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OPTIMALITY CONDITIONS FOR PROBLEMS WITH SET–VALUED OBJECTIVES

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Abstract. Optimality conditions are established for mathematical programming problems where objectives and constraints are given by set–valued mappings. These conditions are stated with Lagrange multipliers associated with the coderivatives of the set-valued data.

Introduction. H.W. Corley [10, 11] and T. Tanino and Y. Sawaragi [34] have developed an important duality theory for mathematical programming problems with convex vector valued data. The dual problem appears in a natural way as a problem whose objective is a set–valued mapping. These authors have also given several applications of this duality in a series of papers [10, 11, 33, 34].

In our knowledge, the first paper establishing necessary optimality conditions for optimization problems where the objective is a set–valued mapping seems to be the one of H.W. Corley [12]. These optimality conditions are

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formulated in terms of derivatives associated to Clarke tangent cones according to the definition introduced by J.-P. Aubin [1]. More precisely, suppose that $\overline{y} \in F(\overline{x})$ is an optimal solution of the problem

$$(P) \qquad \begin{array}{l} \text{Minimize } \; F(x) \\ \text{subject to } x \in S \text{ and } G(x) \cap D \neq \emptyset, \end{array}$$

where F and G are set–valued mappings from a Banach space X into Banach spaces Y and Z (Y being ordered in [12] by a convex cone K with nonempty interior), S is a subset of X and D is a convex cone of Z with nonempty interior. H.W. Corley proved, for $\overline{z} \in G(\overline{x}) \cap D$, the existence of a nonzero pair $(y^*, z^*) \in (-K)^\circ \times D^\circ$ such that $\langle z^*, \overline{z}\rangle = 0$ and

$$\langle y^*, y\rangle + \langle z^*, z\rangle \geq 0 \tag{0.1}$$

for all $x \in \mathrm{dom} D_C(F_S, G_S)(\overline{x}, \overline{y}, \overline{z})$ and $(y, z) \in D_C(F_S, G_S)(\overline{x}, \overline{y}, \overline{z})(x)$. (Here K° denotes the negative polar cone of K and $D_C(F_S, G_S)$ denotes the derivative in the sense of J.-P. Aubin [1] relatively to the Clarke tangent cone of the set-valued mapping from X into $Y \times Z$ defined by

$$(F_S, G_S)(x) = F(x) \times G(x) \;\; \text{if } x \in S \text{ and } (F_S, G_S)(x) = \emptyset \text{ otherwise.})$$

Note that D.T. Luc [26] and D.T. Luc and C. Malivert [27] have also proved for the problem (P) necessary optimality conditions similar to (0.1) but in terms of the contingent derivative of (F_S, G_S) (instead of the Clarke tangent derivative) and with the assumption that the graph of the contingent derivative is convex. In our knowledge there is no other paper (up to now) devoted to optimality conditions for problems with set-valued objective mappings).

The aim of this paper is to establish optimality conditions with Lagrange–Kuhn–Tucker and Lagrange–Fritz–John multipliers for the problem (P) in terms of the coderivatives of the set–valued mappings F and G separately and the normal cone to S. Such conditions are more general then the ones formulated in terms of the derivative of the set-valued mapping (F_S, G_S). Our approach allows to suppose that D is any nonempty closed subset of Z and to use any (sequentially) closed normal cone, for examples the ones by F.H. Clarke [9], A.D. Ioffe [18], B.S. Mordukhovich [28]... . Although all the results also hold for the coderivative with respect to the Mordukhovich normal cone whenever the underlying Banach spaces are Asplund (see [29] for subdifferential calculus of functions in these spaces), we will restrict ourselves (to avoid complications of notations) to Clarke and Ioffe coderivatives. Before concluding this introduction we must also say that a premilinary version (see [15]) of this common work has constitued a chapter of the second thesis of the first author.

1. Preliminaries. In this section we recall some definitions and results which will be needed later.

In all the paper X, Y and Z will be Banach spaces, X^*, Y^* and Z^* their topological duals and $\mathbb{B}_{\mathbb{X}}$ the closed unit ball of X (centered at the origin). Unless otherwise stated the norm on $X \times Y$ will be given by $\|(x, y)\| = \|x\| + \|y\|$.

