Distinct Properties

Heinz H. Bauschke*
Department of Mathematics and Statistics, Okanagan University College, Kelowna, British Columbia V1V 1V7, Canada.
e-mail: bauschke@cecm.sfu.ca

Jonathan M. Borwein†
Centre for Experimental and Constructive Mathematics, Department of Mathematics, Simon Fraser University, Burnaby, British Columbia V5A 1S6, Canada.
e-mail: jborwein@cecm.sfu.ca

Paul Tseng‡
Department of Mathematics, Box 354350, University of Washington, Seattle, Washington 98195-4350, USA.
e-mail: tseng@math.washington.edu

Received July 27, 1998
Revised manuscript received January 8, 2000

Bounded linear regularity, the strong conical hull intersection property (strong CHIP), and the conical hull intersection property (CHIP) are properties of a collection of finitely many closed convex intersecting sets in Euclidean space. It was shown recently that these properties are fundamental in several branches of convex optimization, including convex feasibility problems, error bounds, Fenchel duality, and constrained approximation. It was known that regularity implies strong CHIP, which in turn implies CHIP; moreover, the three properties always hold for *subspaces*. The question whether or not converse implications are true for general convex sets was open.

We show that – even for *convex cones* – the converse implications need not hold by constructing counterexamples in $\mathbb{R}^4$.

Keywords: Bounded linear regularity, linear regularity, normal cone, property CHIP, property strong CHIP, tangent cone

1991 Mathematics Subject Classification: Primary 90C25; Secondary 46C05, 52A15, 52A20

1. Introduction

Consider the following three properties for two given closed convex sets C_1 and C_2 with $C := C_1 \cap C_2 \neq \emptyset$ in some Euclidean space X.

"bounded linear regularity" For every bounded set S in X, there exists $\kappa_S > 0$ such that the *distances to the sets* C_1, C_2, and C are related by

$$d(x, C) \leq \kappa_S \max\{d(x, C_1), d(x, C_2)\}, \quad \text{for every } x \in S.$$

*This research was supported by an NSERC Postdoctoral Fellowship and by the Department of Combinatorics and Optimization at the University of Waterloo.
†Research supported by NSERC.
‡Research supported by the National Science Foundation Grant CCR-9731273.

ISSN 0944-6532 / $ 2.50

"strong CHIP" For every $x \in C$, the *normal cones* satisfy

$$N_C(x) = N_{C_1}(x) + N_{C_2}(x).$$

"CHIP" For every $x \in C$, the *tangent cones* satisfy

$$T_C(x) = T_{C_1}(x) \cap T_{C_2}(x).$$

These geometric properties are key in the study of various areas in convex optimization and bordering fields, such as: *Fenchel duality* [13, 12]; *duality for convex cones* [19]; *systems of convex inequalities and associated error bounds* [22, 23, 9]; *projection algorithms and their rate of convergence* [6, 8, 20]; *constrained interpolation* [15, 14]; *the angle between two subspaces* [11]; *conical open mapping theorems* [7].

It was well-known (see [9, Theorem 3], [23, Proposition 6], [24, Corollary 16.4.2]) that

$$\text{bounded linear regularity} \Rightarrow \text{strong CHIP} \Rightarrow \text{CHIP},$$

and that the three properties always hold when C_1 and C_2 are *subspaces* [9, Section 6]. It was left open whether any of the implications is reversible, even for *convex cones*. Moreover, in this conical setting, the three properties coincide if at least one of the following conditions is satisfied [9]:

- C is either a *singleton* or a *ray*;
- each cone C_i is "*locally smooth*" on $C \setminus \{0\}$;
- C is *linear*;
- each cone C_i is *polyhedral*;
- the *relative interiors* of C_1 and C_2 make a nonempty intersection.
- the space X has dimension 2.

Hence counter-examples are not easy to obtain.

The aim of this paper is to provide counter-examples distinguishing the properties bounded linear regularity, strong CHIP, and CHIP.

Specifically, we provide in Sections 3–5:

(i) two convex cones in $\mathbb{R}^4$ that have strong CHIP but are not boundedly linearly regular (Corollary 3.2);
(ii) two convex cones in $\mathbb{R}^4$ that have CHIP but not strong CHIP (Theorem 4.1);
(iii) the equivalence of CHIP, strong CHIP, and bounded linear regularity for two convex cones in $\mathbb{R}^3$ (Theorem 5.1); and
(iv) two convex sets in $\mathbb{R}^3$ that have CHIP but not strong CHIP (Corollary 4.2).

In view of (iii), the conical counter-examples are minimal (in terms of dimension of the underlying space). It should be mentioned here that two cones even in $\mathbb{R}^3$ may fail to have CHIP, see [9, Example 5]. To make the construction of the counter-examples more transparent, we modularize our work when possible and give a number of results, interesting in their own right, in Section 2.

The original version of this paper appeared as the research report "Metric regularity, strong CHIP, and CHIP are distinct properties," CORR 98-33, University of Waterloo, July, 1998. In September 1998, after the submission of the original version, A. Bakan informed us that he had shown in [2] the existence of an example as in (i) and had shown in [1] the equivalence of bounded linear regularity and a certain "Moreau-Rockafellar equality" for convex cones in $\mathbb{R}^3$. Since the latter equality was shown in [15] to be equivalent to strong CHIP, Bakan's result implies the equivalence of bounded linear regularity and strong CHIP for convex cones in $\mathbb{R}^3$ (see (iii)). Recently F. Deutsch informed us that, in a forthcoming paper [4], he and W. Li and Bakan give two convex sets in $\mathbb{R}^3$ with strong CHIP but without bounded linear regularity. They also give an alternative construction of example as in (i).

We conclude this section by fixing our notation. We adopt mostly standard notation from convex analysis as can be found in the classical [24] or the more recent [16, 17]. The nonnegative (resp. nonpositive) reals are denoted $\mathbb{R}_+$ (resp. $\mathbb{R}_-$). Suppose S is a set in a Euclidean space X. We write B_X for the *unit ball* $\{x \in X : \|x\| \leq 1\}$. The *relative interior* (resp. *closure*, *boundary*, *relative boundary*, *convex hull*, *convex conical hull*, *closed convex conical hull*, *orthogonal complement*) of S is denoted $\mathrm{ri}(S)$ (resp. $\mathrm{cl}(S)$, $\mathrm{bd}(S)$, $\mathrm{rbd}(S)$, $\mathrm{conv}(S)$, $\mathrm{cone}(S)$, $\overline{\mathrm{cone}}(S)$, $S^\perp$). If S is closed and convex, then S induces a *distance function* $d(x, S) := \inf\{\|x - s\| : s \in S\}$ and a corresponding *projection* $P_S(x) \in S$ by $\|x - P_S(x)\| = d(x, S)$. If S is nonempty and $\mathbb{R}_+ \cdot S = S$, then we say that S is a *cone*. The *adjoint* of a linear operator T is denoted by T^*.

Finally, to make the presentation less cluttered, we will loosely write expressions like "$\mathbb{R}^2 \times 0$" rather than "$\mathbb{R}^2 \times \{0\}$".

2. A tool box

Throughout this section, we assume that

$$\boxed{X \text{ is a Euclidean space with inner product } \langle \cdot, \cdot \rangle \text{ and norm } \|\cdot\|.}$$

Polar cones

Definition 2.1. Suppose S is a nonempty set in X. Then the *positive polar cone* (resp. *negative polar cone*) of S, written $S^\oplus$ (resp. $S^\ominus$), is defined by $\{y \in X : \langle y, S \rangle \geq 0\}$ (resp. $\{y \in X : \langle y, S \rangle \leq 0\}$).

Remark 2.2. It is easy to check that $S^\oplus$ is a closed convex cone and that $S^\ominus = -S^\oplus$.

Fact 2.3. *Suppose K and L are closed convex cones in X. Then:*

(i) $(K \cap L)^\oplus = \mathrm{cl}(K^\oplus + L^\oplus)$.
(ii) $(K + L)^\oplus = K^\oplus \cap L^\oplus$.

