Journal of Convex Analysis
Volume 6 (1999), No. 1, 183–194

Denting Points in Bochner Banach Ideal Spaces X(E)

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Received November 17, 1997
Revised manuscript received October 14, 1998

Let $(X, \|.\|_X)$ be an order-continuous Banach ideal space over a $\sigma-$ finite measure space (Ω, Σ, μ) and E a Banach space. We prove that a function f of the vector Banach ideal space $X(E)$ is a denting point of the unit ball of $X(E)$ if and only if : (i) the modulus function $|f| : t \longmapsto \|f(t)\|$ is a denting point of the unit ball of X and (ii) $f(t)/\|f(t)\|$ is a denting point of the unit ball of E for almost all t in supp(f). This gives an answer to the open problem raised in the paper [3].

1. Introduction

Let (Ω, Σ, μ) be a σ- finite measure space. By *Banach ideal space* (shortly Banach *i.s*) over (Ω, Σ, μ) we mean an ideal X of the vector lattice $L^0(\mu)$ of real measurable functions which is equipped with a *monotone* norm $\|.\|_X$ (that is $\|f\|_X \leq \|g\|_X$ whenever $f, g \in X$ and $|f| \leq |g|$) for which it is a Banach space. The notion of Banach *i.s* is an adequate slight generalization of what is known in some literature ([7] 1.b.17) as Köthe function space, that comes from works of the Russian School (c.f. Zabrejko [13], Kantorovich-Akilov [6]). Indeed, it can be proved that Banach *i.s* X over (Ω, Σ, μ) are those spaces for which there are a Probability measure ν on Ω with $\nu \ll \mu$, and a non-negative measurable function α on Ω such that the set inclusions

$$\alpha L^{\infty}(\nu) \subset X \subset \alpha L^{1}(\nu)$$

hold and are topological (the spaces $\alpha L^{\infty}(\nu)$ and $\alpha L^1(\nu)$ being respectively equipped with the norms $\|f\|_{\alpha,\infty} = \|\frac{1}{\alpha} f\|_\infty$ and $\|f\|_{\alpha,1} = \|\frac{1}{\alpha} f\|_1$). For a recent monograph on the subject of ideal spaces, we refer the reader to the book [12] of M. Väth.

Banach ideal spaces can naturally be extended to the vector case. Let E be a Banach space and X be a Banach *i.s.* We denote by $X(E)$ the set of all strongly measurable functions $f : \Omega \to E$ such that the *modulus* function $|f| : t \longmapsto \|f(t)\|$ belongs to X. For more convenience we will identify every function f in $X(E)$ with its equivalence class for the binary relation "equality $\mu - a.e$ ". This leads us to define a norm in $X(E)$ by setting

$$\|f\|_X := \||f|\|_X \quad , \; f \in X(E)$$

and $X(E)$ becomes a Banach space under this norm.

Before going to the main part of our subject let us recall some notations and definitions. If A is a non-empty subset of E, co(A) (resp. $\overline{\text{co}}(A)$) denotes the convex hull (resp. the closed convex hull) of A and $\delta^*(., A)$ will denote the *support function* of A , that is the

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function defined in E^* by $\delta^*(x^*, A) = \sup\{x^*(x) : x \in A\}$. Suppose that A is convex closed and let $x \in A$. We say that x is a *strongly extreme* point of A if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for all $y, z \in A$, the condition $\|1/2(y+z) - x\| < \delta$ implies $\|y - z\| < \varepsilon$. We say that x is a *denting point* of A if for every $\varepsilon > 0$, $x \notin \overline{\text{co}}(A \backslash B(x, \varepsilon))$ where $B(x, \varepsilon)$ is the set of all y of E such that $\|x - y\| < \varepsilon$. It is well known that "x is a denting point of A" $\Longrightarrow$ "x is a strong extreme point of A" $\Longrightarrow$ " x is an extreme point of A" and that the converse implications are false. For a given Banach space Z, we will denote by B_Z the closed unit ball of Z.

The purpose of this paper is to give necessary and sufficient conditions for a function f in $X(E)$ to be a denting point of the unit ball $B_{X(E)}$ of $X(E)$. In the case where $X = L^p(\mu)$ with $1 < p < \infty$, the problem was solved by B. L. Lin and P. Lin [8] and the necessary and sufficient conditions found were $f(t)/\|f(t)\|$ is a denting point of B_E for almost all t in supp(f). Next, Castaing and Pluccienik [3] extended the result in the case of Köthe-Bochner functions spaces $X(E)$ with the assumption X *locally uniformly rotund* (LUR). Moreover, the authors raised in the end of their paper the open question of whether their main result can be true without requiring this assumption. Note that a Banach space which is LUR possesses the property that every unit vector is a denting point of the unit ball. So it is natural to wonder what happens if we replace in Castaing-Pluccienik result the condition "X is LUR" by the less restrictive condition "$|f|$ is a denting point of the unit ball of X". In this paper we give an affirmative answer to this problem. We obtain necessary and sufficient conditions ensuring that function to be denting points of the unit ball. It is worth to mention that our proofs are new and rely on some topological properties of ideal spaces and characterizations of denting points. Nevertheless the paper [3] played an important role in our investigation since several parts of our proof are modifications of arguments given there. The paper contains also other results characterizing denting points of the set of measurable selections of an integrably bounded convex closed valued multifunction.