Let $f : X \longrightarrow \mathbb{R}$ be a locally Lipschitz function and $\overline{x} \in X$.

The Clarke subdifferential $\partial_C f(\overline{x})$ of f at $\overline{x}$ is defined by (see Clarke [9])

$$\partial_C f(\overline{x}) := \{x^* \in X : \langle x^*, v\rangle \leq f^\circ(\overline{x}; v),\ \forall\ v \in X\}$$

where $f^\circ(\overline{x}; v) := \limsup_{(t,x)\longrightarrow(0^+,\overline{x})} t^{-1}[f(x + tv) - f(x)]$.

Let $\mathcal{F}$ be the collection of all finite dimensional subspaces of X. The approximate subdifferential (see Ioffe [18]) is defined by

$$\partial_A f(\overline{x}) := \bigcap_{L\in\mathcal{F}} \limsup_{x\longrightarrow\overline{x}} \partial^- f_{x+L}(x) = \bigcap_{L\in\mathcal{F}} \limsup_{(\varepsilon,x)\rightarrow(0^+,\overline{x})} \partial^-_\varepsilon f_{x+L} f(x)$$

where $f_{x+L}(u) = f(u)$ if $u \in x + L$ and $f_{x+L}(u) = +\infty$ otherwise, for $\varepsilon \geq 0$

$$\partial^-_\varepsilon f_{x+L}(x) := \{x^* \in X^* : \langle x^*, v\rangle \leq \varepsilon\|v\| + \liminf_{t\longrightarrow 0^+} t^{-1}[f_{x+L}(x + tv) - f_{x+L}(x)], \forall v\}$$

and with the convention that $\partial^- = \partial^-_\varepsilon$ for $\varepsilon = 0$. Here, for a set-valued mapping M from (a metric space) U into X^*, $x^* \in \limsup_{u\longrightarrow\overline{u}} M(u)$ means that there exists a net $(u_i, x^*_i) \in Gr\,M := \{(u, x) : u \in U, x^* \in M(u)\}$ converging to $(\overline{u}, x^*)$ with respect to the metric topology in U and the weak–star topology in X^*. Recall (see Ioffe [18]) that (f being locally Lipschitz) one always has

$$\partial_C f(\overline{x}) = w^* - cl\ co\ \partial_A f(\overline{x})$$

(the weak–star closed convex hull of $\partial_A f(\overline{x})$).

When f is the distance function $d(.; S)$ to a subset S and $\overline{x} \in S$, one can require $x \in S$ in the limit above (see [18]), that is

$$\partial_A d(.; S)(\overline{x}) = \bigcap_{L\in\mathcal{F}} \limsup_{\substack{(\varepsilon,x)\longrightarrow(0_+,\overline{x}) \\ x\in S}} \partial^-_\varepsilon d_{x+L}(.; S)(x).$$

So for $0 \leq \alpha \leq \beta$ one has

$$\alpha\partial_A d(.; S)(\overline{x}) \subset \beta\ \partial_A d(.; S)(\overline{x}).$$

In the sequel we will often write $\partial_A d(\overline{x}; S)$ in place of $\partial_A d(.; S)(\overline{x})$.

Consider now a set–valued mapping F from X into Y and the function Δ_F defined on $X \times Y$ by

$$\Delta_F(x, y) = d(y, F(x)) \text{ if } x \in \operatorname{dom} F \quad \text{and} \quad \Delta_F(x, y) = +\infty \text{ otherwise,}$$

where dom $F := \{x \in X : F(x) \neq \emptyset\}$ and note that one always has $d(x,y;GrF) \leq \Delta_F(x,y)$. F.H. Clarke [9] has often used, in optimal control theory, the function Δ_F for locally Lipschitzian set–valued mappings F and L. Thibault [37] has shown that this function can also be crucial in the study of several optimization problems even if F is not (pseudo)–Lipschitzian. Recall that F is γ–pseudo–Lipschitzian around $(\overline{x},\overline{y}) \in Gr\,F$ (see Aubin [2]) if there exists a real number $r > 0$ such that for all $x, x' \in \overline{x} + r\,\mathbb{B}_{\mathbb{X}}$

$$(\overline{y} + r\,\mathbb{B}_{\mathbb{Y}}) \cap F(x) \subset F(x') + \gamma\|x - x'\|\mathbb{B}_{\mathbb{Y}}. \tag{1.1}$$

For such set-valued mappings F one has (see Thibault [37]) for (x,y) near $(\overline{x},\overline{y})$

$$\Delta_F(x,y) \leq (1+\gamma)d(x,y;Gr\,F). \tag{1.2}$$

Note that this has been observed earlier by Clarke [9] for Lipschitzian set-valued mappings.