Proof. See [24, Corollary 16.4.2]. □

Proposition 2.4. *Suppose S is a closed cone in X and A is a nonsingular linear transformation of X. Then $(T(S))^\oplus = (T^*)^{-1}(S^\oplus)$.*

Proof. $y \in (T(S))^\oplus \Leftrightarrow \langle y, T(S) \rangle \geq 0 \Leftrightarrow \langle T^* y, S \rangle \geq 0 \Leftrightarrow T^* y \in S^\oplus \Leftrightarrow y \in (T^*)^{-1}(S^\oplus)$ □

Proposition 2.5. *Suppose L is a closed convex cone in X, and Z is a subspace of X. Then $P_{Z^\perp}((L+Z)\cap\epsilon B_X) = P_{Z^\perp}(L)\cap\epsilon B_X$, for every $\epsilon > 0$.*

Proof. "$\subseteq$": let $y = P_{Z^\perp}(x)$, where $x \in (L+Z)\cap\epsilon B_X$. On the one hand, $\|y\| = \|P_{Z^\perp}(x)\| \leq \|x\| \leq \epsilon$, hence $y \in \epsilon B_X$. On the other hand, $x = l + z$, for some $l \in L$ and $z \in Z$; consequently, $y = P_{Z^\perp}(x) = P_{Z^\perp}(l) \in P_{Z^\perp}(L)$. Altogether, $y \in \epsilon B_X \cap P_{Z^\perp}(L)$.
"$\supseteq$": pick $y \in P_{Z^\perp}(L)\cap\epsilon B_X$, say $y = P_{Z^\perp}(l)$, where $l \in L$. Then $y = l - P_Z l \in (L+Z)\cap\epsilon B_X$. Since $y = P_{Z^\perp}(y)$, we have $y \in P_{Z^\perp}((L+Z)\cap\epsilon B_X)$, as desired. □

Proposition 2.6. *Suppose L is a closed convex cone in X, and Z is a subspace of X. Then the following two statements are equivalent.*

(i) *There exists $\epsilon > 0$ such that $(L+Z)\cap\epsilon B_X \subseteq (L\cap B_X) + (Z\cap B_X)$.*
(ii) *There exists $\epsilon > 0$ such that $P_{Z^\perp}(L)\cap\epsilon B_X \subseteq P_{Z^\perp}(L\cap B_X)$.*

Proof. "(i)⇒(ii)": Suppose $(L+Z)\cap\epsilon B_X \subseteq (L\cap B_X)+(Z\cap B_X)$. Then, using Proposition 2.5, $P_{Z^\perp}(L)\cap\epsilon B_X = P_{Z^\perp}((L+Z)\cap\epsilon B_X) \subseteq P_{Z^\perp}((L\cap B_X)+(Z\cap B_X)) = P_{Z^\perp}(L\cap B_X)$. Hence (ii) holds (with the same ϵ).
"(i)⇐(ii)": Suppose $P_{Z^\perp}(L)\cap\epsilon' B_X \subseteq P_{Z^\perp}(L\cap B_X)$. Fix an arbitrary $x \in (L+Z)\cap\epsilon' B_X$. Then, using Proposition 2.5, $P_{Z^\perp}(x) \in P_{Z^\perp}((L+Z)\cap\epsilon' B_X) = P_{Z^\perp}(L)\cap\epsilon' B_X \subseteq P_{Z^\perp}(L\cap B_X)$. Hence there exists $l \in L\cap B_X$ such that $P_{Z^\perp}(x) = P_{Z^\perp}(l)$. Now $x = P_{Z^\perp}(x)+P_Z(x) = P_{Z^\perp}(l)+P_Z(x) = l+P_Z(x-l)$. Also, $\|P_Z(x-l)\| \leq \|x\|+\|l\| \leq \epsilon'+1$, and clearly $\|l\| \leq \epsilon'+1$. Thus $x \in (L\cap(1+\epsilon')B_X) + (Z\cap(1+\epsilon')B_X)$, which yields $x/(1+\epsilon') \in (L\cap B_X)+(Z\cap B_X)$. Since x was chosen arbitrarily, we conclude

$$(L+Z)\cap\frac{\epsilon'}{1+\epsilon'}B_X \subseteq (L\cap B_X)+(Z\cap B_X).$$

Therefore, (i) holds (with $\epsilon = \epsilon'/(1+\epsilon')$). □

Tangent and normal cones

Definition 2.7. Suppose C is a closed convex nonempty subset of X and $x \in C$. The *tangent cone* (resp. *normal cone*) of C at x is defined by $T_C(x) := \overline{\mathrm{cone}}(C-x)$ (resp. $N_C(x) := (C-x)^\ominus$).

Fact 2.8. *Suppose C is a closed convex subset of X and $x \in C$. Then:*

(i) *y belongs to $T_C(x)$ if and only if there exists a sequence (t_n) of reals tending to $+\infty$ and a sequence (c_n) in C such that $y = \lim_n t_n(c_n - x)$.*
(ii) *$N_C(x)^\ominus = T_C(x)$ and $T_C(x)^\ominus = N_C(x)$.*

Proof. (i): See [16, Definition III.5.1.1 and Proposition III.5.2.1]. (ii): See [16, Proposition III.5.2.4 and Corollary III.5.2.5]. □

Regularities

Definition 2.9. Suppose C_1 and C_2 are two closed convex sets in X with $C := C_1\cap C_2 \neq \emptyset$. Then $\{C_1, C_2\}$ is *linearly regular* if there exists $\kappa > 0$ such that $d(x,C) \leq \kappa\max_i d(x,C_i)$, for every $x \in X$. If for every bounded subset S of X there exists $\kappa_S > 0$ such that $d(x,C) \leq \kappa_S\max_i d(x,C_i)$, for every $x \in S$, then $\{C_1, C_2\}$ is *boundedly linearly regular.*

Remark 2.10. Clearly, linear regularity implies bounded linear regularity. (Also, it is obvious how these definitions extend to more than two sets.) The concept of (bounded) linear regularity is very useful in certain areas of convex optimization, in particular error bounds and projection methods. Bounded linear regularity is guaranteed to hold under the usual constraint qualification $\mathrm{ri}(C_1) \cap \mathrm{ri}(C_2) \neq \emptyset$ (the relative interior can be dropped for polyhedral sets). We refer the reader to [9] and [23, Section 3.4] for further information and references.

Remark 2.11. A comment on the relationship between bounded linear regularity and the well-known notion of *metric regularity* for set-valued maps is in order. To start with, there is no generally agreed upon notion of metric regularity for two sets C_1, C_2. We could define metric regularity as in [9, Remark 5] or as in [18, Section 5]. However, both definitions imply the *constraint qualification* "$0 \in \mathrm{int}(C_1 - C_2)$" and consequently bounded linear regularity [5, Theorem 4.3]. On the other hand, it is easy to obtain two sets that are boundedly linearly regular without the constraint qualification. (Let $C_1 = C_2 = \{0\}$ in $X = \mathbb{R}$, for instance.) Thus, *existing definitions of metric regularity are genuinely stronger than bounded linear regularity.*

For cones, bounded linear regularity implies linear regularity:

Fact 2.12. *Suppose K_1 and K_2 are two closed convex cones in X. Then the following are equivalent.*

(i) *$\{K_1, K_2\}$ is linearly regular.*
(ii) *$\{K_1, K_2\}$ is boundedly linearly regular.*
(iii) *There exists $\epsilon > 0$ such that $(K_1^{\oplus} + K_2^{\oplus}) \cap \epsilon B_X \subseteq (K_1^{\oplus} \cap B_X) + (K_2^{\oplus} \cap B_X)$.*

Remark 2.13. Jameson referred to condition (iii) of Fact 2.12 as "property (G)"; see [19]. If K_1 and K_2 are actually *subspaces*, then these conditions *always* hold [9, Corollary 10].

Strong CHIP and CHIP

Definition 2.14. Suppose C_1 and C_2 are two closed convex sets in X with $C := C_1 \cap C_2 \neq \emptyset$ and let $x \in C$. Then $\{C_1, C_2\}$ has *strong CHIP* (resp. *CHIP*) at x, if $N_C(x) = N_{C_1}(x) + N_{C_2}(x)$ (resp. $T_C(x) = T_{C_1}(x) \cap T_{C_2}(x)$). We say that $\{C_1, C_2\}$ have *strong CHIP* (resp. *CHIP*), if it has strong CHIP (resp. CHIP) at every point in C.

Remark 2.15. It is always true that $T_C(x) \subseteq T_{C_1}(x) \cap T_{C_2}(x)$ and thus (Fact 2.3.(i)) $N_C(x) \supseteq \mathrm{cl}(N_{C_1}(x) + N_{C_2}(x))$. Also, taking negative polars shows that CHIP is less restrictive then strong CHIP. The notions strong CHIP and CHIP — CHIP is an acronym for "conical hull intersection property" — were coined by Deutsch and his co-workers (see [10] and [15]) in their studies of constraint approximation problems. Very recent work on strong CHIP and CHIP are [9], [12], [13], and [14],

We now present the basic relationships.

Fact 2.16. *Bounded linear regularity $\Rightarrow$ strong CHIP $\Rightarrow$ CHIP.*

Proof. The first implication can be shown using the connection between $N_C(x)$ and the subdifferential of $d(x, C)$; see [9, Theorem 3] and [23, Proposition 6]. The second implication follows from [24, Corollary 16.4.2]. □

Remark 2.17. We will construct examples below that show that neither implication is reversible, even for cones. In stark contrast, all three conditions always hold for *subspaces*; see Remark 2.13.