2. Preliminary results

Some results on denting points can be found in the literature [10], [11], [9]. The following criteria are useful

Lemma 2.1 (Lin-Lin-Troyanski [10]). *Let K be a non-empty convex closed subset of E and let $x \in K$. The following assertions are equivalent :*

(i) *x is a denting point of K.*

(ii) *For all sequences $(x_i^n)_{1 \le i \le \nu_n}$ of K and sequences $(\alpha_i^n)_{1 \le i \le \nu_n}$ of positive scalars with $\sum_{i=1}^{\nu_n} \alpha_i^n = 1$, $n \in \mathbb{N}$, the condition $\lim_{n\to\infty} \|\sum_{i=1}^{\nu_n} \alpha_i^n x_i^n - x\| = 0$ implies $\lim_{n\to\infty} \sum_{i=1}^{\nu_n} \alpha_i^n \|x_i^n - x\| = 0$.*

(iii) *There exist no sequences $(x_i^n)_{1 \le i \le \nu_n}$ of K and sequences $(\alpha_i^n)_{1 \le i \le \nu_n}$ of positive scalars with $\sum_{i=1}^{\nu_n} \alpha_i^n = 1 (n \in \mathbb{N})$ such that $\lim_{n\to\infty} \|\sum_{i=1}^{\nu_n} \alpha_i^n x_i^n - x\| = 0$ and $\|x_i^n - x\| > \varepsilon$, $\forall n, 1 \le i \le \nu_n$, for some $\varepsilon > 0$.*

The following result is a variant of the preceding lemma. It allows us to avoid a use of double indexed sequences, which can be useful in some other situations (c.f. Theorem 2.5 below).

Proposition 2.2. *Let K be a non-empty convex closed subset of E and let $x \in K$. The following assertions are equivalent :*

(i) *x is a denting point of K.*

(ii) *For any sequence (x_n) of K, if there exists a sequence (y_n) such that $y_n \in \operatorname{co}\{x_k : k \geq n\}$, $n \in \mathbb{N}$, and $\lim_{n\to\infty} \|y_n - x\| = 0$, then x is a strong cluster point of (x_n).*

(iii) *For any sequence (x_n) of K, the condition $\liminf_{n\to\infty} x^*(x_n) \leq x^*(x)$, $\forall x^* \in E^*$, implies that $\liminf_{n\to\infty} \|x_n - x\| = 0$.*

(iv) *There exists no sequence (x_n) in K such that $\|x_n - x\| > \varepsilon$, $\forall n \in \mathbb{N}$, for some $\varepsilon > 0$ and such that there exists a sequence $y_n \in \operatorname{co}\{x_k : k \geq n\}$ with $\lim_{n\to\infty} \|y_n - x\| = 0$.*

Proof. Let us prove the implication (i)⇒(ii). Let us first remark that the condition "there exists a sequence $y_n \in \operatorname{co}\{x_k : k \geq n\}$, $n \in \mathbb{N}$, such that $\lim_{n\to\infty} \|y_n - x\| = 0$" is equivalent to the condition "$x \in \bigcap_{n\geq 0} \overline{\operatorname{co}}\{x_k : k \geq n\}$". Hence for every $\varepsilon > 0$, since $x \notin \overline{\operatorname{co}}(K \backslash B(x,\varepsilon))$ and $x \in \overline{\operatorname{co}}\{x_k : k \geq n\}$, the set $\{x_k : k \geq n\}$ cannot be contained in the set $K \backslash B(x,\varepsilon)$. Hence we obtain: for every $\varepsilon > 0$ and every $n \geq 0$, there exists $m \geq n$ such that $\|x_m - x\| < \varepsilon$. This is clearly the analytique definition of a cluster point of the sequence (x_n).

The implication (ii)⇒(iv) is obvious. Let us prove (iv)⇒(i). Suppose x is not a denting point of K. Then there exist sequences $(x_i^n)_{1\leq i\leq \nu_n}$ in K and $(\alpha_i^n)_{1\leq i\leq\nu_n}$ in $[0,1]$ with $\sum_{i=1}^{\nu_n} \alpha_i^n = 1$, and $\varepsilon > 0$ such that $\|x_i^n - x\| > \varepsilon$ for all n and $1 \leq i \leq \nu_n$ and $\lim_{n\to\infty} \| \sum_{i=1}^{\nu_n} \alpha_i^n x_i^n - x\| = 0$.

Set $N(1) := 0$ and $N(k) := \sum_{i=1}^{k-1} \nu_i$ for $k \geq 2$. Let $(y_n)_{n\geq 1}$ be the sequence in K defined by

$$y_{N(k)+i} = x_i^k \quad \text{for } k \geq 1 \text{ and } 1 \leq i \leq \nu_k. \tag{2.1}$$

For all $k \geq 1$, the vector $z_k := \sum_{i=1}^{\nu_k} \alpha_i^k x_i^k$ is contained in the convex hull of $\{y_{N(k)+i} : 1 \leq i \leq \nu_k\}$ and since $N(k) + i \geq k$ $(1 \leq i \leq \nu_k)$ we have $z_k \in \operatorname{co}\{y_m : m \geq k\}$. On the other hand, by hypothesis, we have $\lim_{k\to\infty} \|z_k - x\| = 0$ and $\|y_m - x\| > \varepsilon$ for every $m \in \mathbb{N}$. Thus (iv) is not satisfied.

