Journal of Convex Analysis
Volume 6 (1999), No. 1, 115–140

Least Deviation Decomposition with Respect to a Pair of Convex Sets*

D. T. Luc
Université d'Avignon, Département de Mathématiques,
33, rue L. Pasteur, 84000 Avignon, France.
e-mail: dtluc@univ-avignon.fr

J. E. Martínez-Legaz
Universitat Autònoma de Barcelona, CODE
and Departament d'Economia i d'Història Econòmica, 08193 Bellaterra, Spain.
e-mail: JuanEnrique.Martinez@uab.es

A. Seeger
King Fahd University of Petroleum and Minerals, Department of
Mathematical Sciences, Dhahran 31261, Saudi Arabia.
e-mail: alberto.seeger@univ-avignon.fr

Received January 16, 1998
Revised manuscript received December 9, 1998

Let K_1 and K_2 be two nonempty closed convex sets in some normed space $(H, \|\cdot\|)$. This paper is concerned with the question of finding a "good" decomposition, with respect to K_1 and K_2, of a given element of the Minkowski sum $K_1 + K_2$. We introduce and discuss the concept of least deviation decomposition. This concept is an extension of the Moreau orthogonal decomposition with respect to a pair of mutually polar cones. Techniques of convex analysis are applied to obtain some sensitivity and duality results related to the decomposition problem.

Keywords: Least deviation decomposition, convex analysis, Moreau orthogonal decomposition

1991 Mathematics Subject Classification: 41 A 65, 52 A 41

1. Introduction

In a recent work by Martínez-Legaz and Seeger [9], a general formalism has been introduced to discuss the question of finding a "good" decomposition

$$z = \overline{y}_1 + \overline{y}_2, \text{ with } \overline{y}_1 \in K_1 \text{ and } \overline{y}_2 \in K_2, \tag{1.1}$$

of a vector z belonging to the Minkowski sum

$$K_1 + K_2 = \{y_1 + y_2 : y_1 \in K_1,\ y_2 \in K_2\}$$

of two given convex cones K_1 and K_2. The decomposition theory developed in [9] is based on the notion of efficiency. For the sake of completeness, we recall below the definition of

*Research supported by Dirección General de Investigación Científica y Técnica (DGICYT) under Project PB 95-0679, and by Comissionat per Universitats i Recerca de la Generalitat de Catalunya under Grant 1998SGR-00062.

ISSN 0944-6532 / $ 2.50

an efficient decomposition. In what follows we denote

$$D(z) := \{(y_1, y_2) \in K_1 \times K_2 : \quad z = y_1 + y_2\} \tag{1.2}$$

the set of all (admissible) decompositions of z.

Definition 1.1 ([9]). Let K_1 and K_2 be two convex cones in some real vector space H. A decomposition $(\overline{y}_1, \overline{y}_2) \in D(z)$ is said to be *efficient* (or minimal, or nondominated) if $(\overline{y}_1, \overline{y}_2)$ is an efficient point of the set $D(z)$ with respect to the partial order induced by $K_1 \times K_2$, i.e.

$$\left.\begin{array}{l}(y_1, y_2) \in D(z)\\(\overline{y}_1, \overline{y}_2) - (y_1, y_2) \in K_1 \times K_2\end{array}\right\} \Longrightarrow (y_1, y_2) - (\overline{y}_1, \overline{y}_2) \in K_1 \times K_2. \tag{1.3}$$

The above definition yields a powerful and elegant decomposition theory, but unfortunately it relies heavily on the conic structure of the sets K_1 and K_2. The decomposition problem (1.1) is also of interest in the case in which K_1 and K_2 are two arbitrary convex sets. In this paper we will place ourselves in this more general setting. Besides extending some results of [9], we will introduce several new concepts and discuss various aspects of the decomposition problem (1.1). As an alternative to the concept of efficient decomposition, we propose the notion of least deviation decomposition. Perhaps the main merit of this new notion is that of extending the classic concept of Moreau orthogonal decomposition to a much more general setting. Due to space limitation, we will not indulge in discussing the numerous applications of the least deviation decomposition theory. Two simple examples will be appropriate as illustration.

Example 1.2. Denote by $DC(\Omega)$ the class of functions that can be represented as difference of two convex functions defined on some compact convex set $\Omega \subset R^n$. Which is the "best" way of decomposing $f \in DC(\Omega)$ as difference of convex functions ? This is a question which is asked once and over again in the literature. Observe that $DC(\Omega)$ is the Minkowski sum of the convex cones

$$K_1 := \{g : \Omega \to R : g \text{ is convex }\},$$
$$K_2 := \{h : \Omega \to R : h \text{ is concave }\}.$$

To our knowledge, $DC(\Omega)$ has not a Hilbert space structure and therefore the concept of Moreau orthogonal decomposition does not apply. Trying to find an efficient decomposition [9] of a given $f \in DC(\Omega)$, turns out to be a very difficult task in practice. However, $DC(\Omega)$ can be equipped with a norm $\|\cdot\|$ (even with a complete norm; cf. [15]). The concept of least deviation decomposition of f does make sense in this case.

Example 1.3. Let S_n denote the set of symmetric matrices of order n-by-n. Recall that a matrix $Q \in S_n$ is said to be stochastically copositive if for each positive semidefinite matrix $X \in S_n$ with nonnegative entries, and each random vector x with zero mean and covariance matrix X, the expected value of x^TQx is nonnegative. It is known [6, Theorem 3.2.4] that a matrix $Q \in S_n$ is stochastically copositive if and only if it belongs to $K_1 + K_2$, where

$$K_1 = \{A \in S_n : A \text{ is positive semidefinite }\},$$
$$K_2 = \{B \in S_n : B \text{ has only nonnegative entries }\}.$$

In this particular example, K_1 and K_2 are closed convex cones with nonempty interiors, the space S_n being equipped with the standard trace inner-product.

Of course one may consider other examples in which K_1 and K_2 are not necessarily cones (for instance, finding a point in the intersection $A \cap B$ of two convex closed sets amounts to decomposing the origin 0 with respect to $K_1 = A$ and $K_2 = -B$). The purpose of this work is to develop the theoretical aspects of the decomposition problem. Some basic results obtained in this paper are subsequently used by Seeger [13] to design numerical algorithms that solve effectively the decomposition problem (1.1), at least in a Hilbert space setting.

The organization of the paper is as follows:

Section 2. On least deviation decompositions.
Section 3. Topological properties of the admissible decomposition mapping.
Section 4. Continuity of optimal values and optimal solutions.
Section 5. Subdifferential analysis of the least deviation function.
Section 6. Miscellaneous results.
Section 7. Decomposing with respect to moving sets.
Section 8. Decomposing with respect to cones.

2. On Least Deviation Decompositions

Unless otherwise specified, K_1 and K_2 are two nonempty closed convex sets in some common normed space $(H, \|\cdot\|)$. The class of such sets is denoted by $C(H)$.

Definition 2.1. Let $K_1, K_2 \in C(H)$. The pair $(\overline{y}_1, \overline{y}_2) \in H \times H$ is called a *least deviation decomposition* of z if

$$\left.\begin{aligned} &(\overline{y}_1, \overline{y}_2) \in D(z), \\ &\|\overline{y}_1 - \overline{y}_2\| \leq \|y_1 - y_2\| \text{ for all } (y_1, y_2) \in D(z). \end{aligned}\right\} \tag{2.1}$$

The first condition in (2.1) is just an admissibility requirement. The second condition in (2.1) says that the deviation $\|\overline{y}_1 - \overline{y}_2\|$ is smaller than the deviation $\|y_1 - y_2\|$ of any other admissible decomposition $(y_1, y_2) \in D(z)$. As shown in the next proposition, the least deviation problem

$$\text{Minimize}\{\|y_1 - y_2\| : (y_1, y_2) \in D(z)\} \tag{2.2}$$

is equivalent to the *minimal norm problem*

$$\text{Minimize}\{\|v\| : \ v \in [K_1, K_2]_z\}, \tag{2.3}$$

where

$$[K_1, K_2]_z := (-z + 2K_1) \cap (z - 2K_2). \tag{2.4}$$

Proposition 2.2. *Let $K_1, K_2 \in C(H)$ and $z \in K_1 + K_2$. One has:*

(i) *If v is a solution of* (2.3), *then $\left(\frac{z+v}{2}, \frac{z-v}{2}\right)$ is a solution of* (2.2);

(ii) *If (y_1, y_2) is a solution of* (2.2), *then $y_1 - y_2$ is a solution of* (2.3);

(iii) *The optimal values*

$$f(z) := \inf\{\|y_1 - y_2\| : \ (y_1, y_2) \in D(z)\} , \tag{2.5}$$

$$g(z) := \inf\{\|v\| : \ v \in [K_1, K_2]_z\} \tag{2.6}$$

are equal.

Proof. By introducing the variable $v = y_1 - y_2$, one clearly has

$$f(z) = \inf_{\substack{v \in H \\ (y_1, y_2) \in K_1 \times K_2}} \{\|v\| : \ y_1 - y_2 = v, \ \ y_1 + y_2 = z\}.$$

But,

$$\left.\begin{array}{l} y_1 - y_2 = v \\ y_1 + y_2 = z \end{array}\right\} \Leftrightarrow \ y_1 = \frac{z + v}{2} \ \text{ and } \ y_2 = \frac{z - v}{2}. \tag{2.7}$$

Hence,

$$\begin{aligned} f(z) &= \inf_{v \in H}\{\|v\| : \ \frac{z+v}{2} \in K_1 \ \text{ and } \ \frac{z-v}{2} \in K_2\} \\ &= \inf_{v \in H}\{\|v\| : \ v \in (-z + 2K_1) \cap (z - 2K_2)\} \\ &= g(z). \end{aligned}$$

This completes the proof of (iii). Parts (i) and (ii) follow from the equivalence (2.7). □

The above theorem provides us with a method for constructing a least deviation decomposition of z. From a theoretical point of view, the decomposition problem is a matter of projecting the origin $0 \in H$ onto the closed convex set $[K_1, K_2]_z$. Projection techniques can also be applied in the context of the next proposition.