Fact 2.18. *Suppose K_1 and K_2 are two closed convex cones in X. Then $\{K_1, K_2\}$ has strong CHIP if and only if $K_1^{\oplus} + K_2^{\oplus}$ is closed.*

Proof. See [9, Proposition 20]. □

A family of self-dual cones

Definition 2.19. Suppose $1 < p < \infty$. Set $\alpha := \alpha_p := 1/p$, $\beta := \beta_p := 1 - \alpha$, and let $\rho := \rho_p$ be the positive solution of $1/\rho^2 = \alpha^\alpha \beta^\beta$. Define

$$S := S_p := \{(x, y, z) \in \mathbb{R}^3 : |y| \le \rho x^\alpha z^\beta, x \ge 0, z \ge 0\}.$$

Theorem 2.20. $S_p^{\oplus} = S_p$. *In particular, S_p is a closed convex cone.*

Proof. Define α, β, ρ, and S as in Definition 2.19. Fix an arbitrary $(u, v, w) \in S^{\oplus}$. Then

$$ux + vy + wz = \langle (u, v, w), (x, y, z)\rangle \ge 0, \quad \forall (x, y, z) \in S. \qquad (*)$$

Claim 1. $u \ge 0$ and $w \ge 0$.
Indeed, $(1, 0, 0)$ and $(0, 0, 1)$ belong to S and so the claim follows from $(*)$.

Claim 2. $|v| \le \rho u^\alpha w^\beta$.
Case 1: $u = 0$. Then $(x, \pm 1, \rho^{-1/\beta} x^{-\alpha/\beta}) \in S$, $\forall x > 0$. By $(*)$, $0 \le \pm v + w\rho^{-1/\beta} x^{-\alpha/\beta}$. This implies $|v| \le \rho^{-1/\beta} x^{-\alpha/\beta}$, $\forall x > 0$. By letting x tend to $+\infty$, it follows that $v = 0$. This proves Claim 2 for this case. *Case 2:* $w = 0$. Similar to Case 1. *Case 3:* $u > 0$ and $w > 0$. Set momentarily $x := \sqrt{(w\alpha)/(u\beta)}$, $z := \sqrt{(u\beta)/(w\alpha)}$, and $y := \pm\rho x^\alpha z^\beta$. Then $(x, y, z) \in S$. Hence $(*)$ yields $0 \le ux + vy + wz$. Now substitute the values for x, y, z into this inequality and simplify. (Recall that $\alpha + \beta = 1$ and $1/\rho^2 = \alpha^\alpha \beta^\beta$.) The desired inequality follows.

Claims 1 and 2 imply $(u, v, w) \in S$ and hence

$$S^{\oplus} \subseteq S.$$

Claim 3. $\xi + \eta \ge \rho^2 \xi^\alpha \eta^\beta$, $\forall \xi \ge 0, \eta \ge 0$.
Without loss of generality, we may assume that ξ and η are both nonzero. Setting $w := \eta/\xi$, the inequality is then equivalent to $f(w) := w^\alpha + w^{-\beta} \ge \rho^2$, $\forall w > 0$. The function f is differentiable and its range is contained in $(0, +\infty)$. Also, as w tends to either 0^+ or $+\infty$, the value $f(w)$ tends to $+\infty$. Hence f attains its minimum value. Calculus shows that the minimum is ρ^2, attained at β/α. The claim thus holds.

Now fix two arbitrary elements (x, y, z) and (a, b, c) in S. Then x, z, a, c are all nonnegative, $|y| \le \rho x^\alpha z^\beta$, and $|b| \le \rho a^\alpha c^\beta$. Using this and Claim 3 (with $\xi = ax$ and $\eta = cz$), we estimate

$$\begin{aligned}\langle (a,b,c),(x,y,z)\rangle &= ax+by+cz\\ &\geq ax+cz-\rho^2(ax)^\alpha(cz)^\beta\\ &\geq 0.\end{aligned}$$

Therefore, $S \subseteq S^\oplus$ and the proof is complete. □

Proposition 2.21. Suppose K is a closed convex nonempty set in $\mathbb{R}$ and L is a closed convex nonempty set in $\mathbb{R}_+$. Then $S_p + (0 \times K \times L)$ is closed.

Proof. Let α, β, ρ, and S be as in Definition 2.19. Fix a sequence (x_n, y_n, z_n) in S and a sequence $(0, k_n, l_n)$ in $0 \times K \times L$, and assume that

$$(a,b,c) := \lim_n (x_n, y_n + k_n, z_n + l_n).$$

Then $x_n \geq 0$, $z_n \geq 0$, $|y_n| \leq \rho x_n^\alpha z_n^\beta$, $k_n \in K$, and $l_n \in L$, for every n. Hence (x_n) converges to a, which yields $a \geq 0$. Now $(z_n + l_n)$ converges to c, which must be nonnegative. Moreover, after passing to a subsequence if necessary, we assume that $z := \lim_n z_n \geq 0$ and $l := \lim_n l_n \in L$. Hence $z + l = c$ and (y_n) is bounded. After passing to another subsequence, we assume that $y := \lim_n y_n$. Hence $\lim_n k_n = b - y \in K$ and $(a, y, z) \in S$. It follows that $(a,b,c) = (a,y,z) + (0, b-y, l) \in S + (0 \times K \times L)$. □

Remark 2.22. Denote the (real) vector space of 2-by-2 real symmetric matrices by $\mathcal{H}$. Then $\langle x, y\rangle := \mathrm{trace}(xy)$ (for $x, y \in \mathcal{H}$) defines an inner product on $\mathcal{H}$, with induced norm $\|x\| := \sqrt{\langle x, x\rangle}$. The map

$$\Phi : \mathbb{R}^3 \to \mathcal{H} : (x,y,z) \mapsto \begin{pmatrix} x & y/\sqrt{2} \\ y/\sqrt{2} & z \end{pmatrix}$$

is a linear isometry from $\mathbb{R}^3$ onto $\mathcal{H}$. Moreover, the image of $S_2 = \{(x,y,z) : |y| \leq \sqrt{2xz}, x \geq 0, z \geq 0\}$ under Φ, $\mathcal{S} := \Phi(S_2)$, is precisely the cone of all positive semi-definite matrices in $\mathcal{H}$. (This cone has recently received much attention in convex optimization. See, for instance, Lewis's paper [21] for further information.) The projection onto $\mathcal{S}$ of an element $x \in \mathcal{H}$ can be found as follows. Find first an unitary matrix u that diagonalizes x: $x = u^* d u = u^{-1} d u$, where d is diagonal. Then $P_{\mathcal{S}}(x) = u^* d^+ u$, where $(d^+_{ij}) := ((d_{ij})^+)$.

Definition 2.23. Suppose $1 < p < +\infty$ and let α, β, and ρ be as in Definition 2.19. Define

$$\tilde{S}_p := \{(x,y,z) \in \mathbb{R}^3 : |x| \leq \rho y^\alpha z^\beta, y \geq 0, z \geq 0\}.$$

Corollary 2.24. $\tilde{S}_p^\oplus = \tilde{S}_p$.

Proof. Let S_p be as in Definition 2.19. Observe that $\tilde{S}_p$ is just S_p with the first two coordinates interchanged. Hence $\tilde{S}_p = T(S_p)$, where

$$T := \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Since $T = T^* = T^{-1}$, the result follows from Proposition 2.4 and Theorem 2.20. □

A quarter circle

Proposition 2.25. *Define a quarter circle in* $\mathbb{R}^2$ *by*

$$\begin{aligned} Q &:= \{(y,z) \in \mathbb{R}^2 : z^2 + (y-1)^2 \leq 1, 0 \leq y \leq 1, 0 \leq z\} \\ &= \{(y,z) \in \mathbb{R}^2 : 2y \geq y^2 + z^2, y \leq 1, 0 \leq z\}. \end{aligned}$$

Suppose $q = (y,z) \in \mathrm{bd}(Q)$. *Then exactly one of the following two alternatives holds.*

(i) $z = 0$ *and* $0 \leq y \leq 1$. *In this case,*

$$T_Q(q) = \begin{cases} \mathbb{R}_+ \times \mathbb{R}_+, & \textit{if } y = 0; \\ \mathbb{R} \times \mathbb{R}_+, & \textit{if } 0 < y < 1; \\ \mathbb{R}_- \times \mathbb{R}_+, & \textit{if } y = 1. \end{cases}$$

(ii) *Either* $y = 1$ *and* $0 < z \leq 1$, *or* $0 < y < 1$ *and* $z = \sqrt{2y - y^2}$. *In either case,*

$$Q \cap \mathbb{R}_+ \cdot q = [0,1] \cdot q, \quad q \notin T_Q(q), \quad \textit{but} \quad -q \in T_Q(q).$$

Proof. The second description of Q is easy to verify. The various statements on $T_Q(q)$ follow immediately by examining a picture of Q. (This can be made rigorous by calculus, of course.) $\square$

Conification

Proposition 2.26. *Suppose* C *is a compact convex nonempty set in* X. *In* $X \times \mathbb{R}$, *define* $K := \overline{\mathrm{cone}}(C \times 1)$. *Then:*

(i) $K = \mathrm{cone}(C \times 1)$.