Now we only need to prove that (ii)⇔(iii). This follows from the fact that the condition "x is a strong cluster point of (x_n)" is equivalent to the condition "$\liminf_{n\to\infty} \|x_n - x\| = 0$" and that the condition "$x \in \bigcap_{n\geq 0} \overline{\operatorname{co}}\{x_k : k \geq n\}$" is equivalent to the condition "$\liminf_{n\to\infty} x^*(x_n) \leq x^*(x)$, $\forall x^* \in E^*$". The first fact is standard. Let us prove the second one. Suppose that $x \in \bigcap_{n\geq 0} \overline{\operatorname{co}}\{x_k : k \geq n\}$ and let $x^* \in E^*$. Then for all $n \in \mathbb{N}$, we have

$$\begin{aligned} x^*(x) &\leq \delta^*(x^*, \overline{\operatorname{co}}\{x_k : k \geq n\}) = \delta^*(x^*, \{x_k : k \geq n\}) \\ &= \sup_{k\geq n} x^*(x_k) \end{aligned}$$

Hence $x^*(x) \leq \inf_n \sup_{k\geq n} x^*(x_k) = \limsup_{n\to\infty} x^*(x_n)$. A simple change of x^* with $-x^*$ shows then that the converse inequality is also true with lim inf instead of lim sup. Let us prove the converse implication. Suppose for contradiction that $x \notin \bigcap_{n\geq 0} \overline{\operatorname{co}}\,\{x_k : k \geq n\}$. Then $x \notin \overline{\operatorname{co}}\{x_{n_0} : k \geq n_0\}$ for some integer n_0. By the Hahn-Banach Theorem there exists $x^* \in E^* \backslash \{0\}$ such that

$$\sup_{k\geq n_0} x^*(x_k) = \delta^*(x^*, \overline{\operatorname{co}}\,\{x_k : k \geq n_0\}) < x^*(x)$$

Thus $\limsup_{n\to\infty} x^*(x_n) < x^*(x)$ and so $\liminf_{n\to\infty} y^*(x_n) > y^*(x)$ with $y^* = -x^*$. □

Recall that the Banach *i.s* X is said to be *order continuous* if for every decreasing sequence (f_n) of X_+ (the positive cone of X) such that $\inf_{n\geq 0} f_n = 0$ (we write shortly $f_n \downarrow 0$) we have $\lim_{n\to\infty} \|f_n\|_X = 0$.

The following lemma is an analogue of the Lebesgue dominated convergence theorem for the case of order continuous Banach ideal spaces. For a proof of a more general result we refer the reader [12], Theorem 3.3.5.

Lemma 2.3. *Suppose that X is order continuous. Let (f_n) be a sequence in X converging $\mu - a.e$ or in measure to the null function and such that there exists a function $g \in X_+$ such that $|f_n| \leq g$ for all $n \in \mathbb{N}$. Then (f_n) converges in norm to 0 in X.*

Let us now recall the *support* of a Banach ideal space. For a function f of X (or $X(E)$), we define the *support* of f by

$$S_f = \operatorname{supp}(f) := \{t \in \Omega : f(t) \neq 0\}$$

The support of a function f of X (or $X(E)$) is a measurable set defined except for a μ-negligible set since the function f is itself identified with its own equivalence class for the $\mu - a.e$ equality relation. On the other hand, it is known that the $\sigma-$algebra Σ is a complete lattis for the preorder: $A \subset_{\text{ess}} B$ iff $A \backslash B$ is a $\mu-$negligible set. We define the *support* of X as the supremum in Σ (equipped with $\subset_{\text{ess}}$) of the family $\mathcal{F}_X$ of all supports S_f of functions $f \in X$. We denote it by $\operatorname{supp}(X)$.

Lemma 2.4. *There exists a function α in X_+ such that $\alpha(t) > 0$ for all $t \in \operatorname{supp}(X)$.*

Proof. Set $S := \operatorname{supp}(X)$. From the definition of the support, we have $1_S = \operatorname{ess\,sup}\{1_{S_f} : f \in X\}$. Hence, there exists a sequence of functions (h_n) in X_+ such that $1_S = \sup_{n\geq 1} 1_{S_{h_n}}$. For each $n \geq 1$, choose a number $\eta_n > 0$ such that $\eta_n \|h_n\|_X \leq 2^{-n}$ and set $\alpha_n = \sum_{k=1}^{n} \eta_k h_k$. Then (α_n) is a Cauchy sequence in X. So it converges to a function $\alpha \in X_+$. Since (α_n) is increasing, we have necessary $\alpha = \sup_{n\geq 1} \alpha_n$. It follows that

$$S_\alpha = \bigcup_{n\geq 1} S_{\alpha_n} = \bigcup_{n\geq 1} S_{h_n}$$

On the other hand $\cup_{n\geq 1} S_{h_n} \equiv S$. Hence modifying α on a $\mu-$negligible set, we get the required function. □

Without loss of generality it can be assumed that $\operatorname{supp}(X) = \Omega$. In this case, we say that X is a *Banach fundamental space* (shortly *Banach f.s*), c.f. [6], Chap. IV, §3. We will assume this is true in all that follows. Let us recall the *dual i.s* of X, denoted by X', defined as the space of all functions $g \in L^0(\mu)$ such that

$$\operatorname{supp}(g) \underset{\text{ess}}{\subset} \operatorname{supp}(X) \text{ and } f.g \in L^1(\mu), \ \forall f \in X.$$