Proposition 2.3. *Let $K_1, K_2 \in C(H)$ and $z \in K_1 + K_2$. Then the following statements are equivalent:*

(i) $(\overline{y}_1, \overline{y}_2)$ *is a least deviation decomposition of z;*

(ii) $\overline{y}_1$ *and* $\overline{y}_2$ *are projections of $z/2$ onto $K_1 \cap (z - K_2)$ and $K_2 \cap (z - K_1)$, respectively, i.e.*

$$\left\{\begin{array}{l} \overline{y}_1 \in \operatorname{Argmin}\left\{\left\|\frac{z}{2} - y_1\right\| : \ y_1 \in K_1 \cap (z - K_2)\right\}, \\ \overline{y}_2 \in \operatorname{Argmin}\left\{\left\|\frac{z}{2} - y_2\right\| : \ y_2 \in K_2 \cap (z - K_1)\right\}. \end{array}\right. \tag{2.8}$$

Proof. (ii) $\Rightarrow$ (i). Let $\overline{y}_1$ and $\overline{y}_2$ be as in (2.8). A lemma of Martínez-Legaz and Seeger [9] on decomposability through projections shows that the pair $(\overline{y}_1, \overline{y}_2)$ is a decomposition of z. This takes care of the admissibility concern. The least deviation requirement is shown as follows. Starting from the equality

$$\left\|\frac{z}{2} - \overline{y}_1\right\| = \inf\left\{\left\|\frac{z}{2} - y_1\right\| : \ y_1 \in K_1 \cap (z - K_2)\right\},$$

one gets

$$\|z - \overline{y}_1 - \overline{y}_1\| = \inf\{\|z - y_1 - y_1\| : y_1 \in K_1 \cap (z - K_2)\},$$

or equivalently,

$$\|\overline{y}_2 - \overline{y}_1\| = \inf\{\|y_2 - y_1\| : \ (y_1, y_2) \in D(z)\}.$$

(i) ⇒ (ii). Just read the above three equalities in the backward order. □

Corollary 2.4. *Let $K_1, K_2 \in C(H)$. One has:*

(i) *If $(H, \|\cdot\|)$ is a reflexive Banach space, then every $z \in K_1 + K_2$ has a least deviation decomposition;*

(ii) *If the norm $\|\cdot\|$ is strict, then each $z \in K_1 + K_2$ admits at most one least deviation decomposition.*

Proof. If $(H, \|\cdot\|)$ is a reflexive Banach space, then the function $y_1 \in H \mapsto \eta(y_1) := \left\|\frac{z}{2} - y_1\right\|$ attains its minimum over the closed convex set $K_1 \cap (z - K_2)$. Note that this set is nonempty whenever $z \in K_1 + K_2$. If the norm $\|\cdot\|$ is strict, the function η^2 is strictly convex; hence it may have only one minimum point over $K_1 \cap (z - K_2)$. □

The reader may wonder why we have chosen the concept of least deviation as a criterion for selecting a suitable decomposition of $z \in K_1 + K_2$. In fact, there are different reasons for this choice. For instance, from a formal point of view, one can consider the statistical notion of standard deviation

$$\sigma(y_1, y_2) := \left\{ \frac{1}{2} \left\| y_1 - \frac{y_1 + y_2}{2} \right\|^2 + \frac{1}{2} \left\| y_2 - \frac{y_1 + y_2}{2} \right\|^2 \right\}^{1/2}, \tag{2.9}$$

which is a measure of the deviation of the "sample" $\{y_1, y_2\}$ with respect to its average $(y_1 + y_2)/2$. A straightforward calculation shows that the quantity (2.9) reduces to

$$\sigma(y_1, y_2) = \frac{1}{2}\|y_1 - y_2\|,$$

that is to say, the deviation $\|y_1 - y_2\|$ and the standard deviation $\sigma(y_1, y_2)$ differ only by a multiplicative constant. Thus, the concepts of least deviation decomposition and least standard deviation decomposition coincide.

What happens if we replace the deviation $\|y_1 - y_2\|$ by the Euclidean norm

$$N(y_1, y_2) := [\|y_1\|^2 + \|y_2\|^2]^{1/2} \tag{2.10}$$

of the pair $(\|y_1\|, \|y_2\|)$? Is the minimization problem

$$\text{Minimize}\{N(y_1, y_2) : \ (y_1, y_2) \in D(z)\} \tag{2.11}$$

somehow related with the least deviation problem (2.2)? These questions are addressed next. Roughly speaking, the following proposition says that a least deviation decomposition is also a least Euclidean norm decomposition, and vice versa. Observe that a least Euclidean norm decomposition is a projection of $(0, 0) \in H \times H$ onto $D(z)$.

Proposition 2.5. *Let $K_1, K_2 \in C(H)$ and $z \in K_1 + K_2$. Assume that H is a Hilbert space. Then, the optimal solution of* (2.2) *is the same as the optimal solution of* (2.11). *Moreover, the optimal value*

$$e(z) := \min\{N(y_1, y_2) : \ (y_1, y_2) \in D(z)\} \tag{2.12}$$

is related to $f(z)$ by means of the identity

$$[f(z)]^2 = 2[e(z)]^2 - \|z\|^2. \tag{2.13}$$

Proof. From the Hilbert space parallelogram identity

$$\|y_1 + y_2\|^2 + \|y_1 - y_2\|^2 = 2\left[\|y_1\|^2 + \|y_2\|^2\right],$$

one gets

$$\|y_1 - y_2\| = \left\{2\left[\|y_1\|^2 + \|y_2\|^2\right] - \|y_1 + y_2\|^2\right\}^{1/2}.$$

Taking the minimum with respect to $(y_1, y_2) \in D(z)$ on both sides of the above equality, one obtains

$$\begin{aligned} f(z) &= \min_{(y_1,y_2)\in D(z)} \left\{2[\|y_1\|^2 + \|y_2\|^2] - \|z\|^2\right\}^{1/2} \\ &= \left\{2 \min_{(y_1,y_2)\in D(z)} [\|y_1\|^2 + \|y_2\|^2] - \|z\|^2\right\}^{1/2} \\ &= \left\{2[e(z)]^2 - \|z\|^2\right\}^{1/2}. \end{aligned}$$

This proves the identity (2.13), and also the first part of the proposition. □

There is yet another way of looking at the concept of least deviation. In a Hilbert space framework, consider the maximization problem

$$\text{Maximize}\{\langle y_1, y_2\rangle : \ (y_1, y_2) \in D(z)\}. \tag{2.14}$$

Proposition 2.6. *Let H, K_1, K_2, and z be as in* Proposition 2.5. *Then the optimal solution of* (2.2) *is equal to the optimal solution of* (2.14). *Moreover, the optimal value*

$$b(z) := \max\{\langle y_1, y_2\rangle : \ (y_1, y_2) \in D(z)\} \tag{2.15}$$

is related to $f(z)$ as indicated below:

$$[f(z)]^2 = \|z\|^2 - 4\, b(z). \tag{2.16}$$

Proof. Take $z \in K_1 + K_2$. For every $(y_1, y_2) \in D(z)$, one has

$$\|z\|^2 = \|y_1\|^2 + \|y_2\|^2 + 2\langle y_1, y_2\rangle.$$

By combining this with the identity

$$\|y_1\|^2 + \|y_2\|^2 = \|y_1 - y_2\|^2 + 2\langle y_1, y_2\rangle,$$

one gets

$$\|y_1 - y_2\|^2 = \|z\|^2 - 4\langle y_1, y_2\rangle. \tag{2.17}$$

It suffices now to take on both sides the minimum with respect to $(y_1, y_2) \in D(z)$. □

3. Topological Properties of the Admissible Decomposition Mapping

One can see the least deviation problem (2.2) as a particular instance of a parametric optimization problem. From this point of view, it is important to understand the behavior of the optimal value (2.5) as a function of the "parameter" $z \in H$. One would also like to know how the set of optimal solutions depends on this parameter. To answer these questions, it is helpful to study first the topological nature of the admissible decomposition mapping $D : H \to H \times H$ defined by the expression (1.2).

To start with, we record the following trivial result.

Proposition 3.1. *Let $K_1, K_2 \in C(H)$. Then*

(i) *the domain of the mapping D is* $\operatorname{dom} D := \{z \in H : D(z) \neq \phi\} = K_1 + K_2$*;*
(ii) *the graph of D is a closed convex set in $H \times (H \times H)$. In particular, for each $z \in K_1 + K_2$, the set $D(z)$ is closed and convex.*

In general, the mapping D is not bounded-valued. This is due to the fact that no boundedness assumption has been made on the sets K_1 and K_2. In connection with this issue, we recall the following notions:

Definition 3.2. Let C be a nonempty closed set in some normed space $(X, \|\cdot\|)$. Then

(i) C is said to be *boundedly compact* if every bounded sequence in C admits a convergent subsequence;
(ii) (cf. [11]) C is said to be *asymptotically compact* if every sequence $\{x_n/\|x_n\|\}_{n\in N}$, with $x_n \in C$ and $\lim_{n\to\infty} \|x_n\| = \infty$, admits a convergent subsequence.
(iii) (cf. [4]) The *asymptotic cone* of C, denoted by $\operatorname{Rec} C$, is the cone consisting of all limits $\lim_{n\to\infty} t_n c_n$, with $\{t_n\} \downarrow 0$ and $c_n \in C$.

Remark 3.3. If the closed set C is convex, then $\operatorname{Rec} C$ is the cone of all vectors $v \in H$ such that $x + tv \in C$ for every $t \in R_+$ and $x \in C$. Such a cone is sometimes denoted by O^+C (cf. Rockafellar [12, Section 8]). If the closed convex set C is boundedly compact, then it is also asymptotically compact.

In the next lemma we do not look at the mapping $D : H \to H \times H$ itself, but at its "asymptotic" version $z \in H \mapsto [\operatorname{Rec} D](z) := \operatorname{Rec} D(z)$.

Lemma 3.4. *Let $K_1, K_2 \in C(H)$ and $z \in K_1 + K_2$. Then $(v_1, v_2) \in [\operatorname{Rec} D](z)$ if and only if*

$$0 = v_1 + v_2, \quad \text{with } v_1 \in \operatorname{Rec} K_1 \text{ and } v_2 \in \operatorname{Rec} K_2. \tag{3.1}$$

Proof. Take any $(v_1, v_2) \in [\operatorname{Rec} D](z)$. Since $D(z) \subseteq K_1 \times K_2$, one has

$$[\operatorname{Rec} D](z) \subseteq \operatorname{Rec}(K_1 \times K_2) \subseteq \operatorname{Rec} K_1 \times \operatorname{Rec} K_2.$$

Hence, $v_1 \in \operatorname{Rec} K_1$ and $v_2 \in \operatorname{Rec} K_2$. Now, pick any $(y_1, y_2) \in D(z)$. We have $(y_1, y_2) + t(v_1, v_2) \in D(z)$ for all $t \in R_+$, and in particular $(y_1 + v_1) + (y_2 + v_2) = z$. Hence $v_1 + v_2 = 0$. Conversely, let v_1 and v_2 be as in (3.1). If $(y_1, y_2) \in D(z)$, then clearly $(y_1, y_2) + t(v_1, v_2) \in D(z)$ for all $t \in R_+$. This shows that $(v_1, v_2) \in \operatorname{Rec} D(z)$. ☐

The condition (3.1) says that the pair $(v_1, v_2) \in H \times H$ is a decomposition of $0 \in H$ with respect to the cones $\operatorname{Rec} K_1$ and $\operatorname{Rec} K_2$. As an immediate consequence of Lemma 3.4, one has:

Corollary 3.5. *The mapping* $\operatorname{Rec} D : H \to H \times H$ *is constant over its domain, i.e.*

$$[\operatorname{Rec} D](z) = [\operatorname{Rec} D](z') \quad \textit{for all } z, z' \in K_1 + K_2.$$

Corollary 3.6. *Let* $K_1, K_2 \in C(H)$*. Then the following statements are equivalent:*

(i) $[\operatorname{Rec} D](z) = \{(0,0)\}$ *for all* $z \in K_1 + K_2$*;*
(ii) $[\operatorname{Rec} D](z) = \{(0,0)\}$ *for some* $z \in K_1 + K_2$*.*
(iii) $\operatorname{Rec} K_1 \cap -\operatorname{Rec} K_2 = \{0\}$.

Now we can address the question relative to the boundedness of the mapping D.

Proposition 3.7. *Let* $K_1, K_2 \in C(H)$*. Consider the following statements:*

(i) $D(z)$ *is bounded for all* $z \in K_1 + K_2$*;*
(ii) $D(z)$ *is bounded for some* $z \in K_1 + K_2$*;*
(iii) $\operatorname{Rec} K_1 \cap -\operatorname{Rec} K_2 = \{0\}$.