R.T. Rockafellar [32] has proved that F is γ–pseudo–Lipschitzian around $(\overline{x},\overline{y})$ iff there exists $r > 0$ such that for all $x, x' \in \overline{x}+r\,\mathbb{B}_{\mathbb{X}}$ and $y, y' \in \overline{y}+r\,\mathbb{B}_{\mathbb{Y}}$

$$|d(y;F(x)) - d(y';F(x'))| \leq \gamma\|x - x'\| + \|y - y'\|. \tag{1.3}$$

These set–valued mappings will play a crucial role in our approach by using the metric regularity. An important tool, with which we will give concrete verifiable conditions ensuring metric regularity, is the notion of approximate coderivative. It will also be the key in our approach for optimality conditions for the problem (P). So we end this section by recalling that the approximate coderivative $D^*_A F(\overline{x},\overline{y})$ (resp. the Clarke coderivative $D^*_C F(\overline{x},\overline{y})$) of F at $(\overline{x},\overline{y})$ is the set–valued mapping from Y^* into X^* defined by

$$x^* \in D^*_A F(\overline{x},\overline{y})(y^*) \Longleftrightarrow (x^*, -y^*) \in \mathbb{R}_+ \partial_A(\overline{x},\overline{y};Gr\,F)$$

(resp. $x^* \in D^*_C F(\overline{x},\overline{y})(y^*) \Longleftrightarrow (x^*, -y^*) \in \mathbb{R}_+\ \partial_C(\overline{x},\overline{y};Gr\,F)$).

2. Metric regularity. It is well–known that optimality conditions with Lagrange–Kuhn–Tucker multipliers for optimization problems with single–valued objectives require qualification assumptions. In our approach in the next section the qualification conditions satisfied by the constraints are related to the notion of metric regularity. Beginning with the papers by Robinson [30] and Ursescu [39] on set–valued mappings with closed convex graphs, several authors have studied the metric regularity of general set–valued mappings. Here we are going to consider the metric regularity of a set–valued mapping with respect to another one.

Definition 2.1. Let G_1 and G_2 be two set-valued mappings from X into Z and $(\overline{x}, \overline{z}) \in GrG_1 \cap GrG_2$. We will say that G_1 is metrically regular around $(\overline{x}, \overline{z})$ relatively to G_2 if there exists $\gamma \geq 0$ and $r > 0$ such that

$$d(x, z; GrG_1 \cap GrG_2) \leq \gamma\, d(z; G_1(x)) \tag{2.1}$$

for all$(x, z) \in (\overline{x} + r\mathbb{B}_{\mathbb{X}}) \times (\overline{F} + \diagdown\mathbb{B}_{\mathbb{Z}}) \cap \mathbb{G}\diagdown\mathbb{G}_{\not\geq}$.

REMARKS. As (with the convention $d(x; \emptyset) = +\infty$)

$$d(x, z; GrG_1 \cap GrG_2) \leq d(x; G_1^{-1}(z) \cap G_2^{-1}(z))$$

(where $G_1^{-1}(z) := \{x \in X : z \in G_1(x)\}$) the inequality above is fulfilled when several other concepts are satisfied.

1) For $G_2(x) = \{0\}$ if $x \in S$ and $G_2(x) = \emptyset$ otherwise, (2.1) is implied by the assumption

$$d(x; S \cap G_1^{-1}(z)) \leq \gamma\, d(z; G_1(x)) \tag{2.2}$$

for all $(x, z) \in (\overline{x} + r\mathbb{B}_{\mathbb{X}}) \times (\diagdown\mathbb{B}_{\mathbb{Z}}) \cap \mathbb{S} \times \mathbb{Z}$. (Note that Jourani and Thibault [24] have proved that (2.2) ensures the existence of Lagrange-Kuhn-Tucker multipliers for the problem

(P') Minimize $f(x)$ subject to $x \in S$ and $0 \in G_1(x)$

where f is a single-valued mapping from X into Y).