It is proved that X' is a Banach ideal space for the norm

$$\|g\|_{X'} = \sup\left\{\int_\Omega f.g \, d\mu : f \in X \ , \ \|f\|_X \leq 1\right\} \ , \ g \in X'$$

and that $\text{supp}(X') = \text{supp}(X) = \Omega$ (c.f. [6], Chap VI, §1). From Lemma 2.4, choose a function β in X' such that $\beta(t) > 0$ for all $t \in \Omega$. We can suppose further that $\|\beta\|_{X'} = 1$. Consider now the $\sigma-$finite measure $\nu = \beta\mu : A \longmapsto \int_A \beta \, d\mu$ on Σ. From the definition of X' it is clear that the set inclusion $X \subset L^1(\nu)$ holds. Furthermore, we have

$$\|f\|_1 \leq \|f\|_X \quad \text{for all } f \in X \tag{2.2}$$

where $\|.\|_1$ is the L^1-norm in $L^1(\nu)$. This property will us help later. It reduces the study from X to L^1-spaces.

When the Banach space E is not necessarily separable, it is difficult to give an adequate general measurability theory for multifunctions Γ from Ω to subsets of E ([4], Chap III). Fortunately, for the purpose of this paper, we can always reduce to this case. Indeed, let f be a non-zero function in $X(E)$. Then, there exists a non trivial separable Banach subspace E_0 of E such that $f(t) \in E_0$ μ - almost everywhere. The vector Banach *i.s* $X(E_0)$ can no longer be identified with a Banach subspace of $X(E)$ containing the function f. So if f is a denting point of the unit ball of $X(E)$, then it is also a denting point of the unit ball of $X(E_0)$.

Suppose now that the Banach space E is separable. A multifunction Γ from Ω to 2^E is called *measurable* if its *graph* $\text{gr}(\Gamma) = \{(t, x) : x \in \Gamma(t)\}$ belongs to $\hat{\Sigma} \otimes \mathcal{B}(E)$ where $\hat{\Sigma}$ is the μ-completion of Σ and $\mathcal{B}(E)$ is the Borel tribe of E. For more properties on measurability of multifunctions, we refer the reader to the monograph [4], Chap III. In particular, it is found there different kinds of measurability and some relations between them. Our choice of the previous definition of measurability aims at covering the most general case including multifunctions, not necessarily convex or closed valued.

Let Γ be a measurable multifunction from Ω to non-empty convex closed subsets of E. We will say that Γ is *X-bounded* if there exists a function $g \in X_+$ such that $\mu - a.e$, $\Gamma(t) \subset g(t)B_E$. In that case we will denote by $L_\Gamma^X(\mu)$ the set of all selections $f \in X(E)$ of Γ. Note that $L_\Gamma^X(\mu)$ is a non-empty convex closed and bounded subset of $X(E)$. The following theorem generalizes Theorem 3.6 of [1] (Chap II):

Theorem 2.5. *Let Γ be a measurable multifunction from Ω to non-empty convex closed subsets of E. Suppose that Γ is X-bounded and let $f \in L_\Gamma^X(\mu)$. Then f is a denting point of the set $L_\Gamma^X(\mu)$ if and only if $f(t)$ is a denting point of $\Gamma(t)$ for almost all $t \in \Omega$.*

Proof. As we have remarqued above let us choose $\beta \in X'$ such that $\beta > 0$ on Ω. Then we have the set inclusion $X \subset L^1(\nu)$ and the relation (2.2) with $\nu := \beta.\mu$.

Sufficiency: Let us suppose by contradiction that f is not a denting point of the set $L_\Gamma^X(\mu)$. Then, by Proposition 2.2, there exists a sequence (f_n) in $L_\Gamma^X(\mu)$ such that $\|f_n - f\|_X > \varepsilon$, $\forall$ n, for some $\varepsilon > 0$ and such that there exists a sequence of convex combinations $g_n = \sum_{k=n}^{m_n} \alpha_k^n f_k$ ($\alpha_k^n \geq 0$, $\sum_{k=n}^{m_n} \alpha_k^n = 1$) satisfying $\lim_{n\to\infty} \|g_n - f\|_X = 0$. Since the norm-convergence in X implies convergence in measure ([6] Théorème IV.3.1, or [12] Theorem 3.1.1), we can suppose along a subsequence that $g_n(t) \to f(t)$ $\mu - a.e$ in E. On the other hand $f(t)$ is a denting point of $\Gamma(t)$ $\mu - a.e$, thus by Lemma 2.1 we have

$$\lim_{n\to\infty} \sum_{k=n}^{m_n} \alpha_k^n \|f_k(t) - f(t)\| = 0 \quad \mu - a.e.$$

By the Vitali-Lebesgue Theorem, we deduce that the sequence of functions $\sum_{k=n}^{m_n} \alpha_k^n |f_k - f|$ norm-converges to zero in $L^1(\nu)$. So

$$\liminf_{n\to\infty} \|f_n - f\|_1 \leq \liminf_{n\to\infty} \sum_{k=n}^{m_n} \alpha_k^n \|f_k - f\|_1 = 0.$$

Hence there exists a subsequence (f_{n_k}) such that $\lim_{k\to\infty} \|f_{n_k} - f\|_1 = 0$. Thus the sequence $(|f_{n_k} - f|)_k$ converges in measure to 0 and is order bounded by $2g$. Lemma 2.3 implies then that $(f_{n_k} - f)_k$ converges strongly to zero in X, a contradiction with our hypothesis.