Then one has the relationship (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii)*. Moreover, the above three statements are equivalent if one assumes that either* K_1 *or* K_2 *is asymptotically compact.*

Proof. The implication (i) $\Rightarrow$ (ii) is obvious. To prove that (ii) $\Rightarrow$ (iii), suppose that $D(z_0)$ is bounded for some $z_0 \in K_1+K_2$ and that there is a nonzero vector $u \in \operatorname{Rec} K_1 \cap - \operatorname{Rec} K_2$. Now, pick any (y_1, y_2) in $D(z_0)$. In this case one has $(y_1, y_2) + t(u, -u) \in D(z_0)$ for all $t \in R_+$. Obviously, this contradicts the boundedness of $D(z_0)$. To prove that (iii) $\Rightarrow$ (i) one needs a further assumption. Suppose, for instance, that K_1 is asymptotically compact. Take any $z \in K_1 + K_2$. If $D(z)$ was not bounded, one should have a sequence $\{(y_1^n, y_2^n)\}_{n\in N} \subset D(z)$ such that $\lim_{n\to\infty} \{\|y_1^n\| + \|y_2^n\|\} = +\infty$. Because $y_1^n + y_2^n = z$ for all $n \in N$, one must have

$$\lim_{n\to\infty} \|y_1^n\| = \lim_{n\to\infty} \|y_2^n\| = +\infty.$$

The asymptotic compactness of K_1 implies that $\{y_1^n / \|y_1^n\|\}_{n\in N}$ admits a subsequence converging to some limit point $u \neq 0$. Let $P \subset N$ be the index set of this subsequence. Now, observe that

$$\frac{y_2^n}{\|y_1^n\|} = \frac{z}{\|y_1^n\|} - \frac{y_1^n}{\|y_1^n\|}$$

converges to $-u$ as $n \to \infty$, with $n \in P$. In this way we have shown that there is a nonzero vector $u \in H$ such that $u \in \operatorname{Rec} K_1$ and $-u \in \operatorname{Rec} K_2$. This fact contradicts the assumption (iii). □

Remark 3.8. Since $D(z) = \{(y_1, z - y_1) : y_1 \in K_1 \cap (z - K_2)\}$, the boundedness of $D(z)$ is equivalent to the boundedness of $K_1 \cap (z - K_2)$.

Proposition 3.9. *Let $K_1, K_2 \in C(H)$ be such that* $\operatorname{Rec} K_1 \cap -\operatorname{Rec} K_2 = \{0\}$. *Suppose that either K_1 or K_2 is asymptotically compact. Then, D maps bounded sets of H into bounded sets of $H \times H$, i.e.*

$$D(B) := \bigcup_{z \in B} D(z) \text{ is bounded whenever } B \text{ is bounded .}$$

Proof. Suppose that K_1 is asymptotically compact, and let B be any bounded set in H. If $D(B)$ was not bounded, then one should have sequences $\{(y_1^n, y_2^n)\}_{n \in N}$ and $\{z^n\}_{n \in N}$ such that $(y_1^n, y_2^n) \in D(z^n)$, $z^n \in B$, $\lim_{n\to\infty}\{\|y_1^n\| + \|y_2^n\|\} = +\infty$. Since $y_1^n + y_2^n = z^n$ and $\{z^n\}_{n \in N}$ is bounded, one must have

$$\lim_{n\to\infty} \|y_1^n\| = \lim_{n\to\infty} \|y_2^n\| = +\infty.$$

As in Proposition 3.7, one can show that there is a nonzero vector $u \in H$ in the set $\operatorname{Rec} K_1 \cap -\operatorname{Rec} K_2$, a fact which contradicts our assumptions. □

Corollary 3.10. *Let $(H, \|\cdot\|)$ be a Banach space, and let $K_1, K_2 \in C(H)$ be as in* Proposition 3.9. *Then, the mapping D is locally Lipschitz on* $\operatorname{int}(K_1 + K_2)$, *i.e., for each $\overline{z} \in \operatorname{int}(K_1 + K_2)$ there exist a neighborhood $U \subseteq \operatorname{int}(K_1 + K_2)$ of $\overline{z}$ and a constant $L > 0$ such that*

$$D(z) \subseteq D(z') + L\|z - z'\|B \text{ for all } z, z' \in U,$$

where $B = \{(y_1, y_2) \in H \times H : \|y_1\| + \|y_2\| \leq 1\}$ is the unit ball in $H \times H$.

Proof. Suppose $K_1 + K_2$ has a nonempty interior, otherwise there is nothing to prove. The inverse mapping of D is given by

$$D^{-1}(y_1, y_2) = \begin{cases} \{y_1 + y_2\} & \text{if } (y_1, y_2) \in K_1 \times K_2, \\ \phi & \text{otherwise .} \end{cases}$$

The graph of D^{-1} is a closed convex set in $(H \times H) \times H$. Also, by Proposition 3.9 we know that D maps bounded sets of H into bounded sets of $H \times H$. The Lipschitz property of D follows then by applying Corollary 1 in Aubin and Cellina [1, p. 54]. □

Now we will discuss some continuity properties of the set-valued mapping D. The following continuity notions are well known and can be found in standard references (cf. [1], [2]).

Definition 3.11. Let X and Y be two topological spaces. A set-valued mapping $M : X \to Y$ is said to be lower- (resp. upper-) semicontinuous at $\overline{x} \in X$, if for every open set $V \subseteq Y$ with $V \cap M(\overline{x}) \neq \phi$ (resp. $M(\overline{x}) \subseteq V$), there is a neighborhood U of $\overline{x}$ such that $V \cap M(x) \neq \phi$ (resp. $M(x) \subseteq V$) for all $x \in U$. M is said to be closed if its graph is a closed set in the product space $X \times Y$.

Proposition 3.12. *Let $K_1, K_2 \in C(H)$. Then the following properties hold for the mapping D:*

(i) *D is upper-semicontinuous at any $z \in K_1 + K_2$, provided at least one of the sets K_1, K_2 is boundedly compact and* $\operatorname{Rec}K_1 \cap -\operatorname{Rec} K_2 = \{0\}$;

(ii) *D is lower-semicontinuous at any* $z \in \operatorname{int} K_1 + \operatorname{int} K_2$.

Proof. To prove the statement (i), we consider any particular $\overline{z} \in K_1 + K_2$. If D was not upper-semicontinuous at $\overline{z}$, then there should exist an open set $V \subseteq H \times H$ containing $D(\overline{z})$, a sequence $\{z^n\}_{n\in N}$ converging to $\overline{z}$, and a sequence $\{(y_1^n, y_2^n)\}_{n\in N}$ such that $(y_1^n, y_2^n) \in D(z^n)$, $(y_1^n, y_2^n) \notin V$. Suppose, for instance, that K_1 is boundedly compact. In this case, K_1 is also asymptotically compact and we can apply Proposition 3.9 to show that $\{y_1^n\}_{n\in N}$ is bounded. Moreover, we may suppose that a subsequence $\{y_1^n\}_{n\in P}$ converges to some $y_1 \in K_1$. Hence, $y_2^n = z^n - y_1^n$ converges to $y_2 := \overline{z} - y_1 \in K_2$ as $n \to \infty$, with $n \in P$. The convergence of $\{(y_1^n, y_2^n)\}_{n\in P}$ towards $(y_1, y_2) \in D(\overline{z})$ contradicts the fact that $(y_1^n, y_2^n) \notin V$ for all $n \in N$.

To prove the statement (ii), consider any particular $\overline{z} \in \operatorname{int} K_1 + \operatorname{int} K_2$. Let V_1 and V_2 be two open sets in H such that $V_1 \times V_2 \cap D(\overline{z}) \neq \phi$. Take any $(\tilde{y}_1, \tilde{y}_2)$ in this intersection, i.e. $\tilde{y}_1 \in K_1 \cap V_1, \tilde{y}_2 \in K_2 \cap V_2$ and $\tilde{y}_1 + \tilde{y}_2 = \overline{z}$. We need to construct a neighborhood U of $\overline{z}$ such that $V_1 \times V_2 \cap D(z) \neq \phi$ for all $z \in U$. For this purpose, we decompose $\overline{z}$ in the form $\overline{z} = \overline{y}_1 + \overline{y}_2$, with $\overline{y}_1 \in \operatorname{int} K_1$ and $\overline{y}_2 \in \operatorname{int} K_2$. We may suppose that the decomposition $(\overline{y}_1, \overline{y}_2)$ is close enough to $(\tilde{y}_1, \tilde{y}_2)$ so that

$$\{(y_1, y_2) \in H \times H : \|(y_1, y_2) - (\tilde{y}_1, \tilde{y}_2)\| \leq \|(\overline{y}_1, \overline{y}_2) - (\tilde{y}_1, \tilde{y}_2)\|\} \subseteq V_1 \times V_2. \tag{3.2}$$

Indeed, if the inclusion (3.2) does not hold, then we replace $(\overline{y}_1, \overline{y}_2)$ by the decomposition

$$(y_1^t, y_2^t) = (1-t)(\tilde{y}_1, \tilde{y}_2) + t(\overline{y}_1, \overline{y}_2),\ t \in]0, 1[.$$

In such a case, we have

$$\overline{z} = y_1^t + y_2^t, \text{ with } y_1^t \in \operatorname{int} K_1 \text{ and } y_2^t \in \operatorname{int} K_2,$$

and the new ball centered at $(\tilde{y}_1, \tilde{y}_2)$ has a radius

$$\|(y_1^t, y_2^t) - (\tilde{y}_1, \tilde{y}_2)\| = t\|(\overline{y}_1, \overline{y}_2) - (\tilde{y}_1, \tilde{y}_2)\|,$$

that is to say, a radius which can be made as small as desired by choosing t close to 0. So, we suppose that (3.2) holds. In this case there is a positive number ϵ with the property that

$$\{(y_1, y_2) \in H \times H : \|(y_1, y_2) - (\overline{y}_1, \overline{y}_2)\| < \epsilon\} \subseteq (K_1 \times K_2) \cap (V_1 \times V_2).$$

Define a neighborhood U of $\overline{z}$ by $U := \{z \in H : \|z - \overline{z}\| < \epsilon\}$. Now, for every z in this neighborhood, we consider the pair (y_1, y_2) given by

$$y_1 = \overline{y}_1 + \frac{z - \overline{z}}{2}, \quad y_2 = \overline{y}_2 + \frac{z - \overline{z}}{2}.$$

Observe that $y_1 + y_2 = z$, and

$$\|(y_1, y_2) - (\overline{y}_1, \overline{y}_2)\| = \left\| \left(\frac{z - \overline{z}}{2}, \frac{z - \overline{z}}{2} \right) \right\| = \|z - \overline{z}\| < \epsilon.$$

Hence, $(y_1, y_2) \in V_1 \times V_2 \cap D(z)$. This proves the lower-semicontinuity of D at $\overline{z}$. □

Remark 3.13. If $(H, \|\cdot\|)$ is a Banach space, then one can invoke Robinson-Ursescu's theorem to prove that D is lower-semicontinuous at each $z \in \operatorname{int}(K_1 + K_2)$. Indeed, the inverse mapping D^{-1} has a closed convex graph. Hence, D is lower-semicontinuous on the interior of its domain, cf. Theorem 1 in [1, p. 54].