(ii) *Suppose* $p > 0$ *and* $c \in C$. *Then* $(y,s) \in T_K(pc,p)$ *if and only if* $y - sc \in T_C(c)$.

(iii) *If* $0 \in C$, *then* $T_K(0,1) = T_C(0) \times \mathbb{R}$ *and* $N_K(0,1) = N_C(0) \times 0$.

Proof. (i): It is clear that $\mathrm{cone}(C \times 1)$ is contained in K. Conversely, pick $(x,r) \in K$. Then there exists a sequence (c_n) in C and a sequence of nonnegative reals (p_n) such that $(x,r) = \lim_n p_n(c_n, 1)$. Hence $\lim_n p_n = r \geq 0$. Since C is compact, after passing to a subsequence and relabeling if necessary, we assume without loss of generality that $\lim_n c_n =: c \in C$. Hence $rc = \lim_n p_n c_n$ and thus $(x,r) = r(c,1) \in \mathrm{cone}(C \times 1)$.

(ii): "$(y,s) \in T_K(pc,p) \Rightarrow y - sc \in T_C(c)$": There exist sequences (t_n) and (p_n) in $\mathbb{R}_+$, and (c_n) in C such that $(y,s) = \lim_n t_n\big[(p_n c_n, p_n) - (pc,p)\big]$. Hence $s = \lim_n t_n(p_n - p)$ and

$$\begin{aligned} y &= \lim_n t_n(p_n c_n - pc) \\ &= \lim_n t_n p_n (c_n - c) + \lim_n t_n(p_n - p)c \\ &= \lim_n t_n p_n(c_n - c) + sc. \end{aligned}$$

It follows that $y - sc = \lim_n t_n p_n (c_n - c) \in T_C(c)$.

"$(y,s) \in T_K(pc,p) \Leftarrow y - sc \in T_C(c)$": We obtain sequences (t'_n) in $\mathbb{R}_+$ and (c_n) in C such that $\lim_n t'_n = +\infty$ and $y - sc = \lim_n t'_n(c_n - c)$. Define

$$p_n := \frac{t'_n p}{t'_n - s} = \frac{p}{1 - s/t'_n} \to p \quad \text{and} \quad t_n := \frac{t'_n}{p_n} \to +\infty.$$

Then $t_n(p_n - p) = s$, $\forall n$. Hence

$$\begin{aligned}(y,s) &= \big(sc + (y - sc), s\big)\\ &= \big(t_n(p_n - p)c + \lim_n(t_n'(c_n - c)), t_n(p_n - p)\big)\\ &= \big(t_n(p_n - p)c + \lim_n(t_n p_n(c_n - c)), t_n(p_n - p)\big)\\ &= \lim_n\big(t_n(p_n c_n - pc), t_n(p_n - p)\big)\\ &= \lim_n t_n\big[(p_n c_n, p_n) - (pc, p)\big]\\ &\in T_K(pc, p).\end{aligned}$$

(iii): This follows from (ii) and by taking negative polars. ☐

Proposition 2.27. *Suppose K is a closed convex cone in X. In $X \times \mathbb{R}$, define $L := \overline{\operatorname{cone}}(K \times 1)$. Then:*

(i) $L = K \times \mathbb{R}_+$.
(ii) $T_L(k,p) = T_K(k) \times T_{\mathbb{R}_+}(p)$, *for every* $(k,p) \in L$.
(iii) $N_L(k,p) = N_K(k) \times N_{\mathbb{R}_+}(p)$, *for every* $(k,p) \in L$.

Proof. (i): $K \times \mathbb{R}_+$ not only contains $K \times 1$ but also is a closed convex cone. Hence $K \times \mathbb{R}_+ \supseteq L$. Conversely, fix $(k,p) \in K \times \mathbb{R}_+$. Pick a sequence of positive reals (p_n) tending to p. Then $\operatorname{cone}(K \times 1) \ni p_n(k/p_n, 1) = (k, p_n) \to (k,p)$; hence $(k,p) \in \overline{\operatorname{cone}}(K \times 1) = L$. (ii): follows immediately from (i). (iii): Consider (ii) and take negative polars. ☐

3. Strong CHIP $\not\Rightarrow$ bounded linear regularity

Throughout this section, we let

$$\boxed{K := (0 \times \tilde{S}_2) + (S_3 \times 0) \quad \text{and} \quad Y := 0 \times \mathbb{R}^3 \quad \text{in} \quad X := \mathbb{R}^4,}$$

where S_3 (resp. $\tilde{S}_2$) is defined as in Definition 2.19 (resp. Definition 2.23).

Theorem 3.1.
(i) *K is a closed convex cone in X, and Y is a subspace of X.*
(ii) $K \cap Y = 0 \times \tilde{S}_2$.
(iii) $K^\oplus = (\mathbb{R} \times \tilde{S}_2) \cap (S_3 \times \mathbb{R})$ *and* $Y^\oplus = Y^\perp = \mathbb{R} \times 0$.
(iv) $P_Y(K^\oplus) = 0 \times \tilde{S}_2$.
(v) $\mathbb{R} \times \tilde{S}_2 = K^\oplus + Y^\oplus = (K \cap Y)^\oplus$.
(vi) *There is no $\epsilon > 0$ such that $\epsilon B_X \cap P_Y(K^\oplus) \subseteq P_Y(K^\oplus \cap B_X)$.*
(vii) *There is no $\epsilon > 0$ such that $\epsilon B_X \cap (K^\oplus + Y^\oplus) \subseteq (K^\oplus \cap B_X) + (Y^\oplus \cap B_X)$.*
(viii) *For $t > 0$, let $x_t := (1, t^2, t^3, 0)$. Then $d(x_t, K) = 0$, $d(x_t, Y) = 1$, and $d^2(x_t, K \cap Y) = 1 + t^4/(t + \sqrt{t^2+2})^2$.*

Proof. Let $\rho_2 = \sqrt{2}$ and $\rho_3 = 3^{1/2}/2^{1/3}$ be defined as in Definition 2.19.
(i): Clearly, K is a convex cone and Y is a subspace. To show that K is closed, fix sequences (a_n, b_n, c_n) in $\tilde{S}_2$ and (w_n, x_n, y_n) in S_3 and assume

$$\lim_n(w_n, x_n + a_n, y_n + b_n, c_n) =: (w^*, x^*, y^*, z^*).$$

Now all b_n, c_n, w_n, y_n are nonnegative. It follows that $w^* \geq 0$, $y^* \geq 0$, and $z^* \geq 0$. Moreover, (b_n) and (y_n) are bounded. After passing to a subsequence if necessary, we assume that $\bar{b} := \lim_n b_n \geq 0$ and $\bar{y} := \lim_n y_n \geq 0$. Thus $y^* = \bar{b} + \bar{y} \geq 0$. Also, $|a_n| \leq \rho_2 b_n^{1/2} c_n^{1/2}$, $\forall n$. Hence (a_n) is bounded, too. Without loss of generality (subsequence!), assume $\bar{a} := \lim_n a_n$. Hence