Necessity: Suppose that f is a denting point of the set $L_\Gamma^X(\mu)$. Since Γ is $X-$ bounded and $X \subset L^1(\nu)$, the set $L_\Gamma^X(\mu)$ coincides with the set $L_\Gamma^1(\nu)$ of all $\nu-$Bochner integrable selections of the multifunction Γ. Let us prove that the function f is also a denting point of $L_\Gamma^X(\mu) = L_\Gamma^1(\nu)$ relative to the Banach space $L_E^1(\nu)$. We will use for this the criteria of Lemma 2.1. Let $\lambda_i^n \geq 0$, $\sum_{i=1}^{N_n} \lambda_i^n = 1$, and $f_i^n \in L_\Gamma^X(\mu)$, $1 \leq i \leq N_n$, such that $\lim_{n\to\infty} \left\| \sum_{i=1}^{N_n} \lambda_i^n f_i^n - f \right\|_1 = 0$. The sequence $h_n := \sum_{i=1}^{N_n} \lambda_i^n f_i^n$ converges in measure ν, to the function f. Since $\nu = \beta.\mu$ and $\beta > 0$ on Ω, the sequence (h_n) converges also in measure μ to f. On the other hand $|h_n| \leq g \in X_+$, $n \in \mathbb{N}$. Thus Lemma 2.3 implies that $\|h_n - f\|_X \to 0$. It follows from Lemma 2.1 that

$$\lim_{n\to\infty} \sum_{i=1}^{N_n} \lambda_i^n \|f_i^n - f\|_X = 0.$$

Since $\|.\|_1 \leq \|.\|_X$, it follows also that $\lim_{n\to\infty} \sum_{i=1}^{N_n} \lambda_i^n \|f_i^n - f\|_1 = 0$ proving our claim. Now, we have proved that f is a denting point of $L_\Gamma^1(\nu)$. Consequently, by Théorème 3.6 in [1], Chap. II, $f(t)$ is a denting point of $\Gamma(t)$ for almost all $t \in \Omega$. □

Remark 2.6. Using the representation theorem of the strong dual of $X(E)$ which can be found in e.g [2], it is also possible to give a direct proof of the Theorem 2.5 similarly to the proof of Theorem 3.6 in [1].

3. Main result

We can now state the main result of the paper.

Theorem 3.1. *Let $(X, \|.\|_X)$ be an order continuous Banach ideal space over (Ω, Σ, μ). A function f in $X(E)$ is a denting point of the unit ball $B_{X(E)}$ if and only if*

(i) *the modulus function $|f|$ is a denting point of the unit ball B_X of X.*
(ii) *$f(t)/\|f(t)\|$ is a denting point of B_E for almost all $t \in \mathrm{supp}(f)$.*

Proof. Let $f \in X(E)$ satisfy the conditions (i) and (ii) and suppose by contradiction that f is not a denting point of $B_{X(E)}$. Then there are $\varepsilon > 0$, scalars $\alpha_i^n \geq 0$ with $\sum_{i=1}^{m_n} \alpha_i^n = 1$, and functions $f_i^n \in B_{X(E)}$, $i = 1, ..., m_n$, such that $\|f_i^n - f\|_X > \varepsilon$ and

$$\lim_{n\to\infty} \left\| \sum_{i=1}^{m_n} \alpha_i^n f_i^n - f \right\|_X = 0. \tag{3.1}$$

Let us consider the sequences of modulus functions

$$u_n = \left| \sum_{i=1}^{m_n} \alpha_i^n f_i^n \right| \quad \text{and} \quad v_n = \sum_{i=1}^{m_n} \alpha_i^n \left| f_i^n \right|.$$

It is clear that $0 \le u_n \le v_n$ and that $u_n, v_n \in B_X$. Set $u_n' = u_n + 1/2\,(v_n - u_n)$ and $u_n'' = u_n - 1/2\,(v_n - u_n)$. It is easy to check that $u_n \le u_n' \le v_n$ and $-v_n \le u_n'' \le u_n \le v_n$. It follows that $u_n' \in B_X$ and $u_n'' \in B_X$. Now remark that by (3.1),

$$\frac{1}{2}\left(u_n' + u_n''\right) = u_n \to |f| \text{ strongly in } X.$$

Since $|f|$ is a denting point, it is a strongly extreme point of B_X. Consequently, $\|u_n' - |f|\|_X \to 0$. Since $v_n = 2u_n' - u_n$, we deduce that $\|v_n - |f|\|_X \to 0$. Hence we have proved that

$$\lim_{n\to\infty} \left\| \sum_{i=1}^{m_n} \alpha_i^n \left| f_i^n \right| - |f| \right\|_X = 0 \tag{3.2}$$

For $n \in \mathbb{N}$ and $1 \le i \le m_n$, set $A_i^n = \{|f_i^n| \le |f|\}$ and

$$g_i^n(t) := \begin{cases} f_i^n(t) & \text{if } t \in A_i^n \\ \dfrac{\|f(t)\|}{\|f_i^n(t)\|} f_i^n(t) & \text{if } t \in \Omega \backslash A_i^n \end{cases}$$