Remark 3.14. Proposition 3.12 (ii) is of interest only if the sets K_1 and K_2 have interior points. For instance, in Example 1.3 one has

$$\operatorname{int} K_1 = \{A \in S_n : A \text{ is positive definite }\},$$
$$\operatorname{int} K_2 = \{B \in S_n : B \text{ has only positive entries }\}.$$

Example 1.2 is more involved. If $DC(\Omega)$ is equipped with the norm

$$\|f\|_\infty = \operatorname*{Max}_{x\in\Omega}|f(x)| ,$$

then $\operatorname{int} K_1 = \operatorname{int} K_2 = \emptyset$. One may think also of the norm (cf. [15])

$$\|f\| = \inf_{g+h=f} \operatorname{Max}\{\|g\|_\infty , \|h\|_\infty\} ,$$

but once again K_1 and K_2 fail to have interior points. If one defines

$$K_1 := \{g : \Omega \longrightarrow R : g \text{ is convex and of class } C^2\} ,$$
$$K_2 := \{h : \Omega \longrightarrow R : h \text{ is concave and of class } C^2\} ,$$

then the vector space $K_1 + K_2$ can be equipped with the usual C^2-norm

$$\|f\|_{C^2(\Omega)} := \|f\|_\infty + \|f'\|_\infty + \|f''\|_\infty ,$$

the meaning of each term being clear from the context. In this favorable setting, both cones K_1 and K_2 have interior points.

The interiority condition in Proposition 3.12 (ii) can not be omitted: notice that, according to Definition 3.11, $D : H \to H \times H$ is not lower-semicontinuous at any point in $(K_1 + K_2)\setminus \operatorname{int}(K_1 + K_2)$. Even the restriction of D to $\operatorname{dom} D = K_1 + K_2$ may fail to be lower-semicontinuous at boundary points of $\operatorname{dom} D$ (in this case, of course, $\operatorname{dom} D$ is equipped with the topology induced by the normed space H). To see this, consider the case when K_1 is the convex hull of $\{(x, 0, z) \in R^3 : x^2 + (z + 1)^2 = 1, x \geq 0, z \geq -1\} \cup \{(0, 1, 0)\}$, and K_2 is the segment with endpoints $(0, 0, 0)$ and $(0, -1, 0)$. The restriction of D to K_1+K_2 is not lower-semicontinuous at $(0, 0, 0)$, because $D(0, 0, 0) = \{(0, t, 0), (0, -t, 0) : t \in [0, 1]\}$, while $D(t, -t, \sqrt{1 - t^2} - 1) = \{(t, 0, \sqrt{1 - t^2} - 1), (0, -t, 0)\}$ for t close to 0.

According to the statement (i) in Proposition 3.12, the upper-semicontinuity of D requires some additional assumptions on the sets $K_1, K_2 \in C(H)$. The next example shows that the upper-semicontinuity of D can no longer be ensured if none of the sets K_1, K_2 is boundedly compact.

Example 3.15. Let us consider the space of sequences $H = \ell^2$ equipped with the norm

$$\|x\| = \left[\sum_{i=1}^{\infty} |x_i|^2\right]^{1/2} , \ x = (x_1, x_2, \dots) \in H,$$

and $K := \{x \in H : \|x\| \leq 1 \text{ and } x_i \geq 0, \; i = 1, 2, \ldots\}$. The sets $K_1 = K$ and $K_2 = -K$ are closed convex and bounded. Hence $\operatorname{Rec} K_1 = \operatorname{Rec} K_2 = \{0\}$. Observe that $\operatorname{Rec} K_1 \cap -\operatorname{Rec} K_2 = \{0\}$, but neither K_1 nor K_2 is boundedly compact. We will show that D is not upper-semicontinuous at the point $\overline{z} = (1/2, 0, 0, \ldots) \in K_1 + K_2$. For each $(y_1, y_2) \in D(\overline{z})$ one has $(y_1)_i \leq \sqrt{3}/2$ and $(y_2)_i \geq -\sqrt{3}/2$ for $i \geq 2$. Let us define

$$V_1 = \left\{x \in H : -\frac{1}{2} < x_1 < \frac{3}{2}, \; -\frac{1}{2} < x_i < \frac{\sqrt{3}}{2} + \left(\frac{1}{2}\right)^i, \quad i = 2, 3, \ldots \right\},$$

$$V_2 = \left\{x \in H : -\frac{3}{2} < x_1 < \frac{1}{2}, \; -\frac{\sqrt{3}}{2} - \left(\frac{1}{2}\right)^i < x_i < \frac{1}{2}, \quad i = 2, 3, \ldots \right\},$$

so that $V_1 \times V_2$ is an open set in $H \times H$ which contains $D(\overline{z})$. Now consider the sequence $\{z^n\}_{n \geq 2}$ whose general term is $z^n = \left(\frac{1}{2} - \frac{1}{n}, 0, 0, \ldots\right)$. This sequence converges to $\overline{z}$, and every set $D(z^n)$ contains a pair $\{(y_1^n, y_2^n)\}$ which is not in $V_1 \times V_2$. To see this, observe that for each $n \geq 2$ one can find a large i such that

$$s_n := \left[1 - \left(\frac{1}{2} - \frac{1}{n}\right)^2\right]^{1/2} > \frac{\sqrt{3}}{2} + \left(\frac{1}{2}\right)^i.$$

So, we define $y_1^n \in K_1$ as the vector having $\frac{1}{2} - \frac{1}{n}$ as first coordinate, s_n as i-th coordinate, and 0 elsewhere. The coordinates of $y_2^n \in K_2$ are all 0, except the i-th coordinate which is $-s_n$.

The next example shows that the upper-semicontinuity of D can no longer be ensured if one drops the assumption $\operatorname{Rec} K_1 \cap -\operatorname{Rec} K_2 = \{0\}$.

Example 3.16. In $H = R^2$, consider the sets $K_1 = R_+ \times \{0\}$ and $K_2 = R_- \times R_-$. Both sets are boundedly compact, but $\operatorname{Rec} K_1 \cap -\operatorname{Rec} K_2 = R_+ \times \{0\} \neq \{(0, 0)\}$. The mapping D fails to be upper-semicontinuous at $z = (0, 0)$. To see this, consider the open sets

$$V_1 = \left\{(x_1, x_2) \in R^2 : |x_2| < \frac{1}{x_1 + 2}, \; x_1 > -1\right\}, \; V_2 = -V_1.$$

One can see that $D(z) = \{((\alpha, 0), (-\alpha, 0)) : \alpha \in R_+\}$ is contained in $V_1 \times V_2$. Consider now the sequence $\{z^n\}_{n \in N}$ given by $z^n = \left(0, -\frac{1}{n}\right)$. This sequence converges to z but $D(z^n) = \left\{((\alpha, 0), \left(-\alpha, -\frac{1}{n}\right) : \alpha \in R_+\right\}$ is not contained in $V_1 \times V_2$.

In many applications the condition $\operatorname{Rec} K_1 \cap -\operatorname{Rec} K_2 = \{0\}$ does not hold, and therefore the mapping D may fail to be upper-semicontinuous. However, a property slightly weaker than upper-semicontinuity can be expected.

Definition 3.17 (cf. [3], [8]). Let X and Y be two normed spaces. A set-valued mapping $M : X \to Y$ is said to be upper-hemicontinuous at $\overline{x} \in X$ if for all $\epsilon > 0$ there is a neighborhood U of $\overline{x}$ such that $M(x) \subseteq M(\overline{x}) + \epsilon B_Y$ for all $x \in U$, where B_Y denotes the closed unit ball in Y.

Proposition 3.18. *Let $K_1, K_2 \in C(H)$. Suppose there are a compact set $A \subseteq K_1 \times K_2$ and a neighborhood U of $\overline{z} \in K_1 + K_2$ such that*

$$D(z) \subseteq A + \operatorname{Rec} D(z) \quad \textit{for all } z \in U. \tag{3.3}$$

Then D is upper-hemicontinuous at $\overline{z}$.

Proof. Suppose to the contrary that D is not upper-hemicontinuous at $\overline{z}$. Then, there exist a positive number ϵ, a sequence $\{z^n\}_{n\in N}$ converging to $\overline{z}$, and a sequence $\{(y_1^n, y_2^n)\}_{n\in N}$ such that

$$(y_1^n, y_2^n) \in D(z^n), \text{ and } (y_1^n, y_2^n) \notin D(\overline{z}) + \epsilon B \quad \text{for all } n \in N,$$

where B is the unit ball in $H \times H$. It follows from the condition (3.3) that

$$(y_1^n, y_2^n) = (a_1^n, a_2^n) + (v_1^n, v_2^n), \text{ with } (a_1^n, a_2^n) \in A \text{ and } (v_1^n, v_2^n) \in \operatorname{Rec} D(z^n).$$

By using Lemma 3.4 one gets $z^n = y_1^n + y_2^n = a_1^n + a_2^n$. We may assume that $\{(a_1^n, a_2^n)\}_{n\in N}$ converges to some $(a_1, a_2) \in A$. Hence, $\overline{z} = a_1 + a_2$, with $a_1 \in K_1$ and $a_2 \in K_2$, that is to say, $(a_1, a_2) \in D(\overline{z})$. Now, consider the set

$$Q := \{(a_1, a_2) + (v_1^n, v_2^n) : n \in N\}.$$

Since the mapping $\operatorname{Rec} D$ is constant over its domain (cf. Corollary 3.5), each (v_1^n, v_2^n) belongs to $\operatorname{Rec} D(\overline{z})$. Consequently, the set Q is contained in $D(\overline{z})$, and the distance from (y_1^n, y_2^n) to $D(\overline{z})$ can be estimated as follows:

$$\operatorname{dist}[(y_1^n, y_2^n); D(\overline{z})] \leq \operatorname{dist}[(y_1^n, y_2^n), (a_1 + v_1^n, a_2 + v_2^n)] \leq \operatorname{dist}[(a_1^n, a_2^n), (a_1, a_2)].$$

So, $\operatorname{dist}[(y_1^n, y_2^n); D(\overline{z})]$ converges to 0, contradicting the fact that $\{(y_1^n, y_2^n)\}_{n\in N}$ remains away from $D(\overline{z}) + \epsilon B$. □

Next on our agenda is a discussion on the continuity properties of the optimal value function f and the set of optimal solutions to the least deviation problem (2.2). Such a discussion is the object of the coming section.

4. Continuity of Optimal Values and Optimal Solutions

The purpose of this section is to study the continuity of the least deviation function $f : H \to [0, +\infty]$. By definition, f is the optimal value function associated to the parametric program (2.2), i.e.

$$f(z) := \begin{cases} \inf\{\|y_1 - y_2\| : (y_1, y_2) \in D(z)\} & \text{if } z \in K_1 + K_2, \\ +\infty & \text{otherwise .} \end{cases} \tag{4.1}$$

We will also study the behavior of the set of optimal solutions

$$S(z) := \{(y_1, y_2) \in D(z) : \|y_1 - y_2\| = f(z)\} \tag{4.2}$$

as a function of the parameter $z \in H$.