$$(\bar{a}, \bar{b}, z^*) \in \tilde{S}_2.$$

This implies $\lim_n x_n = x^* - \bar{a}$ and so

$$(w^*, x^* - \bar{a}, \bar{y}) \in S_3.$$

Altogether, $(w^*, x^*, y^*, z^*) = (0, \bar{a}, \bar{b}, z^*) + (w^*, x^* - \bar{a}, \bar{y}, 0) \in K$.
(ii): It is clear that $0 \times \tilde{S}_2$ is a subset of $K \cap Y$. Now take an element of K, say $(w, x + a, y + b, c)$, where $(a, b, c) \in \tilde{S}_2$ and $(w, x, y) \in S_3$. Assume further that $(w, x + a, y + b, c)$ belongs also to Y. Hence $w = 0$, which implies $x = 0$. Since $y \geq 0$, the vector $(a, b + y, c)$ clearly belongs to $\tilde{S}_2$ and the reverse inclusion is established.
(iii): Follows from Fact 2.3.(ii), Theorem 2.20, and Corollary 2.24.
(iv): By (iii), $K^{\oplus} \subseteq \mathbb{R} \times \tilde{S}_2$. Hence $P_Y(K^{\oplus}) \subseteq P_Y(\mathbb{R} \times \tilde{S}_2) = 0 \times \tilde{S}_2$. Conversely, fix $(a, b, c) \in \tilde{S}_2$. Then $b \geq 0$, $c \geq 0$, and $|a| \leq \rho_2 b^{1/2} c^{1/2}$. If $b = 0$, then $a = 0$. So we can always pick $w \geq 0$ such that $|a| \leq \rho_3 w^{1/3} b^{2/3}$. Thus $(w, a, b) \in S_3$. Using (iii), $(w, a, b, c) \in (\mathbb{R} \times \tilde{S}_2) \cap (S_3 \times \mathbb{R}) = K^{\oplus}$. Hence $(0, a, b, c) = P_Y(w, a, b, c) \in P(K^{\oplus})$.
(v): Since $Y^{\oplus} = \mathbb{R} \times 0$ (by (iii)) and $P_Y(K^{\oplus}) = 0 \times \tilde{S}_2$ (by (iv)), we have $K^{\oplus} + Y^{\oplus} = \mathbb{R} \times \tilde{S}_2$. This proves the first equality. In particular, $K^{\oplus} + Y^{\oplus}$ is closed. The second equality now follows from Fact 2.3.(i).
(vi): By contradiction. Assume there exists $\epsilon > 0$ such that $\epsilon B_X \cap P_Y(K^{\oplus}) \subseteq P_Y(K^{\oplus} \cap B_X)$. Multiplication by $\delta := \rho_3^{-3} > 0$ yields

$$\delta\epsilon B_X \cap P_Y(K^{\oplus}) \subseteq P_Y(K^{\oplus} \cap \delta B_X). \tag{$*$}$$

Now choose $t > 0$ so small such that $y^* := (0, t^2, t^3, t) \in \delta\epsilon B_X$. Let $x^* := (\rho_3^{-3}, t^2, t^3, t)$. Then it can be checked that $x^* \in (\mathbb{R} \times \tilde{S}_2) \cap (S_3 \times \mathbb{R})$. By (iii), $x^* \in K^{\oplus}$ and hence $y^* = P_Y(x^*) \in P_Y(K^{\oplus})$. Thus $y^* \in \delta\epsilon B_X \cap P_Y(K^{\oplus})$. By $(*)$, there exists $x := (w, t^2, t^3, t) \in K^{\oplus} \cap \delta B_X$ such that $P_Y(x) = y^*$. On the one hand, $|w| < \delta$. On the other hand, by (iii), $x \in (\mathbb{R} \times \tilde{S}_2) \cap (S_3 \times \mathbb{R})$. In particular, $(w, t^2, t^3) \in S_3$, which is equivalent to $w \geq \rho_3^{-3}$. Altogether, we get $w \geq \rho_3^{-3} = \delta > |w|$, which is the desired contradiction.
(vii): This follows immediately from (vi) and Proposition 2.6 (with $L = K^{\oplus}$ and $Z = Y^{\perp}$).
(viii): It is clear that $d(x_t, Y) = 1$. Because $\rho_3 > 1$, it is easy to verify that $(1, t^2, t^3) \in S_3$. Hence $x_t \in S_3 \times 0 \subseteq K$ and further $d(x_t, K) = 0$. By (ii), $K \cap Y = 0 \times \tilde{S}_2$. Hence

$$d(x_t, K \cap Y) = \sqrt{1 + d^2\big((t^2, t^3, 0), \tilde{S}_2\big)}.$$

Let T be as in the proof of Corollary 2.24. Then $T(S_2) = \tilde{S}_2$ and $T = T^* = T^{-1}$. Thus if s is an arbitrary element of S_2, then $\|(t^2, t^3, 0) - Ts\| = \|T(t^2, t^3, 0) - s\| = \|(t^3, t^2, 0) - s\|$. Borrowing notation from Remark 2.22, we thus have

$$d\big((t^2, t^3, 0), \tilde{S}_2\big) = d\Big(\begin{pmatrix} t^3 & t^2/\sqrt{2} \\ t^2/\sqrt{2} & 0 \end{pmatrix}, \mathcal{S}\Big).$$

The eigenvalues of the matrix $\begin{pmatrix} t^3 & t^2/\sqrt{2} \\ t^2/\sqrt{2} & 0 \end{pmatrix}$ are $t^2(t \pm \sqrt{t^2+2})/2$. Hence the distance from this matrix to $\mathcal{S}$ is $t^2(\sqrt{t^2+2}-t)/2 = t^2/(t+\sqrt{t^2+2})$. Altogether,

$$d(x_t, K \cap Y) = \sqrt{1 + \frac{t^4}{(t+\sqrt{t^2+2})^2}},$$

which completes the proof of the theorem. □

Corollary 3.2. *$\{K, Y\}$ has strong CHIP, but is not boundedly linearly regular.*

Proof. On the one hand, $\{K, Y\}$ has strong CHIP, because of Theorem 3.1.(v) and Fact 2.18. On the other hand, Theorem 3.1.(vii) and Fact 2.12 show that $\{K, Y\}$ is not boundedly linearly regular. □

Remark 3.3. The intuition for the example $\{K, Y\}$ can be seen by noting that $K^{\oplus} = \overline{\mathrm{cone}}(C \times 1)$ and $Y^{\oplus} = Z \times 0$, where $C := S_3 \cap \tilde{C}$, with $\tilde{C} := \mathbb{R} \times \{x \in \mathbb{R}^2 : (x, 1) \in \tilde{S}_2\}$, and $Z := \mathbb{R} \times 0 \subset \mathbb{R}^3$. The pair $\{C, Z\}$ has properties analogous to those given in Theorem 3.1.(v) and (vii). In particular, $\mathrm{bd}(\tilde{C}) = \{(x, y, z) \in \mathbb{R}^3 : y^2 = 2z\}$ intersects $\mathrm{bd}(S_3) = \{(x, y, z) \in \mathbb{R}^3 : |y|^3 = \rho_3^3 x z^2, z \geq 0\}$ in two curves that have Z as an asymptote. This in turn ensures that $C + Z$ is closed and there is no $\epsilon > 0$ such that $\epsilon B_X \cap (C + Z) \subseteq (C \cap B_X) + (Z \cap B_X)$. [The latter can be seen by taking any sequence of points (x_t, y_t, z_t) in $\mathrm{bd}(S_3) \cap \mathrm{bd}(\tilde{C})$ that asymptotically approaches Z as $t \to 0$, such as $(x_t, y_t, z_t) = (4/(\rho_3^3 t), t, t^2/2)$. Then, $(0, y_t, z_t)$ belongs to $C + Z$ and is bounded as $t \to 0$, but its unique decomposition as sum of elements of C and Z, namely (x_t, y_t, z_t) and $(-x_t, 0, 0)$, is not bounded due to $x_t \to \infty$.] Notice that the particular choice of S_3 and $\tilde{C}$ is not essential. What is essential is that the intersection of their boundary has Z as an asymptote.

We outline, using Theorem 3.1.(viii), a different and more explicit proof that $\{K, Y\}$ is not boundedly linearly regular. Indeed, since K and Y are cones, it suffices to show that $\{K, Y\}$ is not linearly regular (Fact 2.12). If $\{K, Y\}$ were linearly regular, then there would exist $\kappa > 0$ such that $d^2(x, K \cap Y) \leq \kappa^2 \max\{d^2(x, K), d^2(x, Y)\}$, for every $x \in X$. Letting x_t be as in Theorem 3.1.(viii) and setting $z_t := (1/t)x_t = (1/t, t, t^2, 0)$, we would conclude that

$$\begin{aligned}
\frac{1}{t^2} + \frac{1}{\left(1 + \sqrt{1 + 2/t^2}\right)^2} &= d^2(z_t, K \cap Y) \\
&\leq k^2 \max\{d^2(z_t, K), d^2(z_t, Y)\} \\
&= \frac{k^2}{t^2},
\end{aligned}$$

for every $t > 0$. But this is absurd, since, as t tends to $+\infty$, the first expression in this chain of inequalities tends to $1/4$, whereas the last expression tends to 0. Therefore, $\{K, Y\}$ is not boundedly linearly regular.

Also, the example from Corollary 3.2 can be embedded into any higher-dimensional space.

Remark 3.4. The earlier result of Bakan [2] asserts the existence of two convex cones in $\mathbb{R}^4$ with strong CHIP but without bounded linear regularity. Our result gives an explicit construction of such cones. Another result of Bakan [1], together with a result in [15], asserts that these concepts coincide for convex cones in $\mathbb{R}^3$.

Also, a somewhat simpler example (residing in $\mathbb{R}^7$) is described in a subsequent paper [7, Remark 3.12].