2) Consider G_2 given as above and $G_1(x) = -g(x) + D$ if $x \in S$ and $G_1(x) = \emptyset$ otherwise. Then (2.1) is equivalent to the relation

$$d(x; S \cap g^{-1}(D)) \leq \gamma\, d(g(x); D) \tag{2.3}$$

for all $x \in (\overline{x} + r\,\mathbb{B}_{\mathbb{X}}) \cap \mathbb{S}$. This relation has been used in Jourani and Thibault [22] to establish optimality conditions with Lagrange-Kuhn-Tucker multipliers for the problem

(P'') Minimize $f(x)$ subject to $x \in S$ and $g(x) \in D$.

3) Suppose now that $g : X \longrightarrow Z$ is continuously differentiable at $\overline{x} \in S \cap g^{-1}(D)$ and that S and D are closed convex subsets of X and Z respectively. Under the condition

$$0 \in \text{core}\ [\nabla g(\overline{x})(S - \overline{x}) - (D - g(\overline{x}))] \tag{2.4}$$

necessary optimiality conditions have been proved in Borwein [5] and Penot [30] for the problem (P") when f is continuously differentiable. It is known that (2.4) ensures (2.3) which is equivalent in this case to (2.1).

We are going to consider some conditions in terms of coderivatives ensuring (2.1). First we will need the following proposition using ideas, introduced in metric regularity theory by Ioffe [17] and applied later by Auslender [4] and Borwein [5]. The method will be also inspired by techniques in Theorem 2.2. of Borwein and Zhuang [7], Lemma 1.2 of Jourani [20] and Theorem 3.1 of Kruger [25]. Here we will follow the approaches in [20] and [7].

Proposition 2.2. *Let G_1 and G_2 be two set–valued mappings with closed graphs from X into Z and let $(\overline{x},\overline{z}) \in Gr\, G_1 \cap Gr\, G_2$. If G_1 is not metrically regular at $(\overline{x},\overline{z})$ relatively to G_2, then there exist sequences $(x_n, y_n, z_n) \longrightarrow (\overline{x},\overline{z},\overline{z})$ and $s_n \downarrow 0$ such that*

i) $y_n \in G_1(x_n), z_n \in G_2(x_n)$ and $z_n \notin G_1(x_n)$

ii) $\|z_n - y_n\| \leq \|z - y\| + s_n(\|x - x_n\| + \|y - y_n\| + \|z - z_n\|)$

for all $(x, y, z) \in A := \{(x, y, z) \in X \times Z \times Z : (x, y) \in Gr\, G_1, (x, z) \in Gr\, G_2\}$.

Proof. By definition 2.1 there exists $(a_n, b_n) \longrightarrow (\overline{x},\overline{z})$ with $(a_n, b_n) \in Gr\, G_2$ such that

$$d(a_n, b_n; Gr\, G_1 \cap Gr\, G_2) > nd(b_n; G_1(a_n)).$$

It follows that $b_n \notin G_1(a_n)$ and there exists $c_n \in G_1(a_n)$ satisfying

$$d(a_n, b_n; Gr\, G_1 \cap Gr\, G_2) > n\|b_n - c_n\|. \tag{2.5}$$

Since $(a_n, b_n)_n$ converges, one has $c_n \longrightarrow \overline{z}$. If one puts $f(x, y, z) := \|z - y\|$ and $\varepsilon_n^2 := f(a_n, b_n, c_n)$ one has $\varepsilon_n^2 > 0$ and for all $(x, y, z) \in A$

$$f(a_n, b_n, c_n) \leq f(x, y, z) + \varepsilon_n^2.$$

Applying the Ekeland variational principle (see [13]) on A with $\lambda_n := \min\{n\varepsilon_n^2, \varepsilon_n\}$ one obtains $(x_n, y_n, z_n) \in A$ satisfying for all $(x, y, z) \in A$

$$\begin{aligned} &\|x_n - a_n\| + \|y_n - c_n\| + \|z_n - b_n\| \leq \lambda_n \\ &f_n(x_n, y_n, z_n) \leq f_n(x, y, z) + s_n(\|x - x_n\| + \|y - y_n\| + \|z - z_n\|), \end{aligned} \tag{2.6}$$

where $s_n := \lambda_n^{-1}\varepsilon_n^2 \longrightarrow 0$. Moreover (2.6) and (2.5) ensure that $z_n \notin G_1(x_n)$ and hence the proof is complete. □

Before proving our first theorem, let us introduce the following notion.