Proposition 4.1. *Let $K_1, K_2 \in C(H)$. Then,*

(i) *f is upper-semicontinuous at any $z \in \operatorname{int} K_1 + \operatorname{int} K_2$;*
(ii) *f is lower-semicontinuous at any point $\overline{z} \in K_1 + K_2$ at which D is upper-hemicontinuous (cf.* Proposition 3.18*).*

Proof. The first assertion is by the classical Berge's theorem and by Proposition 3.12 (ii). For the second assertion, let D be upper-hemicontinuous at $\overline{z} \in K_1 + K_2$ and let $\{z^n\}_{n\in N}$ be a sequence in $K_1 + K_2$ converging to $\overline{z}$. We have to show that

$$f(\overline{z}) \leq \liminf_{n\to\infty} f(z^n). \tag{4.3}$$

Let ϵ be an arbitrary positive number. Due to the upper-hemicontinuity of D, there exists an integer n_0 such that $D(z^n) \subseteq D(\overline{z}) + \epsilon B \times B$ for all $n \geq n_0$, where B is the closed unit ball in H. A simple calculation shows that

$$\begin{aligned} f(z^n) &= \inf\{\|y_1 - y_2\| : (y_1, y_2) \in D(z^n)\} \\ &\geq \inf\{\|y_1 - y_2\| : (y_1, y_2) \in D(\overline{z}) + \epsilon B \times B\} \\ &\geq \inf\{\|y_1 - y_2\| - \epsilon\|v_1 - v_2\| : (y_1, y_2) \in D(\overline{z}), (v_1, v_2) \in B \times B\} \\ &= f(\overline{z}) - \epsilon \cdot \operatorname{diam} B, \end{aligned}$$

where $\operatorname{diam} B = \sup\{\|v_1 - v_2\| : v_1, v_2 \in B\} = 2 < +\infty$. Since ϵ is arbitrary, we obtain (4.3) and complete the proof. □

Proposition 4.2. *Under the same assumptions as in* Proposition 3.9*, the least deviation function f maps bounded subsets of $K_1 + K_2$ into bounded subsets of R_+.*

Proof. It is immediate from Proposition 3.9. □

Remark 4.3. The assumption $\operatorname{Rec} K_1 \cap -\operatorname{Rec} K_2 = \{0\}$ is essential in the statement of Proposition 4.2. To see this, consider the case in which $H = R^2$, $K_1 = \{(x_1, x_2) \in R^2 : x_1 > 0,\ x_1 \cdot x_2 \geq 1\}$, and $K_2 = \{0\} \times R_-$. One can check that $]0, 1] \times \{0\}$ is a bounded subset of $K_1 + K_2$, but $f(z_1, 0)$ goes to $+\infty$ as $z_1 > 0$ decreases to 0. The lack of boundedness of $f(]0, 1] \times \{0\})$ is due to the fact that $\operatorname{Rec} K_1 \cap -\operatorname{Rec} K_2 = \{0\} \times R_+ \neq \{(0, 0)\}$.

We now turn our attention to the set of optimal solutions to the problem (2.2). The elements of $S(z)$ are nothing else but the least deviation decompositions of z.

In some special instances, it may happen that S is empty-valued at a given point $z \in K_1 + K_2$. For this reason it helps sometimes to consider an enlargement of the set $S(z)$, namely

$$S_\epsilon(z) := \{(y_1, y_2) \in D(z) : \|y_1 - y_2\| \leq f(z) + \epsilon\}. \tag{4.4}$$

Here ϵ is any nonnegative real number. With the choice $\epsilon = 0$ one recovers, of course, the set $S(z)$. Each element of $S_\epsilon(z)$ is called an ϵ-least deviation decomposition of z.

Proposition 4.4. *Let $K_1, K_2 \in C(H)$ and $\epsilon \in R_+$. Then, for each $z \in K_1 + K_2$, $S_\epsilon(z) \subseteq H \times H$ is a convex closed bounded set.*

Proof. Clearly $S_\epsilon(z)$ is convex and closed. If $(y_1, y_2) \in S_\epsilon(z)$, then

$$\|2y_1 - z\| = \|y_1 - y_2\| \leq f(z) + \epsilon,$$

$$\|z - 2y_2\| = \|y_1 - y_2\| \leq f(z) + \epsilon,$$

and consequently

$$\max\{\|y_1\|, \|y_2\|\} \leq \frac{1}{2}\{\|z\| + f(z) + \epsilon\}.$$

This proves that

$$\|(y_1, y_2)\| := \|y_1\| + \|y_2\| \leq \|z\| + f(z) + \epsilon, \tag{4.5}$$

that is to say, $S_\epsilon(z)$ is contained in a ball of radius $r(\epsilon, z) := \|z\| + f(z) + \epsilon$. □

Proposition 4.5. *Let $K_1, K_2 \in C(H)$. Then, the set-valued mapping $(\epsilon, z) \in R_+ \times H \mapsto S_\epsilon(z)$ is closed at any $(\overline{\epsilon}, \overline{z}) \in R_+ \times (\operatorname{int} K_1 + \operatorname{int} K_2)$. In particular, for any fixed $\epsilon \in R_+$, the mapping $z \in H \mapsto S_\epsilon(z)$ is closed over int $K_1 + \operatorname{int} K_2$.*

Proof. Consider any particular $(\overline{\epsilon}, \overline{z}) \in R_+ \times (\operatorname{int} K_1 + \operatorname{int} K_2)$. Let the sequences $\{(y_1^n, y_2^n)\}_{n\in N}, \{z^n\}_{n\in N}$, and $\{\epsilon_n\}_{n\in N}$ converge to $(\overline{y}_1, \overline{y}_2), \overline{z}$, and $\overline{\epsilon}$, respectively, and be such that $(y_1^n, y_2^n) \in S_{\epsilon_n}(z^n)$ for all $n \in N$. From Proposition 3.1 (ii) it follows that $(\overline{y}_1, \overline{y}_2) \in D(\overline{z})$. From the inequality $\|y_1^n - y_2^n\| \leq f(z^n) + \epsilon_n$, one gets $\|\overline{y}_1 - \overline{y}_2\| \leq \limsup_{n\to\infty} f(z^n) + \overline{\epsilon}$. But, according to Proposition 4.1 (i), the function f is upper-semicontinuous at $\overline{z}$. Thus, $\|\overline{y}_1 - \overline{y}_2\| \leq f(\overline{z}) + \overline{\epsilon}$.This proves that $(\overline{y}_1, \overline{y}_2) \in S_{\overline{\epsilon}}(\overline{z})$. □

Proposition 4.6. *Suppose that at least one of the sets $K_1, K_2 \in C(H)$ is boundedly compact. Then, the set-valued mapping $(\epsilon, z) \in R_+ \times H \mapsto S_\epsilon(z)$ is upper-semicontinuous at any $(\overline{\epsilon}, \overline{z}) \in R_+ \times (\operatorname{int} K_1 + \operatorname{int} K_2)$. In particular, for any fixed $\epsilon \in R_+$, the mapping $z \in H \mapsto S_\epsilon(z)$ is upper-semicontinuous over* $\operatorname{int} K_1 + \operatorname{int} K_2$.

Proof. Consider any particular $(\overline{\epsilon}, \overline{z}) \in R_+ \times (\operatorname{int} K_1 + \operatorname{int} K_2)$. Suppose to the contrary that S is not upper-semicontinuous at $(\overline{\epsilon}, \overline{z})$. In this case one can find an open set $V \subseteq H \times H$ containing $S_{\overline{\epsilon}}(\overline{z})$, a sequence $\{(\epsilon_n, z^n)\}_{n\in N}$ converging to $(\overline{\epsilon}, \overline{z})$, and a sequence $\{(y_1^n, y_2^n)\}_{n\in N}$ such that

$$(y_1^n, y_2^n) \in S_{\epsilon_n}(z^n),\ (y_1^n, y_2^n) \notin V.$$

By using the inequality (4.5) and the upper-semicontinuity of f at $\overline{z} \in \operatorname{int} K_1 + \int K_2$, one concludes that $\{(y_1^n, y_2^n)\}_{n\in N}$ is bounded. Due to the bounded compactness assumption, we may suppose that $\{(y_1^n, y_2^n)\}_{n\in N}$ admits a subsequence $\{(y_1^n, y_2^n)\}_{n\in P}$ which converges to some $(\overline{y}_1, \overline{y}_2)$. Proposition 4.5 implies that $(\overline{y}_1, \overline{y}_2) \in S_{\overline{\epsilon}}(\overline{z})$, contradicting the fact that $\{(y_1^n, y_2^n)\}_{n\in N}$ lies outside the open set V. □

5. Subdifferential Analysis of the Least Deviation Function

Continuity properties of the function f have been examined in Section 4. Now we explore the first-order behavior of f around a reference point, say $\overline{z} \in K_1 + K_2$. To carry out this analysis we rely on the following representation of the least deviation function:

$$f(z) = \inf_{A(y_1,y_2)=z} \{\|L(y_1, y_2)\| + \psi_{K_1 \times K_2}(y_1, y_2)\} . \tag{5.1}$$

Here ψ_K stands for the indicator function of the set K, i.e.

$$\psi_K(p) = \begin{cases} 0 & \text{if } p \in K, \\ +\infty & \text{otherwise} . \end{cases}$$

The continuous linear operators $A : H \times H \to H$ and $L : H \times H \to H$ are defined by $A(y_1, y_2) = y_1 + y_2$ and $L(y_1, y_2) = y_1 - y_2$, respectively.

Directly from the representation (5.1) it follows that:

Proposition 5.1. *The least deviation function $f : H \to [0, +\infty]$ is convex.*

Proof. f is the image of the convex function

$$(y_1, y_2) \in H \times H \mapsto m(y_1, y_2) := \|L(y_1, y_2)\| + \psi_{K_1 \times K_2}(y_1, y_2)$$

under the linear operator A; cf. [12, p. 38]. □

First-order information on the behavior of the convex function f is captured by the subdifferential mapping $\partial f : H \to H^*$. The subdifferential of f at a given point $\overline{z} \in K_1 + K_2$ is defined by

$$\partial f(\overline{z}) := \{w \in H^* : \ f(z) \geq f(\overline{z}) + \langle w, z - \overline{z}\rangle \text{ for all } z \in H\}. \tag{5.2}$$

Some comments on the notation are in order. In this section we suppose that $(H, \|\cdot\|)$ is a reflexive Banach space, H^* is the topological dual of H, and $\langle \cdot, \cdot \rangle : H^* \times H \to R$ stands for the duality pairing between H and H^*. The notation $\|\cdot\|_*$ refers to a norm in H^* which is dual to $\|\cdot\|$, i.e.

$$\|w\|_* = \sup_{\|h\|\leq 1} \langle w, h\rangle \quad \text{and} \quad \|h\| = \sup_{\|w\|_*\leq 1} \langle w, h\rangle. \tag{5.3}$$

It is our intention to derive a formula for computing the subdifferential $\partial f(\overline{z})$. In fact, we will obtain a formula which applies to a whole family $\{\partial_\epsilon f(\overline{z}) : \epsilon \in R_+\}$ of sets including $\partial f(\overline{z})$ as a particular case. Given a number $\epsilon \in R_+$, one defines the ϵ–subdifferential of f at $\overline{z}$ as the set

$$\partial_\epsilon f(\overline{z}) := \{w \in H^* : f(z) \geq f(\overline{z}) + \langle w, z - \overline{z}\rangle - \epsilon \quad \text{for all } z \in H\}. \tag{5.4}$$

Recall that if $\overline{p}$ belongs to the set $K \in C(H)$, then

$$\partial_\epsilon \psi_K(\overline{p}) = \{w \in H^* : \langle w, p - \overline{p}\rangle \leq \epsilon \ \text{ for all } p \in K\}$$

is referred to as the set of ϵ-normal directions to K at $\overline{p}$; see, for instance, Hiriart-Urruty [5].