It is easy to check that the "regularized " functions g_i^n satisfies the following properties

$$|g_i^n| \le |f| \quad , \quad |g_i^n| \le |f_i^n| \quad \text{and} \quad |f_i^n - g_i^n| \le \left| |f_i^n| - |f| \right|. \tag{3.3}$$

It follows that

$$\sum_{i=1}^{m_n} \alpha_i^n \left\| f_i^n - g_i^n \right\|_X \le \sum_{i=1}^{m_n} \alpha_i^n \left\| |f_i^n| - |f| \right\|_X$$

Since $|f|$ is a denting point of B_X, condition (3.2) implies, by virtue of Lemma 2.1, that

$$\lim_{n\to\infty} \sum_{i=1}^{m_n} \alpha_i^n \left\| |f_i^n| - |f| \right\|_X = 0$$

and hence

$$\lim_{n\to\infty} \sum_{i=1}^{m_n} \alpha_i^n \left\| f_i^n - g_i^n \right\|_X = 0. \tag{3.4}$$

Now, we have the following inequality

$$\left\| \sum_{i=1}^{m_n} \alpha_i^n g_i^n - f \right\|_X \le \sum_{i=1}^{m_n} \alpha_i^n \left\| g_i^n - f_i^n \right\|_X + \left\| \sum_{i=1}^{m_n} \alpha_i^n f_i^n - f \right\|_X$$

from which we deduce that

$$\lim_{n\to\infty}\left\|\sum_{i=1}^{m_n}\alpha_i^n g_i^n - f\right\|_X = 0. \tag{3.5}$$

It is known that the norm convergence in X implies convergence in measure ([6],Chap. IV, §3) and hence $\mu - a.e$ convergence along a subsequence. Thus condition (3.5) implies the existence of an increasing sequence of integers $n_1 < n_2 < \cdots$ such that

$$\lim_{k\to\infty}\sum_{i=1}^{m_{n_k}}\alpha_i^{n_k}\left\|g_i^{n_k}(t) - f(t)\right\| = 0 \quad \mu - a.e \tag{3.6}$$

On the other hand, the sequence of functions $\sum_{i=1}^{m_{n_k}}\alpha_i^{n_k}\left|g_i^{n_k} - f\right|$ is order dominated by the function $2|f|$. So by Lemma 2.3, condition (3.6) implies that

$$\lim_{k\to\infty}\left\|\sum_{i=1}^{m_{n_k}}\alpha_i^{n_k}\left|g_i^{n_k} - f\right|\right\|_X = 0. \tag{3.7}$$

We will prove now that there exists $\varepsilon' > 0$ such that

$$\|g_i^n - f\|_X > \varepsilon' \text{ for all } n \in \mathbb{N} \text{ and } 1 \le i \le m_n. \tag{3.8}$$

Set $I_n = \{1 \le i \le m_n : \|f_i^n - g_i^n\|_X < \varepsilon/2\}$. Then if $i \in I$, we have

$$\|g_i^n - f\|_X \ge \|f_i^n - f\|_X - \|f_i^n - g_i^n\|_X > \varepsilon - \frac{\varepsilon}{2} = \frac{\varepsilon}{2}.$$

Let now $1 \le i \le m_n$ such that $i \notin I$. Then

$$\|f_i^n - g_i^n\|_X \ge \frac{\varepsilon}{2}. \tag{3.9}$$

Since $|f|$ is a strongly extreme point of B_X, there exists $\delta > 0$ (depending only on $|f|$ and ε) such that for all $g, h \in B_X$,

$$\|g - h\|_X \ge \frac{\varepsilon}{2} \Rightarrow \left\|\frac{1}{2}(g+h) - |f|\right\|_X > \delta. \tag{3.10}$$

Set $g = |g_i^n| + 1/2\left(|f_i^n| - |g_i^n|\right)$ and $h = |g_i^n| - 1/2\left(|f_i^n| - |g_i^n|\right)$. As in the beginning of the proof , we can easily check that $g \in B_X$ and $h \in B_X$. Furthermore $\|g - h\|_X = \left\||f_i^n| - |g_i^n|\right\|_X$ and from the definition of g_i^n, we can easily show that the identity $\left||f_i^n| - |g_i^n|\right| = |f_i^n - g_i^n|$ holds. So, by (3.9), $\|g - h\|_X = \|f_i^n - g_i^n\|_X \ge \varepsilon/2$. Applying (3.10), we get

$$\left\||g_i^n| - |f|\right\|_X = \left\|\frac{1}{2}(g+h) - |f|\right\| > \delta$$

It follows that $\|g_i^n - f\|_X \ge \left\||g_i^n| - |f|\right\|_X > \delta$. Now, by setting $\varepsilon' = \inf(\varepsilon/2, \delta) > 0$, it is clear that condition (3.8) is satisfied.