Definition 2.3. Let G be a set–valued mapping from X into Z and $(\overline{x},\overline{z}) \in Gr\, G$. We will say that G is partially normally stable at $(\overline{x},\overline{z})$ (with respect to the second variable) if for any sequence $(x_n, z_n) \longrightarrow (\overline{x},\overline{z})$ with $z_n \in G(x_n)$ and any $(x_n^*, z_n^*) \in \partial_A d(x_n, z_n; Gr\, G)$ with $\lim \|z_n^*\| \neq 0$, one has $(x^*, z^*) \neq (0, 0)$ for the limit (x^*, z^*) of any w^*–convergent subnet of (x_n^*, z_n^*).

When $z^* \neq 0$ for the limit (x^*, z^*) of any w^*–convergent subnet, we will say that G is *partially uniformly normally stable* at $(\overline{x}, \overline{z})$.

Obviously any set–valued mapping is partially uniformly normally stable (hence partially normally stable) whenever the range space Z is finite dimensional. More generally, a typical example of such mappings is that of partially compactly epi–Lipschitzian.

Recall (see Jourani and Thibault [24]) that G is *partially compactly epi–Lipschitzian* at $(\overline{x}, \overline{z})$ if there exist a real number $r > 0$ and two compact subsets H and K in X and Z respectively such that

$$(\overline{x} + r\mathbb{B}_X) \times (\overline{z} + r\mathbb{B}_Z) \cap Gr\,G + \varepsilon(\{0\} \times r\mathbb{B}_Z) \subset Gr\,G - \varepsilon(H \times K).$$

This is a slight adaptation of the definition of compactly epi–Lipchitzian sets by Borwein and Strojwas [6] to set-valued mappings.

Jourani and Thibault [24] showed that, for such a set–valued mapping, there exists $\gamma > 0$ such that for any $\varepsilon \in]0,1]$ there are vectors $h_1, \dots, h_m \in H$ and $k_1, \dots, k_m \in K$ satisfying

$$\varepsilon \|x^*\| + \|z^*\| \leq 3\varepsilon + \gamma \max_{1 \leq i \leq m} |\langle x^*, h_i \rangle| + \gamma \max_{1 \leq i \leq m} |\langle z^*, k_i \rangle|$$

for all (x, z) near $(\overline{x}, \overline{z})$ and $(x^*, z^*) \in \partial_A d(x, z; Gr\,G)$. According to this inequality, G is partially normally stable at $(\overline{x}, \overline{z})$ whenever it is partially compactly epi–Lipschitzian at $(\overline{x}, \overline{z})$.

Theorem 2.4. *Let G_1 and G_2 be two set–valued mappings with closed graphs from X into Z with $(\overline{x}, \overline{z}) \in Gr\,G_1 \cap Gr\,G_2$ and let A be the subset as given in the statement of Proposition 2.2. Assume that for some $\alpha > 0$*

$$(2.7) \qquad d(x, y, z; A) \leq \alpha\ [d(x, y; Gr\,G_1) + d(x, z; Gr\,G_2)]$$

for all (x, y, z) near $(\overline{x}, \overline{y}, \overline{z})$, that G_1 or G_2 is partially normally stable at $(\overline{x}, \overline{z})$ and

$$(2.9) \qquad D_A^* G_1(\overline{x}, \overline{z})(0) \cap (-D_A^* G_2(\overline{x}, \overline{z})(0)) = \{0\}.$$

If for all $z^* \neq 0$ in Z^*

$$(2.10) \qquad 0 \notin D_A^* G_1(\overline{x}, \overline{z})(z^*) + D_A^* G_2(\overline{x}, \overline{z})(-z^*),$$

then the set-valued mapping G_1 is metrically regular at $(\overline{x}, \overline{z})$ relatively to G_2.