Theorem 5.2. *Let $(H,\|\cdot\|)$ be a reflexive Banach space and let $K_1, K_2 \in C(H)$. Suppose $\overline{z} \in K_1 + K_2$ admits $(\overline{y}_1, \overline{y}_2)$ as a least deviation decomposition. Then, for every $\epsilon \in R_+$, one has*

$$\partial_\epsilon f(\overline{z}) = \bigcup_{\substack{\epsilon_1 \geq 0, \epsilon_2 \geq 0 \\ \epsilon_1 + \epsilon_2 = \epsilon}} \bigcup_{\substack{u \in M(\epsilon_1) \\ \alpha_1 \geq 0, \alpha_2 \geq 0 \\ \alpha_1 + \alpha_2 = \epsilon_2}} \{[u + \partial_{\alpha_1} \psi_{K_1}(\overline{y}_1)] \cap [-u + \partial_{\alpha_2} \psi_{K_2}(\overline{y}_2)]\},$$

where $M(\epsilon_1) := \{u \in H^* : \|u\|_* \leq 1, \ \langle u, \overline{y}_1 - \overline{y}_2 \rangle \geq f(\overline{z}) - \epsilon_1\}$.

Proof. Let $(\overline{y}_1, \overline{y}_2) \in S(\overline{z})$. The ϵ-subdifferential at $\overline{z}$ of the function

$$z \in H \mapsto f(z) = \inf\{m(y_1, y_2) : A(y_1, y_2) = z\}$$

is given by

$$\partial_\epsilon f(\overline{z}) = \{w \in H^* : A^* w \in \partial_\epsilon m(\overline{y}_1, \overline{y}_2)\}, \qquad \text{(cf. [5], [16])}$$

where $A^* : H^* \to H^* \times H^*$ denotes the adjoint operator of A, i.e. $A^* w = (w, w)$. By applying general calculus rules on ϵ-subdifferentials, one gets

$$\partial_\epsilon m(\overline{y}_1, \overline{y}_2) = \bigcup_{\substack{\epsilon_1 \geq 0, \epsilon_2 \geq 0 \\ \epsilon_1 + \epsilon_2 = \epsilon}} \{L^* \partial_{\epsilon_1} \|\cdot\|(L(\overline{y}_1, \overline{y}_2)) + \partial_{\epsilon_2} \psi_{K_1 \times K_2}(\overline{y}_1, \overline{y}_2)\}.$$

But,

$$\begin{aligned} L^* \partial_{\epsilon_1} \|\cdot\|(L(\overline{y}_1, \overline{y}_2)) &= \{(u, -u) \in H^* \times H^* : u \in \partial_{\epsilon_1} \|\cdot\|(\overline{y}_1 - \overline{y}_2)\} \\ &= \{(u, -u) \in H^* \times H^* : \|u\|_* \leq 1, \ \langle u, \overline{y}_1 - \overline{y}_2 \rangle \geq f(\overline{z}) - \epsilon_1\}, \end{aligned}$$

and

$$\partial_{\epsilon_2} \psi_{K_1 \times K_2}(\overline{y}_1, \overline{y}_2) = \bigcup_{\substack{\alpha_1 \geq 0, \alpha_2 \geq 0 \\ \alpha_1 + \alpha_2 = \epsilon_2}} \{\partial_{\alpha_1} \psi_{K_1}(\overline{y}) \times \partial_{\alpha_2} \psi_{K_2}(\overline{y}_2)\}.$$

Therefore, $w \in \partial_\epsilon f(\overline{z})$ if and only if there exist $(\epsilon_1, \epsilon_2) \in R_+ \times R_+$, $(\alpha_1, \alpha_2) \in R_+ \times R_+$, and $u \in H^*$ such that

$$\epsilon_1 + \epsilon_2 = \epsilon, \ \ \alpha_1 + \alpha_2 = \epsilon_2, \ \ u \in M(\epsilon_1), \ \ (w - u, w + u) \in \partial_{\alpha_1} \psi_{K_1}(\overline{y}_1) \times \partial_{\alpha_2} \psi_{K_2}(\overline{y}_2).$$

To complete the proof it suffices to write the last condition in the form

$$w \in [u + \partial_{\alpha_1} \psi_{K_1}(\overline{y}_1)] \cap [-u + \partial_{\alpha_2} \psi_{K_2}(\overline{y}_2)].$$

□

Remark 5.3. The formula stated in Theorem 5.2 can be written in several equivalent ways; for instance,

$$\partial_\epsilon f(\overline{z}) = \bigcup_{\substack{\alpha_1 \geq 0, \alpha_2 \geq 0 \\ \alpha_1 + \alpha_2 \leq \epsilon \\ u \in M(\epsilon - \alpha_1 - \alpha_2)}} \{[u + \partial_{\alpha_1} \psi_{K_1}(\overline{y}_1)] \cap [-u + \partial_{\alpha_2} \psi_{K_2}(\overline{y}_2)]\}. \tag{5.5}$$

The choice $\epsilon = 0$ yields a formula for the (exact) subdifferential of f at $\overline{z}$. This case deserves to be recorded as a separate result. The notation $N_K(\overline{p}) := \partial \psi_K(\overline{p})$ refers to the normal cone to $K \in C(H)$ at the point $\overline{p} \in K$.

Corollary 5.4. *With the same assumptions as in* Theorem 5.2, *one has the formula*

$$\partial f(\overline{z}) = \bigcup_{u \in M} \{[u + N_{K_1}(\overline{y}_1)] \cap [-u + N_{K_2}(\overline{y}_2)]\}\,, \tag{5.6}$$

where

$$M := \{u \in H^* : \|u\|_* \leq 1,\ \ \langle u, \overline{y}_1 - \overline{y}_2 \rangle = f(\overline{z})\}\,. \tag{5.7}$$

Further simplifications in the formula (5.7) occur in the cases described below.

Corollary 5.5. *With the same assumptions as in* Theorem 5.2, *one has:*

(i) *If* $\overline{z} \in 2(K_1 \cap K_2)$, *then* $f(\overline{z}) = 0$ *and*

$$\partial f(\overline{z}) = \bigcup_{\|u\|_* \leq 1} \left\{ \left[u + N_{K_1}\left(\frac{\overline{z}}{2}\right)\right] \cap \left[-u + N_{K_2}\left(\frac{\overline{z}}{2}\right)\right] \right\};$$

(ii) *If* $\overline{z} \notin 2(K_1 \cap K_2)$ *and if the norm* $\|\cdot\|_*$ *is strict, then* M *contains a single element, say* $\overline{u}$, *and*

$$\partial f(\overline{z}) = [\overline{u} + N_{K_1}(\overline{y}_1)] \cap [-\overline{u} + N_{K_2}(\overline{y}_2)]\,. \tag{5.8}$$

6. Miscellaneous Results

6.1. Optimality Conditions

Techniques of convex analysis have been used in the previous section to derive a formula for the subdifferential of the least deviation function. In fact, convex analysis provides us with useful tools for handling several questions related to the least deviation problem (2.2). By way of example, we record below a characterization of the set of optimal solutions to (2.2).

Theorem 6.1. *Let* $(H, \|\cdot\|)$ *be a reflexive Banach space. Let* $K_1, K_2 \in C(H)$ *and* $\overline{z} \in \operatorname{int} K_1 + \operatorname{int} K_2$. *Then, the following statements are equivalent:*

(i) $(\overline{y}_1, \overline{y}_2)$ *is a least deviation decomposition of* $\overline{z}$*;*

(ii) $\overline{y}_1 + \overline{y}_2 = \overline{z}$ *and there is some* $u \in M$ *such that*

$$[u + N_{K_1}(\overline{y}_1)] \cap [-u + N_{K_2}(\overline{y}_2)] \neq \phi.$$

Proof. We use the same notation as in the proof of Theorem 5.2. Under the "constraint qualification" assumption $\overline{z} \in \operatorname{int} K_1 + \operatorname{int} K_2$, the standard optimality conditions for the convex minimization problem

$$\text{Minimize}\{m(y_1, y_2) :\ A(y_1, y_2) = \overline{z}\} \tag{6.1}$$

are as follows: $(\overline{y}_1, \overline{y}_2)$ is an optimal solution to (6.1) if and only if

$$\begin{cases} A(\overline{y}_1, \overline{y}_2) = \overline{z}, \\ \text{Range}\, A^* \cap \partial m(\overline{y}_1, \overline{y}_2) \neq \phi. \end{cases} \tag{6.2}$$

The second condition in (6.2) says that the set

$$\partial m(\overline{y}_1, \overline{y}_2) = \{(u, -u) : u \in M\} + N_{K_1}(\overline{y}_1) \times N_{K_2}(\overline{y}_2)$$

intersects the diagonal Range $A^* = \{(w, w) : \ w \in H^*\}$. This amounts to saying that

$$(w - u, w + u) \in N_{K_1}(\overline{y}_1) \times N_{K_2}(\overline{y}_2) \tag{6.3}$$

for some $w \in H^*$ and $u \in M$. Now it suffices to observe that (6.3) can be written in the form $w \in [u + N_{K_1}(\overline{y}_1)] \cap [-u + N_{K_2}(\overline{y}_2)]$. □

Corollary 6.2. *Let* $H, K_1, K_2,$ *and* $\overline{z}$ *be as in* Theorem 6.1. *Then the least deviation function* f *is subdifferentiable at* $\overline{z}$ *in the sense that* $\partial f(\overline{z}) \neq \phi$.

6.2. Duality

The next topic addressed in this section also lies within the realm of convex analysis. Recall that the Legendre-Fenchel conjugate of f is the function $f^* : H^* \to R \cup \{+\infty\}$ given by

$$f^*(w) := \sup_{z \in H} \{\langle w, z\rangle - f(z)\} \quad \text{for all } w \in H^*.$$

The following proposition provides a formula for f^* when f is given by (4.1).

Proposition 6.3. *Let* H, K_1, K_2 *be as in* Theorem 6.1. *Then, for all* $w \in H^*$, *one has*

$$f^*(w) = \inf_{\|u\|_* \leq 1} \left\{\psi^*_{K_1}(w - u) + \psi^*_{K_2}(w + u)\right\}, \tag{6.4}$$

where $\psi^*_{K_1}$ *and* $\psi^*_{K_2}$ *denote the support functions of* K_1 *and* K_2, *respectively.*

Proof. For all $w \in H^*$, one has

$$\begin{aligned} f^*(w) &= \sup_{z \in K_1 + K_2} \left\{\langle w, z\rangle - \inf_{(y_1, y_2) \in D(z)} \|y_1 - y_2\|\right\} \\ &= \sup_{z \in K_1 + K_2} \sup_{\substack{(y_1, y_2) \in K_1 \times K_2 \\ y_1 + y_2 = z}} \{\langle w, z\rangle - \|y_1 - y_2\|\} \\ &= \sup_{(y_1, y_2) \in K_1 \times K_2} \{\langle w, y_1 + y_2\rangle - \|y_1 - y_2\|\}. \end{aligned}$$

Since $\|y_1 - y_2\| = \sup_{\|u\|_* \leq 1} \langle u, y_1 - y_2\rangle$, one obtains

$$f^*(w) = \sup_{(y_1, y_2) \in K_1 \times K_2} \inf_{\|u\|_* \leq 1} \{\langle w, y_1 + y_2\rangle - \langle u, y_1 - y_2\rangle\}.$$

But, in the above expression, it is possible to exchange the order of the supremum and the infimum (cf. [14]). Thus

$$\begin{aligned} f^*(w) &= \inf_{\|u\|_* \leq 1} \sup_{\substack{y_1 \in K_1 \\ y_2 \in K_2}} \{\langle w, y_1 + y_2\rangle - \langle u, y_1 - y_2\rangle\} \\ &= \inf_{\|u\|_* \leq 1} \left\{\sup_{y_1 \in K_1} \langle y_1, w - u\rangle + \sup_{y_2 \in K_2} \langle y_2, w + u\rangle\right\}. \end{aligned}$$

This completes the proof of formula (6.4). □

6.3. Minimization of f

Consider now the question of the minimization of the least deviation function f. In the next proposition we derive a simple expression for computing the infimal value of f.