For $p \geq 1$, $n \in \mathbb{N}$ and $1 \leq i \leq m_n$, set

$$A_{i,p}^n := \left\{ t \in \Omega : \frac{1}{p} \|f(t)\| \leq \|g_i^n(t) - f(t)\| \right\}.$$

Let us prove that there exist $\eta > 0$ and $p \geq 1$ such that

$$\left\| 1_{A_{i,p}^n} f \right\|_X > \eta \text{ for all } n \text{ and } 1 \leq i \leq m_n. \tag{3.11}$$

Suppose by contradiction that the opposite of (3.11) holds. Then for every $p \geq 1$ there exist $n(p) \in \mathbb{N}$ and $1 \leq i(p) \leq m_{n(p)}$ such that the set $A(p) := A_{i(p),p}^{n(p)}$ satisfies $\left\| 1_{A(p)} f \right\|_X < \varepsilon'/4$. For $p > 2/\varepsilon'$, we have

$$\begin{aligned} \left\| f - g_{i(p)}^{n(p)} \right\|_X &\leq \left\| 1_{A(p)} \left(f - g_{i(p)}^{n(p)} \right) \right\|_X + \left\| 1_{\Omega \setminus A(p)} \left(f - g_{i(p)}^{n(p)} \right) \right\|_X \\ &\leq \left\| 1_{A(p)} f \right\|_X + \left\| 1_{A(p)} g_{i(p)}^{n(p)} \right\|_X + \left\| 1_{\Omega \setminus A(p)} \left(f - g_{i(p)}^{n(p)} \right) \right\|_X \\ &\leq 2 \left\| 1_{A(p)} f \right\|_X + \frac{1}{p} \|f\|_X \\ &< 2.\frac{\varepsilon'}{4} + \frac{1}{p} < \varepsilon'. \end{aligned}$$

This contradicts (3.8). Hence condition (3.11) is true. Now, in virtue of the definition of $A_{i,p}^n$, we have

$$\frac{1}{p} 1_{A_{i,p}^n} |f| \leq |g_i^n - f| \ , \ n \in \mathbb{N}, 1 \leq i \leq m_n.$$

Hence

$$0 \leq \frac{1}{p} \sum_{i=1}^{m_n} \alpha_i^n 1_{A_{i,p}^n} |f| \leq \sum_{i=1}^{m_n} \alpha_i^n |g_i^n - f|$$

In virtue of (3.7), we deduce that

$$\lim_{k \to \infty} \left\| \sum_{i=1}^{m_{n_k}} \alpha_i^{n_k} 1_{A_{i,p}^{n_k}} |f| \right\|_X = 0.$$

Hence

$$\lim_{k \to \infty} \left\| \sum_{i=1}^{m_{n_k}} \alpha_i^{n_k} 1_{\Omega \setminus A_{i,p}^{n_k}} |f| - |f| \right\|_X = 0.$$

Since $1_{\Omega \setminus A_{i,p}^{n_k}} |f| \in B_X$ and $|f|$ is a denting point of B_X, it follows from Lemma 2.1 that

$$\lim_{k \to \infty} \sum_{i=1}^{m_{n_k}} \alpha_i^{n_k} \left\| 1_{\Omega \setminus A_{i,p}^{n_k}} |f| - |f| \right\|_X = 0.$$

On the other hand, by (3.11), $\left\|1_{\Omega\backslash A^{n_k}_{i,p}}|f| - |f|\right\|_X = \left\|1_{A^{n_k}_{i,p}} f\right\|_X > \eta$ for all k and $1 \leq i \leq m_{n_k}$. We obtain hence a contradiction which proves the direct part of the theorem.

Let us prove the converse implication. Let f be a denting point of the unit ball $B_{X(E)}$ of $X(E)$ and let us prove that conditions (i) and (ii) of the theorem hold. As remaked in section 2, we may suppose w.l.o.g that $\mathrm{supp}(X) = \Omega$ and that the Banach space E is separable. For each $t \in \Omega$, set $\Gamma(t) = \|f(t)\| B_E$. Then Γ is a measurable $X-$bounded multifunction from Ω to the non-empty convex closed subsets of E and $f \in L^X_\Gamma(\mu) \subset B_{X(E)}$. Hence f is also a denting point of the closed convex bounded set $L^X_\Gamma(\mu)$. By theorem 2.5 this implies that $\mu-a.e$, $f(t)$ is a denting point of $\Gamma(t)$ and so condition (i) is satisfied. It remains to prove (ii). Let $(h^n_i)_{1\leq i\leq N_n}$ be a sequence of B_X and $(\lambda^n_i)_{1\leq i\leq N_n}$ a sequence of positive scalars with $\sum_{i=1}^{N_n} \lambda^n_i = 1$, such that $\lim_{n\to\infty} \left\|\sum_{i=1}^{N_n} \lambda^n_i h^n_i - |f|\right\|_X = 0$. To complete the proof of theorem it remains to check that

$$\lim_{n\to\infty} \sum_{i=1}^{N_n} \lambda^n_i \left\|h^n_i - |f|\right\|_X = 0. \tag{3.12}$$

Let us define the function

$$u(t) := \begin{cases} \dfrac{f(t)}{\|f(t)\|} & \text{if } t \in S_f \\ 0 & \text{if } t \in S^c_f = \Omega\backslash S_f \end{cases}$$

Then we have $f = |f|\, u : t \mapsto |f|(t)\, u(t) = f(t)$. Set $\bar{h}^n_i := h^n_i u$ for $n \in \mathbb{N}$ and $1 \leq i \leq N_n$. Then $\bar{h}^n_i \in B_{X(E)}$ and

$$\begin{aligned} \left\|\sum_{i=1}^{N_n} \lambda^n_i \bar{h}^n_i - f\right\|_X &= \left\|\left(\sum_{i=1}^{N_n} \lambda^n_i h^n_i - |f|\right) u\right\|_X \\ &\leq \left\|\sum_{i=1}^{N_n} \lambda^n_i h^n_i - |f|\right\|_X \to 0 \end{aligned}$$