Proof. Assume that the metric regularity is not satisfied. By (2.7), the Clarke penalization (Proposition 2.4.3 in [9]) and Proposition 2.2, the point

(x_n, y_n, z_n) is, for some $\beta > 0$, an unconstrained local minimizer of the function

$$(x, y, z) \longmapsto f(x, y, z) + \beta\, d(x, y; Gr\, G_1) + d(x, z; Gr\, G_2).$$

An easy calculus of the subdifferential of the function $(y, z) \longmapsto \|z - y\|$ ensures that there exist $(x^*_{1,n}, z^*_{1,n}) \in \beta\, \partial_A d(x_n, y_n; Gr\, G_1)$, $(x^*_{2,n}, z^*_{2,n}) \in \beta\, \partial_A\, d(x_n, z_n; Gr\, G_2)$ and $y^*_n \in Z^*$ with $\|y^*_n\| = 1$ satisfying

$$\|x^*_{1,n} + x^*_{2,n}\| \le s_n, \quad \|y^*_n + z^*_{1,n}\| \le s_n \quad \text{and} \quad \| - y^*_n + z^*_{2,n}\| \le s_n.$$

Extracting subnets if necessary we may suppose

$$(x^*_{1,n}, x^*_{2,n}, y^*_n, z^*_{1,n}, z^*_{2,n}) \xrightarrow{w^*} (x^*, -x^*, z^*, -z^*, z^*)$$

and hence $(x^*, -z^*) \in \beta\, \partial_A\, d(\overline{x}, \overline{z}; Gr\, G_1)$ and $(-x^*, z^*) \in \beta\, \partial_A\, d(\overline{x}, \overline{z}; Gr\, G_2)$. The partial normal stability assumption ensures that $(x^*, z^*) \neq (0, 0)$. So considering $x^* \neq 0$ (resp. $z^* \neq 0$) we arrive at a contradiction with (2.9) (resp. 2.10) and the proof is complete. □

The proof of the following theorem is similar to the one of Theorem 2.4.

Theorem 2.5. *Under the notations of Theorem 2.4, assume that (2.7) holds, that G_1 or G_2 is partially uniformly normally stable at $(\overline{x}, \overline{z})$ and that (2.10) holds. Then G_1 is metrically regular at $(\overline{x}, \overline{z})$ relatively to G_2.*

Before giving some corollaries, let us consider an important case where condition (2.7) is automatically satisfied.

Proposition 2.6. *Assume that $G_1 : X \rightrightarrows Y$ is γ–pseudo–Lipschitzian at $(\overline{x}, \overline{y}) \in Gr\, G_1$. Then for any $G_2 : X \rightrightarrows Z$ with $\overline{z} \in G_2(\overline{x})$ and for*

$$\Lambda := \{(x, y, z) \in X \times Y \times Z : y \in G_1(x), z \in G_2(x)\},$$

one has for $k := 1 + \gamma$ and (x, y, z) near $(\overline{x}, \overline{y}, \overline{z})$

$$d(x, y, z; \Lambda) \le k[d(x, y; Gr\, G_1) + d(x, z; Gr\, G_2)].$$

Proof. Fix $r > 0$ given by (1.1), (1.2) and (1.3) and $(x, y, z) \in V := (\overline{x}, \overline{y}, \overline{r}) + (r/3)\mathbb{B}$. Then for any $(a, b) \in ((\overline{x}, \overline{z}) + r\,\mathbb{B}) \cap \mathbb{G} \setminus \mathbb{G}_{\not\models}$ and any $c \in G_1(a)$, we have

$$d(x, y, z; \Lambda) \le \|x - a\| + \|y - c\| + \|z - b\|$$

and hence

$$d(x, y; z; \Lambda) \le \|x - a\| + \|z - b\| + d(y; G_1(a)).$$

So (1.3) ensures that

$$d(x,y,z;\Lambda) \le (1+\gamma)\|x-a\| + \|z-b\| + d(y;G_1(x))$$

and hence by (1.2)

$$\begin{aligned} d(x,y,z;\Lambda) &\le (1+\gamma)\ d(x,z;Gr\,G_2 \cap ((\overline{x},\overline{z}) + r\,\mathbb{B})) + (\nVdash + \gamma)(\curvearrowleft, \curvearrowright; \mathbb{G} \setminus \mathbb{G}_{\nVdash}) \\ &= (1+\gamma)[d(x,y;Gr\,G_1) + d(x,z;Gr\,G_2)]. \end{aligned}$$

□

We can now state the following corollary. It is a direct consequence of Theorems 2.4 and 2.5 and Proposition 2.6 since, as easily seen, $\|x^*\| \le \gamma\|z^*\|$ for any $x^* \in D^*_A G_1(\overline{x},\overline{z})(z^*)$ whenever G_1 is pseudo-Lipschitzian around $(\overline{x},\overline{z})$ and Z is finite dimensional.