Proposition 6.4. *Let $K_1, K_2 \in C(H)$. Then, the infimal value of f over H is equal to the gap between the sets K_1 and K_2, i.e.*

$$\inf_{z\in H} f(z) = \delta(K_1, K_2) := \inf\{\|y_1 - y_2\| : y_1 \in K_1,\ y_2 \in K_2\}. \tag{6.5}$$

Proof. As a matter of direct computation, one has:

$$\begin{aligned}\inf_{z\in H} f(z) &= \inf_{z\in K_1+K_2} \inf_{\substack{y_1\in K_1\\ y_2\in K_2}} \left\{\|y_1 - y_2\| + \psi_{\{0\}}(y_1 + y_2 - z)\right\}, \\ &= \inf_{\substack{y_1\in K_1\\ y_2\in K_2}} \inf_{z\in K_1+K_2} \left\{\|y_1 - y_2\| + \psi_{\{0\}}(y_1 + y_2 - z)\right\}.\end{aligned}$$

Now, observe that the inner infimum is attained at $z = y_1 + y_2$. This shows that $\inf_H f$ is equal to $\delta(K_1, K_2)$. □

A different expression for $\inf_H f$ is obtained by using the duality formula (6.4).

Proposition 6.5. *Let H, K_1, K_2 be as in* Theorem 6.1. *Denote by*

$$W_{\min}(K_1, K_2) := \inf_{\|u\|_*\leq 1} \left\{\psi^*_{K_1}(-u) + \psi^*_{K_2}(u)\right\}$$

the minimal width of the pair (K_1, K_2). Then

$$\inf_{z\in H} f(z) = -W_{\min}(K_1, K_2). \tag{6.6}$$

Proof. The result follows from Proposition 6.3. It suffices to take $w = 0$ in the formula (6.4). □

7. Decomposing With Respect to Moving Sets

This rather technical section deals with the algorithmic aspect of the decomposition problem. In many applications one has to consider an "approximate" version of the decomposition problem, namely

$$z = y_1 + y_2, \text{ with } y_1 \in K_1^n \text{ and } y_2 \in K_2^n. \tag{7.1}$$

Here $\{K_1^n\}_{n\in N}$ and $\{K_2^n\}_{n\in N}$ are sequences of simple structured sets which serve to approximate K_1 and K_2, respectively. In what follows, the notation

$$D^n(z) \ : \ = \{(y_1, y_2) \in K_1^n \times K_2^n : y_1 + y_2 = z\}, \tag{7.2}$$

$$f^n(z) \ : \ = \inf\{\|y_1 - y_2\| : (y_1, y_2) \in D^n(z)\}, \tag{7.3}$$

$$S^n(z) \ : \ = \{(y_1, y_2) \in D^n(z) : \|y_1 - y_2\| = f^n(z)\}, \tag{7.4}$$

will refer, respectively, to the admissible set, the optimal value, and the set of optimal solutions to the approximate problem

$$\text{Minimize}\{\|y_1 - y_2\| : (y_1, y_2) \in D^n(z)\}. \tag{7.5}$$

The specific goal of this section is to examine the limiting behavior of $D^n(z)$, $f^n(z)$, and $S^n(z)$, as n goes to $+\infty$. Here z is no longer regarded as a parameter, but as a fixed element in H. The qualitative behavior of the sequences $\{K_1^n\}_{n\in N}$ and $\{K_2^n\}_{n\in N}$ will be expressed in terms of standard convergence notions.

Definition 7.1 (cf. [7], [3]). Let $\{K^n\}_{n\in N}$ be a sequence of sets in $C(H)$. The lower and the upper limits of $\{K^n\}_{n\in N}$ are the sets defined by

$$\liminf K^n := \{p \in H : p = \lim p^n, \quad p^n \in K^n \text{ for all } n \in N\},$$

and

$$\limsup K^n := \{p \in H : p = \lim_{\substack{n\to\infty \\ n\in N'}} p^n, \quad p^n \in K^n \text{ for all } n \in N' \subset N\},$$

respectively. If $\liminf K^n = \limsup K^n$, then the common limit is denoted by $\lim K^n$ and one says that $\{K^n\}_{n\in N}$ is convergent in the sense of Painlevé-Kuratowski.

Proposition 7.2. *Let $K_1, K_2, K_1^n, K_2^n \in C(H)$ be such that*

$$\limsup K_1^n \subseteq K_1 \text{ and } \limsup K_2^n \subseteq K_2. \tag{7.6}$$

Then,

$$\limsup D^n(z) \subseteq D(z) \quad \text{for all } z \in K_1 + K_2. \tag{7.7}$$

Proof. It is immediate. □

Proposition 7.3. *Let $K_1, K_2, K_1^n, K_2^n \in C(H)$ be such that*

$$\operatorname{int}(K_1 \times K_2) \subseteq \bigcup_{m\in N} \bigcap_{n\geq m} \operatorname{int}(K_1^n \times K_2^n). \tag{7.8}$$

Then,

$$D(z) \subseteq \liminf D^n(z) \quad \text{for all } z \in \operatorname{int} K_1 + \operatorname{int} K_2. \tag{7.9}$$

Proof. Suppose that $\operatorname{int}(K_1 \times K_2)$ is nonempty, and take $z \in \operatorname{int} K_1 + \operatorname{int} K_2$. Pick up any pair (y_1, y_2) in $D(z)$. To prove that $(y_1, y_2) \in \liminf D^n(z)$, we consider first the case $(y_1, y_2) \in \operatorname{int}(K_1 \times K_2)$. The assumption (7.8) implies that there is some integer $m \in N$ such that

$$(y_1, y_2) \in \operatorname{int}(K_1^n \times K_2^n) \quad \text{for all } n \geq m.$$

Hence $(y_1, y_2) \in D^n(z)$ for all $n \geq m$, and, consequently, $(y_1, y_2) \in \liminf D^n(z)$. Consider now the case $(y_1, y_2) \notin \operatorname{int}(K_1 \times K_2)$. Decompose z in the form $z = \overline{y}_1 + \overline{y}_2$ with $\overline{y}_1 \in \operatorname{int} K_1$ and $\overline{y}_2 \in \operatorname{int} K_2$. For each $t \in]0, 1[$, the new pair

$$(y_1^t, y_2^t) := (1-t)(y_1, y_2) + t(\overline{y}_1, \overline{y}_2)$$

belongs to $D(z) \cap \operatorname{int}(K_1 \times K_2)$. As we have seen before, this implies that $(y_1^t, y_2^t) \in \liminf D^n(z)$. Since the set $\liminf D^n(z)$ is closed, it follows that

$$(y_1, y_2) = \lim_{t \to 0}(y_1^t, y_2^t) \in \liminf D^n(z).$$

□

Remark 7.4. Proposition 7.3 can be proved also by using general results on the intersection of two lower-semicontinuous set-valued mappings.

Corollary 7.5. *Let H be a finite dimensional space. Let $K_1, K_2, K_1^n, K_2^n \in C(H)$ be such that*

$$K_1 \subseteq \liminf K_1^n \quad \textit{and} \quad K_2 \subseteq \liminf K_2^n. \tag{7.10}$$

Then,

$$D(z) \subseteq \liminf D^n(z) \quad \textit{for all } z \in \operatorname{int} K_1 + \operatorname{int} K_2.$$

Proof. In view of the preceding proposition, we need only to show that the inclusion (7.8) holds. Take any $(\overline{y}_1, \overline{y}_2) \in \operatorname{int}(K_1 \times K_2)$. If $(\overline{y}_1, \overline{y}_2)$ were not in the set appearing on the right hand side of (7.8), then for all $m \in N$ there would be some $n \geq m$ such that

$$(\overline{y}_1, \overline{y}_2) \notin \operatorname{int}(K_1^n \times K_2^n). \tag{7.11}$$

Let $n(m)$ be the smallest integer greater than m such that (7.11) holds. For each n in the index set $N' := \{n(m) : m \in N\}$ one can separate $(\overline{y}_1, \overline{y}_2)$ from the convex set $K_1^n \times K_2^n$, that is to say, one can find a normalized vector $w^n = (w_1^n, w_2^n) \in H^* \times H^*$ such that

$$\langle w^n, (\overline{y}_1, \overline{y}_2)\rangle \geq \langle w^n, (y_1, y_2)\rangle \quad \text{for all } (y_1, y_2) \in K_1^n \times K_2^n.$$

We may assume that $\{w^n\}_{n \in N'}$ converges to some $w = (w_1, w_2) \neq (0, 0)$. Hence,

$$\langle w, (\overline{y}_1, \overline{y}_2)\rangle \geq \langle w, (y_1, y_2)\rangle \tag{7.12}$$

for all $y_1 \in \liminf K_1^n$ and $y_2 \in \liminf K_2^n$. Due to the assumption (7.10), the inequality (7.12) holds, in particular, for all $y_1 \in K_1$ and $y_2 \in K_2$. This contradicts the fact that $(\overline{y}_1, \overline{y}_2) \in \operatorname{int}(K_1 \times K_2)$. □

The next proposition describes the limiting behavior of the sequence $\{f^n(z)\}_{n \in N}$. Recall that $\{K^n\}_{n \in N}$ is said to be uniformly compact if all the sets K^n, $n \in N$, are contained in some common compact set.

Proposition 7.6. *Let $K_1, K_2, K_1^n, K_2^n \in C(H)$. Then,*

(i) *If* (7.6) *holds, and if either $\{K_1^n\}_{n \in N}$ or $\{K_2^n\}_{n \in N}$ is uniformly compact, then*

$$f(z) \leq \liminf f^n(z) \quad \textit{for all } z \in K_1 + K_2;$$

(ii) *If* (7.8) *holds, then*

$$\limsup f^n(z) \leq f(z) \quad \textit{for all } z \in \operatorname{int} K_1 + \operatorname{int} K_2.$$

Proof. Part (ii) follows from Proposition 7.3. The proof of the part (i) is based on Proposition 7.2 and the following fact: If $\limsup D^n(z) \subseteq D(z)$, and if either $\{K_1^n\}_{n\in N}$ or $\{K_2^n\}_{n\in N}$ is uniformly compact, then for every $\epsilon > 0$, there is an integer $n_0 \in N$ such that $D^n(z) \subseteq D(z) + \epsilon B \times B$ for all $n \geq n_0$, where B is the closed unit ball in H. Thus, we are in the same kind of situation as in the proof of Proposition 4.1 (ii). □

Finally, we discuss the limiting behavior of the sequence $\{S^n(z)\}_{n\in N}$.