Since f is a denting point of $B_{X(E)}$, we deduce that

$$\lim_{n\to\infty} \sum_{i=1}^{N_n} \lambda^n_i \left\|\bar{h}^n_i - f\right\|_X = 0. \tag{3.13}$$

Let us now remark the following equality

$$\left|\bar{h}^n_i - f\right| = \left|\left(h^n_i - |f|\right) u\right| = \left|1_{S_f}\left(h^n_i - |f|\right)\right|.$$

Hence condition (3.13) becomes

$$\lim_{n\to\infty} \sum_{i=1}^{N_n} \lambda^n_i \left\|1_{S_f}\left(h^n_i - |f|\right)\right\|_X = 0. \tag{3.14}$$

We define now a new sequence

$$\tilde{h}_i^n(t) := \begin{cases} h_i^n(t)\, u(t) & \text{if } t \in S_f \\ h_i^n(t)\, x_0 & \text{if } t \in S_f^c \end{cases}$$

where $x_0 \in E$, $\|x_0\| = 1$, a fixed vector. For $n \in \mathbb{N}$ and $1 \le i \le N_n$, we have $\tilde{h}_i^n \in B_{X(E)}$ since $\left|\tilde{h}_i^n\right| = |h_i^n|$. Furthermore,

$$\begin{aligned} \left\| \sum_{i=1}^{N_n} \lambda_i^n \tilde{h}_i^n - f \right\|_X &= \left\| 1_{S_f} \left(\sum_{i=1}^{N_n} \lambda_i^n h_i^n u - |f|\, u \right) + 1_{S_f^c} \left(\sum_{i=1}^{N_n} \lambda_i^n h_i^n \right) x_0 \right\|_X \\ &\le \left\| 1_{S_f} \left(\sum_{i=1}^{N_n} \lambda_i^n h_i^n - |f| \right) \right\|_X + \left\| 1_{S_f^c} \left(\sum_{i=1}^{N_n} \lambda_i^n h_i^n \right) \right\|_X . \end{aligned} \tag{3.15}$$

Let us remark now that

$$1_{S_f^c} \left(\sum_{i=1}^{N_n} \lambda_i^n h_i^n \right) = \sum_{i=1}^{N_n} \lambda_i^n h_i^n - 1_{S_f} \left(\sum_{i=1}^{N_n} \lambda_i^n h_i^n \right)$$

and that $\sum_{i=1}^{N_n} \lambda_i^n h_i^n \to |f|$, $1_{S_f} \left(\sum_{i=1}^{N_n} \lambda_i^n h_i^n \right) \to |f|$ strongly in X . Hence

$$\lim_{n\to\infty} \left\| 1_{S_f^c} \left(\sum_{i=1}^{N_n} \lambda_i^n h_i^n \right) \right\|_X = 0,$$

so condition (3.15) implies that

$$\lim_{n\to\infty} \left\| \sum_{i=1}^{N_n} \lambda_i^n \tilde{h}_i^n - f \right\|_X = 0.$$

Since f is a denting point of $B_{X(E)}$, we deduce that

$$\lim_{n\to\infty} \sum_{i=1}^{N_n} \lambda_i^n \left\| \tilde{h}_i^n - f \right\|_X = 0. \tag{3.16}$$

From the definition of $\tilde{h}_i^n$, it can now be easily checked that

$$\left| \tilde{h}_i^n - f \right| = 1_{S_f} \left| h_i^n - |f| \right| + 1_{S_f^c} \left| h_i^n \right| .$$

Hence

$$\sum_{i=1}^{N_n} \lambda_i^n \left\| 1_{S_f^c} |h_i^n| \right\|_X \le \sum_{i=1}^{N_n} \lambda_i^n \left\| \tilde{h}_i^n - f \right\|_X + \sum_{i=1}^{N_n} \lambda_i^n \left\| 1_{S_f} \left| h_i^n - |f| \right| \right\|_X .$$

It follows from (3.14) and (3.16) that

$$\lim_{n\to\infty} \sum_{i=1}^{N_n} \lambda_i^n \left\| 1_{S_f^c} \left| h_i^n \right| \right\|_X = 0. \tag{3.17}$$

Finally we have the following inequality

$$\sum_{i=1}^{N_n} \lambda_i^n \left\| h_i^n - |f| \right\|_X \leq \sum_{i=1}^{N_n} \lambda_i^n \left\| 1_{S_f} \left(h_i^n - |f| \right) \right\|_X + \sum_{i=1}^{N_n} \lambda_i^n \left\| 1_{S_f^c} \left| h_i^n \right| \right\|_X$$

which together with conditions (3.14) and (3.17) implies our claim (3.12). □

Remark 3.2. Using the tools developed in [2], concerning the representation of the strong dual of $X(E)$, most results obtained in this paper can be transposed without essential changes to the study of denting points in the strong dual Banach space $X(E)^*$. Perhaps it is also possible to give by means of these tools new proofs of the results obtained in [5] relative to weak* denting points in $L^p(\mu, X)^*$. This would be shown in a further work.

Note. While writing this paper I received from Professor C. Castaing the preprint [9] of P-K. Lin and H. Sun where the authors study similar problems given in this work. I wish to thank C. Castaing for sending me this preprint and also for the paper [3].

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