4. CHIP $\not\Rightarrow$ strong CHIP

Throughout this section, we let

$$\boxed{A := S_2 \cap \{(x,y,z) \in \mathbb{R}^3 : (x-1)^2 + z^2 \leq 1, 0 \leq x \leq 1, 0 \leq z\}}$$

and

$$\boxed{B := 0 \times Q, \quad C := \operatorname{conv}(A \cup B), \quad Y := 0 \times \mathbb{R}^2,}$$

where S_2 (resp. Q) is as in Definition 2.19 (resp. Proposition 2.25).

Theorem 4.1.
(i) *The sets A, B, C are compact, convex, and nonempty.*
(ii) $T_A(0) = S_2$.
(iii) $A \cap B = A \cap Y = \{0\}$.
(iv) $C \cap Y = B$.
(v) $T_C(b) \cap Y \subseteq T_B(b)$, *for every* $b \in B$.
(vi) $T_C(b) \cap T_Y(b) = T_B(b)$, *for every* $b \in B$.
(vii) $(0,-1,0) \in N_B(0) \setminus \big[N_C(0) + N_Y(0)\big]$.

Proof. (i): If $(x,y,z) \in A$, then both x and z belong to $[0,1]$. Hence $y^2 \leq 2xz \leq 2$ and so A is bounded. Using Theorem 2.20, we see that A is convex. Also, A is nonempty and closed. Clearly, B is compact, convex, and nonempty. The compactness of C follows from [24, Theorem 17.2].
(ii): $T_A(0) \subseteq S_2$, because $A \subseteq S_2$. Conversely, fix an arbitrary $(x,y,z) \in S_2$. Then $x \geq 0$, $z \geq 0$, and $y^2 \leq 2xz$. *Case 1*: $x > 0$. Pick $\alpha \geq x$ large enough so that $x^2 + z^2 \leq 2x\alpha$. Set $u := x/\alpha$, $v := y/\alpha$, $w := z/\alpha$. Then $(u,v,w) \in S_2$, $0 \leq u \leq 1$, and $0 \leq w$. Also, $2x\alpha \geq x^2 + z^2 \Leftrightarrow 2u \geq u^2 + w^2 \Leftrightarrow 1 \geq (u-1)^2 + w^2$. Hence $(u,v,w) \in A$ and so $(x,y,z) = \alpha(u,v,w) \in \operatorname{cone}(A) \subseteq T_A(0)$. *Case 2*: $x = 0$. Then $y = 0$ and $z \geq 0$. For small $\epsilon > 0$, consider $a := (\epsilon, 0, \sqrt{\epsilon(2-\epsilon)})$. It is easy to check that $a \in A$. Hence

$$\frac{z}{\sqrt{2\epsilon}}a = \frac{z}{\sqrt{2}}(\sqrt{\epsilon}, 0, \sqrt{2-\epsilon}) \in \operatorname{cone}(A).$$

We conclude that $(z/\sqrt{2})(0,0,\sqrt{2}) = (0,0,z) = (x,y,z) \in \overline{\operatorname{cone}}(A) = T_A(0)$, by letting ϵ tend to 0 from above. Hence (ii) holds.
(iii): It is straightforward to check that $0 \in A \cap B \subseteq A \cap Y = \{0\}$.
(iv): Clearly, $B \subseteq C \cap Y$. To prove the reverse inclusion, note first that an arbitrary element in C, say v, can be written as (see [24, Theorem 3.3])

$$v = \lambda(a,b,c) + (1-\lambda)(0,y,z) = (\lambda a, \lambda b + (1-\lambda)y, \lambda c + (1-\lambda)z),$$

where $\lambda \in [0,1]$, $(a,b,c) \in A$, and $(0,y,z) \in B$. Hence $1 \geq a \geq 0$, $c \geq 0$, $b^2 \leq 2ac$, $2a \geq a^2 + c^2$, $0 \leq y \leq 1$, $0 \leq z$, and $2y \geq y^2 + z^2$. In addition, assume that v belongs to Y. Then

$$\lambda a = 0.$$

If $\lambda = 0$, then $v = (0, y, z) \in B$, as required. Otherwise, $\lambda > 0$. Then $a = 0$, which implies $b = 0$ and $c = 0$. Hence $v = \lambda(0, 0, 0) + (1 - \lambda)(0, y, z) \in B$, since B is convex and both $(0, 0, 0)$ and $(0, y, z)$ belong to B. In either case, $v \in B$ and (iv) thus holds.
(v): The inclusion is trivial when $b \in \mathrm{ri}(B)$. Hence we fix arbitrarily $b \in \mathrm{rbd}(B) = 0 \times \mathrm{bd}(Q)$ and $d \in T_C(b) \cap Y$. By Fact 2.8.(i), we obtain a sequence of positive reals (t_n) with $\lim_n t_n = +\infty$, a sequence (a_n) in A, a sequence (b_n) in B, and a sequence (λ_n) in $[0, 1]$ such that

$$d = \lim_n t_n \big[\lambda_n a_n + (1 - \lambda_n) b_n - b\big].$$

By compactness of A and B (see (i)), we assume without loss of generality that

$$\bar{a} := \lim_n a_n \in A, \quad \bar{b} := \lim_n b_n \in B, \quad \text{and} \quad \lambda := \lim_n \lambda_n \in [0, 1].$$

Claim 1: $\lambda \bar{a} = 0$ and $b = (1 - \lambda)\bar{b}$.
Because t_n tends to $+\infty$, we must have $\lim_n \lambda_n a_n + (1 - \lambda_n) b_n = b$. Taking limits yields $\lambda \bar{a} + (1 - \lambda)\bar{b} = b$. On the one hand, $\lambda \bar{a} \in A$ (since A is convex and contains both 0 and $\bar{a}$). On the other hand, $\lambda \bar{a} = b - (1 - \lambda)\bar{b} \in Y$ (since both b and $\bar{b}$ belong to $B \subseteq Y$). Altogether, $\lambda \bar{a} \in A \cap Y = \{0\}$ (by (iii)). Claim 1 thus follows.

Claim 2: $d \in Y \cap \mathrm{cl}(S_2 + T_B(b))$.
Clearly, $d \in Y$. Also, $t_n\big[\lambda_n a_n + (1 - \lambda_n) b_n - b\big] = t_n\big[\lambda_n a_n - 0\big] + t_n\big[(1 - \lambda_n) b_n - b\big] \in T_A(0) + T_B(b)$. Hence $d \in \mathrm{cl}(T_A(0) + T_B(b))$. Now Claim 2 follows from (ii).

Now write $b = (0, q)$, where $q = (y, z) \in \mathrm{bd}(Q)$. Then

$$T_B(b) = 0 \times T_Q(q)$$

(by [16, Proposition III.5.3.1.(ii)], for instance). We now complete the proof of (v) by considering two alternatives.

Case 1: $q = (y, z)$ is as in Proposition 2.25.(i). By Proposition 2.25.(i), the tangent cone $T_Q(q)$ is of the form $K \times \mathbb{R}_+$, where $K \in \{\mathbb{R}_+, \mathbb{R}, \mathbb{R}_-\}$. Now by Proposition 2.21, $S_2 + T_B(b) = S_2 + (0 \times T_Q(q)) = S_2 + (0 \times K \times \mathbb{R}_+)$ is closed. Hence

$$d \in Y \cap (S_2 + T_B(b)).$$

On the one hand, since $T_B(b) \subseteq Y$, we have $Y \cap (S_2 + T_B(b)) = (Y \cap S_2) + (Y \cap T_B(b)) = (Y \cap S_2) + T_B(b)$. On the other hand, $Y \cap S_2 = 0 \times 0 \times \mathbb{R}_+$. Altogether, $d \in (0 \times 0 \times \mathbb{R}_+) + T_B(b)$. However, $(0 \times 0 \times \mathbb{R}_+) + T_B(b) = (0 \times 0 \times \mathbb{R}_+) + (0 \times K \times \mathbb{R}_+) = (0 \times K \times \mathbb{R}_+) = T_B(b)$, and (v) holds for Case 1.

Case 2: $q = (y, z)$ is as in Proposition 2.25.(ii).
Since $q \neq 0$, we have $b \neq 0$ and so (by Claim 1) $\lambda < 1$.

Claim 3: $\lambda = 0$, $b \notin T_B(b)$, but $-b \in T_B(b)$.
Write $\bar{b} = (0, \bar{q})$, with $\bar{q} \in Q$. Then Claim 1 yields $\bar{q} = q/(1 - \lambda)$. Were $\lambda > 0$, then Claim 1 would yield $\bar{q} = q/(1 - \lambda)$, with $1/(1 - \lambda) > 1$. Recall that $q \in \mathrm{bd}(Q)$. Altogether, we would contradict Proposition 2.25.(ii). So $\lambda = 0$. Again by Proposition 2.25.(ii), we have $q \notin T_Q(q)$ and $-q \in T_Q(q)$. But this is equivalent to $b \notin T_B(b)$ and $-b \in T_B(b)$. Claim 3 thus holds.