Proposition 7.7. *Let the sets $K_1, K_2, K_1^n, K_2^n \in C(H)$ be such that (7.6) and (7.8) hold. Assume that either $\{K_1^n\}_{n\in N}$ or $\{K_2^n\}_{n\in N}$ is uniformly compact. Then,*

$$\limsup S^n(z) \subseteq S(z) \quad \text{for all } z \in \operatorname{int} K_1 + \operatorname{int} K_2. \tag{7.13}$$

Proof. Let $z \in \operatorname{int} K_1 + \operatorname{int} K_2$ and $(y_1, y_2) \in \limsup S^n(z)$. Then, for some index set $N' \subseteq N$, one can write

$$(y_1, y_2) = \lim_{\substack{n\to\infty \\ n\in N'}} (y_1^n, y_2^n),$$

with

$$(y_1^n, y_2^n) \in D^n(z) \text{ and } f^n(z) = \|y_1^n - y_2^n\| \text{ for all } n \in N' \subseteq N.$$

But, Proposition 7.2 shows that $(y_1, y_2) \in D(z)$, and Proposition 7.6 yields

$$f(z) = \lim_{n\in N} f^n(z) = \lim_{n\in N'} f^n(z) = \|y_1 - y_2\|.$$

In this way one proves that $(y_1, y_2) \in S(z)$. □

8. Decomposing With Respect to Cones

In a wide range of decomposition problems, the underlying sets K_1 and K_2 turn out to be cones. This section examines this particular case more closely, and establishes some links with the earlier work by Martínez-Legaz and Seeger [9].

In what follows we denote by $Q(H)$ the class of closed convex cones in the space H. Our first two results concern the mapping D.

Proposition 8.1. *If K_1 and K_2 belong to $Q(H)$, then the graph of D is a closed convex cone in $H \times (H \times H)$.*

Proof. It is immediate. □

Proposition 8.2. *Let $(H, \|\cdot\|)$ be a Banach space, and let $K_1, K_2 \in Q(H)$ be such that $K_1 + K_2 = H$. Then, the set-valued mapping D is Lipschitz, i.e. there is a constant $L > 0$ such that*

$$D(z) \subseteq D(z') + L\|z - z'\|B \quad \text{for all } z, z' \in H,$$

where B is the unit ball in $H \times H$.

Proof. Apply Corollary 3 in Aubin and Cellina [1, p. 55]. □

Next we state some results related to the least deviation function f.

Proposition 8.3. *Let K_1, K_2 belong to $Q(H)$. Then, the least deviation function f is positively homogeneous.*

Proof. It is immediate. ☐

By combining Propositions 5.1 and 8.3, one sees that f is a sublinear function. Therefore, the lower-semicontinuous hull $c\ell f$ of f is a support function. More precisely, if $K^- := \{\xi \in H^* : \langle \xi, p \rangle \leq 0 \text{ for all } p \in K\}$ denotes the negative dual cone of K, then one has:

Proposition 8.4. *Let $(H, \|\cdot\|)$ be a reflexive Banach space, and let $K_1, K_2 \in Q(H)$. Then,*

$$[c\ell\, f](z) = \psi^*_\Omega(z) := \sup_{w \in \Omega} \langle w, z \rangle \quad \text{for all } z \in H,$$

where

$$\Omega := \bigcup_{\|u\|_* \leq 1} \{[u + K_1^-] \cap [-u + K_2^-]\}. \tag{8.1}$$

Hence, the Legendre-Fenchel conjugate f^* of f is equal to the indicator function of the set Ω.

Proof. The lower-semicontinuous hull of the sublinear function f coincides with the support function of the set $\Omega = \partial f(0)$. Now, observe that $(\overline{y}_1, \overline{y}_2) = (0, 0)$ is a least deviation decomposition of $\overline{z} = 0$. Formula (8.1) follows directly from Corollary 5.4. Finally, the Legendre-Fenchel conjugate f^* is given by $f^* = [c\ell\, f]^* = [\psi^*_\Omega]^* = \psi_\Omega$. ☐

Further information on f can be derived under additional assumptions regarding the relative positioning of the cones K_1 and K_2.

Proposition 8.5. *Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space, and let $K_1, K_2 \in Q(H)$. If the cones K_1 and K_2 are mutually obtuse in the sense that*

$$\langle y_1, y_2 \rangle \leq 0 \quad \text{for all } y_1 \in K_1 \text{ and } y_2 \in K_2,$$

then

$$\|z\| \leq f(z) \quad \text{for all } z \in K_1 + K_2. \tag{8.2}$$

The converse is also true.

Proof. Let $z \in K_1 + K_2$. Mutual obtusity implies that

$$b(z) := \max_{(y_1, y_2) \in D(z)} \langle y_1, y_2 \rangle \leq 0.$$

It suffices now to apply the identity (2.16). To prove the converse, suppose that (8.2) holds, and that $\langle y_1, y_2 \rangle > 0$ for some pair $(y_1, y_2) \in K_1 \times K_2$. In this case,

$$f(y_1 + y_2) \leq \|y_1 - y_2\| < \|y_1 + y_2\|,$$

which is clearly a contradiction. ☐

Proposition 8.6. *Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space, and let $K_1, K_2 \in Q(H)$. Then z admits an acute decomposition, i.e. there is some $(y_1, y_2) \in D(z)$ such that $\langle y_1, y_2\rangle \geq 0$, if and only if* $f(z) \leq \|z\|$.

Proof. The existence of an acute decomposition of z is equivalent to $b(z) \geq 0$. The inequality $f(z) \leq \|z\|$ follows then from the identity (2.16). □

The next theorem deals with the case in which K_1 and K_2 are two mutually polar cones in a Hilbert space $(H, \langle\cdot,\cdot\rangle)$, i.e. $K_1 = K_2^-$ and $K_2 = K_1^-$. This is a particular instance of a pair of mutually obtuse closed convex cones.

Theorem 8.7. *Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space, and let $K_1, K_2 \in Q(H)$. Consider the following assertions:*

(i) *K_1 and K_2 are mutually polar;*
(ii) *$f = \|\cdot\|$;*
(iii) *for each $z \in H$, there is a decomposition $(y_1, y_2) \in D(z)$ such that $\|y_1 - y_2\| = \|z\|$;*
(iv) *each $z \in H$ admits a decomposition $(y_1, y_2) \in D(z)$ satisfying the orthogonality condition $\langle y_1, y_2\rangle = 0$.*

Then, one has the relationship (i) $\Leftrightarrow$ (ii) $\Rightarrow$ (iii) $\Leftrightarrow$ (iv).

Proof. (i) $\Rightarrow$ (iv). This implication corresponds to a celebrated theorem due to Moreau [10]. In fact, the decomposition mentioned in (iv) is unique, and it is known as the Moreau orthogonal decomposition of the given $z \in H$.

(iii) $\Leftrightarrow$ (iv). It follows from the identity (2.17). We note, incidentally, that a decomposition like in (iii) is necessarily unique. In fact, it coincides with the Moreau orthogonal decomposition.

(i) $\Rightarrow$ (ii). The Minkowski sum of the cones K_1 and K_2 is the whole space H. Take any $z \in H$. From Proposition 8.5 we know that $\|z\| \leq f(z)$. To prove the reverse inequality, it suffices to show that z admits an acute decomposition (cf. Proposition 8.6). But, this is clearly the case since the Moreau orthogonal decomposition is acute.

(ii) $\Rightarrow$ (i). According to Proposition 8.5, the condition $f(z) = \|z\|$ implies that $K_1 \subset K_2^-$ and $K_2 \subset K_1^-$. Let us prove, for instance, the reverse inclusion $K_2^- \subset K_1$. Take any $z \in K_2^-$ and let (y_1, y_2) be a least deviation decomposition of z. Then

$$\|y_1 - y_2\| = f(z) = \|z\| = \|y_1 + y_2\|.$$

Hence, $\langle y_1, y_2\rangle = 0$ and

$$0 \leq \|y_2\|^2 = \langle y_2, y_2\rangle = \langle y_1 + y_2, y_2\rangle = \langle z, y_2\rangle \leq 0.$$

It follows that $y_2 = 0$ and $z = y_1 \in K_1$. This proves the inclusion $K_2^- \subset K_1$. □

Remark 8.8. That K_1 and K_2 are cones has not been used in the previous three results. In fact, Theorem 8.7 and Propositions 8.5 and 8.6 apply to arbitrary sets in $C(H)$.

We end this paper by mentioning in an explicit way one of the most important conclusions of Theorem 8.7:

> "If K_1 and K_2 are mutually polar cones in a Hilbert space H, then for any $z \in H$, the least deviation decomposition of z coincides with its Moreau orthogonal decomposition."

The above statement should not be underestimated. It says, in particular, that the concept of least deviation decomposition is a natural extension of the Moreau decomposition to the case in which H is not necessarily a Hilbert space, or the sets $K_1, K_2 \in C(H)$ are not necessarily cones.

Acknowledgements. Thanks are due to Jean-Paul Penot for stimulating criticism.

References

[1] J. P. Aubin, A. Cellina: Differential Inclusions, Springer-Verlag, Berlin et al., 1984.

[2] J. P. Aubin, I. Ekeland: Applied Nonlinear Analysis, J. Wiley & Sons, New York, 1984.

[3] J. P. Aubin, H. Frankowska: Set-valued Analysis, Birkhäuser, Boston, 1990.

[4] J. P. Dedieu: Cône asymptote d'un ensemble non convexe, C. R. Acad. Sci. Paris 285 (1977) 501–503.

[5] J. B. Hiriart-Urruty: ϵ-subdifferential calculus, In: Convex Analysis and Optimization, J. P. Aubin, R. B. Vinter (eds.), Research Notes in Mathematics 57, Pitman, New York (1982) 43–92.

[6] D. H. Jacobson: Extensions of Linear Quadratic Control, Optimization and Matrix Theory, Academic Press, London, 1977.

[7] K. Kuratowski: Topology, Vol. I, Transl. J. Jaworowski, Academic Press, New York, 1966.

[8] D. T. Luc, J. P. Penot: Convergence of asymptotic directions, Preprint 1994, University of Pau, France.

[9] J. E. Martínez-Legaz, A. Seeger: A general cone decomposition theory based on efficiency, Math. Programming 65 (1994) 1–20.

[10] J. J. Moreau: Décomposition orthogonale d'un espace Hilbertien selon deux cônes mutuellement polaires, C. R. Acad. Sci. Paris 225 (1962) 238–240.

[11] J. P. Penot: Compact nets, filters and relations, J. Math. Anal. Appl. 93 (1983) 400–417.

[12] R. T. Rockafellar: Convex Analysis, Princeton Univ. Press, Princeton, 1970.

[13] A. Seeger: Alternating projection and decomposition with respect to two convex sets, Mathematica Japonica 47 (1998) 273-280.

[14] M. Sion: On general minimax theorems, Pacific J. Math. 8 (1958) 171–176.

[15] X. D. H. Truong: Banach spaces of d.c. functions and quasidifferentiable functions, Acta Math. Vietnamica 13 (1988) 55-70.

[16] C. Zalinescu: Stability for a class of nonlinear optimisation problems and applications, In: Nonsmooth Optimization and Related Topics, F. H. Clarke, V. F. Demyanov, F. Giannessi (eds.), Plenum Press (1989) 437–458.