Now define $\mu_n := t_n\lambda_n$, $\forall n$. Then

$$d = \lim_n \mu_n(a_n - b) + t_n(1 - \lambda_n)(b_n - b). \qquad (*)$$

After passing to a subsequence if necessary, we assume that

$$\mu := \lim_n \mu_n \in [0, +\infty].$$

We consider now three subcases.

Case 2.1: $\mu = 0$.
Since A is compact by (i), the sequence (a_n) is bounded. Hence $\lim_n \mu_n(a_n - b) = 0$. From $(*)$, $d = \lim_n t_n(1 - \lambda_n)(b_n - b) \in T_B(b)$, as required.

Case 2.2: $\mu = +\infty$.
We assume without loss of generality that $\lambda_n > 0$, $\forall n$. Dividing $(*)$ by μ_n yields $b - \bar{a} = \lim_n((1 - \lambda_n)/\lambda_n)(b_n - b) \in T_B(b) \subseteq Y$. Since $b \in Y$, this implies $\bar{a} \in Y$. Clearly, $\bar{a} \in A$. Hence, using (iii), $\bar{a} = 0$. This implies $b = b - \bar{a} \in T_B(b)$, which is impossible by Claim 3.

Case 2.3: $\mu \in (0, +\infty)$.
Then $(*)$ yields $d - \mu(\bar{a} - b) = \lim_n t_n(1 - \lambda_n)(b_n - b) =: t \in T_B(b)$. Note that we chose $d \in Y$, and that both b and t are in Y. Thus $\bar{a} \in Y$. Using (iii) once again, we have $\bar{a} = 0$. By Claim 3, $-b \in T_B(b)$. Altogether, $d = -\mu b + t \in T_B(b) + T_B(b) = T_B(b)$, and (v) is finally established.
(vi): The inclusion $T_C(b) \cap T_Y(b) \supseteq T_B(b)$ is always true; see Remark 2.15. Also, since Y is a subspace, $T_Y(b) = Y$. Hence (vi) follows from (v).
(vii): Using Proposition 2.25.(i), we have $T_B(0) = 0 \times T_Q(0) = 0 \times \mathbb{R}_+ \times \mathbb{R}_+$. Hence, by Fact 2.8.(ii), $N_B(0) = \mathbb{R} \times \mathbb{R}_- \times \mathbb{R}_- \ni (0, -1, 0)$. Since $A \subseteq C$, (ii) yields $S_2 = T_A(0) \subseteq T_C(0)$. Taking negative polars and Theorem 2.20 gives $N_C(0) \subseteq N_A(0) = S_2^{\ominus} = -S_2^{\oplus} = -S_2$. Also, $N_Y(0) = Y^{\ominus} = Y^{\perp} = \mathbb{R} \times 0 \times 0$. To prove (vii), it thus suffices to verify the following

Claim: $(0, 1, 0) \notin S_2 + (\mathbb{R} \times 0 \times 0)$.
Suppose the claim were false, say $(0, 1, 0) = (x, y, z) + (r, 0, 0)$, for some $(x, y, z) \in S_2$ and $r \in \mathbb{R}$. Then $z = 0$, which results in $y = 0$ and we obtain the contradiction $0 = y = 1$. Hence the claim is true and the entire theorem is proven. $\square$

Corollary 4.2. *$\{C, Y\}$ has CHIP, but does not have strong CHIP at* 0.

Proof. By Theorem 4.1.(iv) and (vi), $\{C, Y\}$ has CHIP. Now Theorem 4.1.(iv) and (vii) imply that $N_B(0) \neq N_C(0) + N_Y(0)$. Hence $\{C, Y\}$ does not have strong CHIP. $\square$

Remark 4.3. The intuition for the example $\{C, Y\}$ can be seen by first considering the simpler example $\{S_2, Y\}$. This example does not have strong CHIP at 0, but it also does not have CHIP at any point in $S_2 \cap Y = 0 \times \mathbb{R}_+$ except 0. By intersecting S_2 with a quarter-cylinder to obtain A, we cut away these latter points and still maintain the property that strong CHIP fails at 0. Although $\{A, Y\}$ does not have CHIP at 0, this is the only point in the intersection $A \cap Y = 0$ where CHIP fails. To restore CHIP at 0, we need the intersection to have a curved boundary asymptotic to $S_2 \cap Y$ at 0, such as B. Taking the convex hull of A and B yields the set C which, at least intuitively, has the desired properties.

Finally, define two closed convex cones in $\mathbb{R}^4$ by

$$\boxed{K := \overline{\mathrm{cone}}(C \times 1) \quad \text{and} \quad L := \overline{\mathrm{cone}}(Y \times 1).}$$

Theorem 4.4. *$\{K, L\}$ has CHIP, but does not have strong CHIP.*

Proof. Recall that $C \cap Y = B$ (by Theorem 4.1.(iv)).

Claim 1: $K \cap L = \mathrm{cone}(B \times 1) = \overline{\mathrm{cone}}(B \times 1)$.
It suffices to prove the first equality. By Proposition 2.26.(i) and Proposition 2.27.(i), $K = \mathrm{cone}(C \times 1)$ and $L = Y \times \mathbb{R}_+$. First, let $(x, r) \in K \cap L$. Then $r \geq 0$, $x \in Y$, and $(x, r) \in \mathrm{cone}(C \times 1)$. If $r = 0$, then $x = 0$ (since all nonzero elements in $\mathrm{cone}(C \times 1)$ have a nonzero last component); thus, $(x, r) = (0, 0) \in \mathrm{cone}(B \times 1)$. Otherwise, $r > 0$, in which case $x/r \in C \cap Y = B$; consequently, $(x, r) = r(x/r, 1) \in \mathrm{cone}(B \times 1)$. The reverse inclusion is even simpler. Hence Claim 1 holds.

Claim 2: $\{K, L\}$ has CHIP.
In view of Remark 2.15 and Claim 1, it suffices to show that $T_K(pb, p) \cap T_L(pb, p) \subseteq T_{K\cap L}(pb, p)$, for every $p \geq 0$ and $b \in B$. This inclusion is obvious when $p = 0$. Hence we assume $p > 0$. So take $(y, s) \in T_K(pb, p) \cap T_L(pb, p)$. Proposition 2.26.(ii) and Proposition 2.27.(ii) yield $y - sb \in T_C(b)$, $s \in \mathbb{R}$, and $y \in Y$. Since $b \in B \subseteq Y$, this is equivalent to $y - sb \in T_C(b) \cap Y = T_C(b) \cap T_Y(b)$. But the last set is equal to $T_B(b)$ by Corollary 4.2. Hence $y - sb \in T_B(b)$. Altogether, by Proposition 2.26.(ii) and Claim 1, $(y, s) \in T_{K\cap L}(pb, b)$. Claim 2 thus holds.

Claim 3: $\{K, L\}$ does not have strong CHIP at $(0, 1)$.
Corollary 4.2 and Remark 2.15 yield $N_B(0) \supsetneqq N_C(0) + N_Y(0)$. By Claim 1 and Proposition 2.26.(iii), $N_{K\cap L}(0, 1) = N_B(0) \times 0$ and $N_K(0, 1) = N_C(0) \times 0$. By Proposition 2.27.(iii), $N_L(0, 1) = N_Y(0) \times 0$. Altogether, $N_{K\cap L}(0, 1) \supsetneqq N_K(0, 1) + N_Y(0, 1)$. Hence Claim 3 holds and the entire theorem is proven. $\square$

Remark 4.5. It is clear that the two examples from Corollary 4.2 and Theorem 4.4 can be embedded into any higher dimensional space.

5. Equivalence for cones in $\mathbb{R}^3$

For completeness, we include below a result (obtained while the paper was undergoing revision) showing that the above conical examples cannot reside in spaces of dimension smaller than 4. The equivalence of (i) and (ii) in this result can also be inferred from a result of Bakan in 1988, see [1].

Theorem 5.1. *Suppose K_1 and K_2 are two closed convex cones in $\mathbb{R}^3$. Then the following are equivalent.*

(i) *$\{K_1, K_2\}$ is boundedly linearly regular.*
(ii) *$\{K_1, K_2\}$ has strong CHIP.*
(iii) *$\{K_1, K_2\}$ has CHIP.*

Proof. Let $K := K_1 \cap K_2$. In view of Fact 2.16, it suffices to show that (iii) implies (i). We will argue by contradiction. So suppose $\{K_1, K_2\}$ is CHIP but not boundedly linearly regular. We assume without loss of generality that $\mathrm{int}\, K = \emptyset$ (by [5, Corollary 4.5], for instance). Hence K lies in some hyperplane H. Since K is a convex cone, it is polyhedral. Also, H separates K_1 from K_2. Since $\{K_1, K_2\}$ is not boundedly linearly regular, there exists a bounded sequence (x_n) in $\mathbb{R}^3 \setminus K$ such that

$$\frac{\|x_n - y_{1,n}\| + \|x_n - y_{2,n}\|}{\|x_n - y_n\|} \to 0, \tag{$*$}$$

where $y_{i,n} := P_{K_i}(x_n)$ $(i = 1, 2)$ and $y_n := P_K(x_n)$.

Case 1: K has dimension of 2 and y_n lies in $\operatorname{ri} K$ for an infinite number of n.
After passing to a subsequence and relabelling if necessary, we may assume that always $y_n \in \operatorname{ri} K$ and that x_n is on the same side as K_1. Then $\|x_n - y_{1,n}\| \leq \|x_n - y_n\|$ and $y_{2,n} = y_n$, for all n. But this contradicts $(*)$.

Case 2: Either K has dimension of less than 2, or K has dimension of 2 and $y_n \in \operatorname{rbd} K$ eventually. In this case, there are four possibilities for K: • $K = \{0\}$; • K is a line; • K is a ray; or • $\operatorname{rbd} K$ is the union of two rays. After passing to a subsequence if necessary, we must be in the setting of one of the following two subcases.

Subcase 2.1: $y_n = 0$, for all n.
Then x_n belongs to $K^{\ominus}$, as does $x_n/\|x_n\|$. Let z be a cluster point of $(x_n/\|x_n\|)$. Then $z \in K^{\ominus}$. On the other hand, for $i = 1, 2$:

$$\frac{x_n}{\|x_n\|} - \frac{x_n - y_{i,n}}{\|x_n - y_n\|} = \frac{y_{i,n}}{\|x_n\|} \in K_i,$$

which together with $(*)$ implies $z \in K_i$. Altogether, $z \in K^{\ominus} \cap K_1 \cap K_2 = \{0\}$, which is absurd since $\|z\| = 1$. Thus it remains to consider

Subcase 2.2: each y_n is nonzero and lies in some fixed ray.
Then $y_n/\|y_n\| = \tilde{y}$, for some nonzero $\tilde{y} \in K$. Set $\tilde{x}_n := x_n/\|y_n\|$ and $\tilde{y}_{i,n} := y_{i,n}/\|y_n\|$ for all n and each $i = 1, 2$. Now $(*)$ implies

$$\frac{\|\tilde{x}_n - \tilde{y}_{1,n}\| + \|\tilde{x}_n - \tilde{y}_{2,n}\|}{\|\tilde{x}_n - \tilde{y}\|} \to 0. \tag{$**$}$$

Let d be a cluster point of $(\tilde{x}_n - \tilde{y})/\|\tilde{x}_n - \tilde{y}\|$. Because $\tilde{x}_n - \tilde{y} \in N_K(\tilde{y})$ (the projection onto K is positively homogeneous), d belongs to $N_K(\tilde{y})$. Since $d \neq 0$, we have

$$d \notin T_K(\tilde{y}).$$

On the other hand, $(**)$ implies that d is a cluster point of $(\tilde{y}_{i,n} - \tilde{y})/\|\tilde{x}_n - \tilde{y}\|$, for $i = 1, 2$. Since $\tilde{y}_{i,n} \in K_i$, we learn that $d \in T_{K_i}(\tilde{y})$, for each $i = 1, 2$. Altogether,

$$d \in \big(T_{K_1}(\tilde{y}) \cap T_{K_2}(\tilde{y})\big) \setminus T_K(\tilde{y}).$$

But this contradicts CHIP and the proof is thus complete. $\square$

Acknowledgements. We wish to thank two anonymous referees for their helpful comments. Thanks are also due to A. Bakan for referring us to [1, 2, 3] and to F. Deutsch for referring us to [4].

References

[1] A. G. Bakan: Normal pairs of cones in finite-dimensional spaces (Russian), In: Some problems in the theory of the approximation of functions, and their applications (Russian), Akad. Nauk Ukrain. SSR, Inst. Mat., Kiev (1988) 3–11.

[2] A. G. Bakan: Nonemptiness of classes of normal pairs of cones of transfinite order (Russian), Ukrain. Mat. Zh. 41(4) (1989) 531–536, 576; translation in: Ukrainian Mathematical Journal 41(4) (1989) 462-466.

[3] A. G. Bakan: The Moreau-Rockafellar equality for sublinear functionals (Russian), Ukrain. Mat. Zh. 41(8) (1989) 1011–1022, 1149; translation in Ukrainian Mathematical Journal 41(8) (1990) 861–871.

[4] A. G. Bakan, F. Deutsch, W. Li: CHIP, strong CHIP, and linear regularity of convex sets and their conifications, forthcoming.

[5] H. H. Bauschke, J. M. Borwein: On the convergence of von Neumann's alternating projection algorithm for two sets, Set-Valued Analysis 1(2) (1993) 185–212.

[6] H. H. Bauschke, J. M. Borwein: On projection algorithms for solving convex feasibility problems, SIAM Review 38(3) (1996) 367–426.

[7] H. H. Bauschke, J. M. Borwein: Conical open mapping theorems and regularity; in: J. Giles, B. Ninness (eds.) Functional Analysis, Optimization and Applications, Proceedings of the Centre for Mathematics and its Applications 36, Australian National University (1999) 1–10; Proceedings of the National Symposium on Functional Analysis, Optimization and Applications, March 1998.

[8] H. H. Bauschke, J. M. Borwein, A. S. Lewis: The method of cyclic projections for closed convex sets in Hilbert space; in: Y. Censor, S. Reich (eds.), Optimization and Nonlinear Analysis, American Mathematical Society, Contemporary Mathematics 204 (1997) 1–38; Proceedings on the Special Session on Optimization and Nonlinear Analysis, Jerusalem, May 1995.

[9] H. H. Bauschke, J. M. Borwein, W. Li: Strong conical hull intersection property, bounded linear regularity, Jameson's property (G), and error bounds in convex optimization, Mathematical Programming 86(1) (1999) 135–160.

[10] C. K. Chui, F. Deutsch, J. D. Ward: Constrained best approximation in Hilbert space II, Journal of Approximation Theory 71(2) (1992) 213–238.

[11] F. Deutsch: The angle between subspaces of a Hilbert space; in: S. P. Singh (ed.), Approximation Theory, Wavelets and Applications, Kluwer Academic Publishers (1995) 107–130.

[12] F. Deutsch: The role of the strong conical hull intersection property in convex optimization and approximation; in: C. K. Chui, L. L. Schumaker (eds.), Approximation Theory IX, 1998.

[13] F. Deutsch, W. Li, J. Swetits: Fenchel duality and the strong conical hull intersection property, Journal of Optimization Theory and Applications 102(3) (1999) 681–695.

[14] F. Deutsch, W. Li, J. D. Ward: Best approximation from the intersection of a closed convex set and a polyhedron in Hilbert space, weak Slater conditions, and the strong conical hull intersection property, SIAM J. Optim. 10(1) (1999) 252–268.

[15] F. Deutsch, W. Li, J. D. Ward: A dual approach to constrained interpolation from a convex subset of Hilbert space, Journal of Approximation Theory 90 (1997) 385–414.

[16] J.-B. Hiriart-Urruty, C. Lemaréchal: Convex Analysis and Minimization Algorithms I, Grundlehren der mathematischen Wissenschaften 305, Springer-Verlag, 1993.

[17] J.-B. Hiriart-Urruty, C. Lemaréchal: Convex Analysis and Minimization Algorithms II, Grundlehren der mathematischen Wissenschaften 306, Springer-Verlag, 1993.

[18] A. D. Ioffe: Codirectional compactness, metric regularity and subdifferential calculus, preprint.

[19] G. J. O. Jameson: The duality of pairs of wedges, Proceedings of the London Mathematical Society 24(3) (1972) 531–547.

[20] K. C. Kiwiel, B. Lopuch: Surrogate projection methods for finding fixed points of firmly nonexpansive mappings, SIAM Journal on Optimization 7(4) (1997) 1084–1102.

[21] A. S. Lewis: Convex analysis on the Hermitian matrices, SIAM Journal on Optimization 6(1) (1996) 164–177.

[22] A. S. Lewis, J.-S. Pang: Error bounds for convex inequality systems; in: J.-P. Crouzeix (ed.), Generalized Convexity Proceedings of the Fifth Symposium on Generalized Convexity, Luminy-Marseille (1997) 75–110.

[23] J.-S. Pang: Error bounds in mathematical programming, Mathematical Programming 79 (1997) 299–332.

[24] R. T. Rockafellar: Convex Analysis, Princeton University Press, Princeton, NJ, 1970.