Corollary 2.7. *Let G_1 and G_2 be two set-valued mappings with closed graphs from X into Z and let $\overline{z} \in G_1(\overline{x}) \cap G_2(\overline{x})$ such that*

$$0 \notin D^*_A G_1(\overline{x},\overline{z})(z^*) + D^*_A G_2(\overline{x},\overline{z})(-z^*) \quad \textit{for all nonzero } z^* \textit{ in } Z^*.$$

Consider

i) *G_1 or G_2 is pseudo-Lipschitzian at $(\overline{x},\overline{z})$ and (2.9) holds ;*
ii) *G_1 or G_2 is pseudo-Lipschitzian at $(\overline{x},\overline{z})$ and Z is finite dimensional ;*
iii) *Z is finite dimensional and (2.7) is satisfied.*

Then under one of the three assertions i), ii) and iii), G_1 is metrically regular at $(\overline{x},\overline{z})$ relatively to G_2.

3. Necessary optimality conditions. In all the sequel, K will be a *convex subset* of Y with $\operatorname{int}(K) \neq \emptyset$ and $0 \in K \setminus \operatorname{int}_Y K$, F and G will be two set-valued mappings with closed graphs from X into Y and Z and S and D will be two subsets of X and Z.

We recall that $\overline{x} \in S \cap G^-(D)$ is a weak local Pareto solution for the problem

(P) Minimize $F(x)$ subject to $x \in S$ and $G(x) \cap D \neq \emptyset$

if there exist a neighborhood V of $\overline{x}$ in X and a point $\overline{y} \in F(\overline{x})$ such that for all $x \in V \cap S \cap G^-(D)$ and $y \in F(x)$ one has $\overline{y} - y \notin \operatorname{int}_Y K$.

In this case we will say that $\overline{x}$ solves locally (P) in $\overline{y}$ with respect to K. Note that the order above is not so general than the nontransitive relations considered by L. Gajek and D. Zagrodny [16] for which they established existence results for maximal points.

We start by proving the following lemma whose proof is largely inspired by a similar result in Jahn [19] (see also Thibault [38]) where the Pareto notion is considered with respect to a convex *cone*.

Lemma 3.1. *Let $\overline{y}$ be a weak local Pareto minimum of a subset $L \subset Y$. For each point $b \in \overline{y} - \frac{1}{2}\mathrm{int}_Y K$, there exist a continuous seminorm p on Y and a neighborhood W of $\overline{y}$ such that*

$$1 = p(\overline{y} - b) \leq p(y - b) \text{ for all } y \in L \cap W$$

and

$$p(\overline{y} - b - u) \leq p(\overline{y} - b) \quad \text{for all } u \in K \cap (2\overline{y} - 2b - \mathrm{int}_Y(K)).$$

Proof. Put

$$p(y) = \inf\{t \in \mathbb{R} : \approx > \nvdash, \approx^{-\nVdash} \curvearrowright \in (- \overline{\curvearrowright} + \mathbb{K}) \cap (\overline{\curvearrowright} - \ - \mathbb{K})\}.$$

Then p is a continuous seminorm (since $Q := (b - \overline{y} + K) \cap (\overline{y} - b - K)$ is a convex neighborhood of zero) and $p(\overline{y} - b) = 1$ (since $\overline{y} - b$ is a boundary point of Q). If W denotes a neighborhood of $\overline{y}$ such that $L \cap W \cap (\overline{y} - \mathrm{int}_Y K) = \emptyset$, then $((L \cap W) - b) \cap \mathrm{int}_Y Q = \emptyset$ and hence

$$1 \leq p(y - b) \text{ for all } y \in L \cap W.$$

To prove the second inequality of the lemma, it is enough to see that for any $u \in K \cap (2\overline{y} - 2b - \mathrm{int}_Y K)$ one has $\overline{y} - b - u \in Q$ and hence

$$p(\overline{y} - b - y) \leq 1 = p(\overline{y} - b).$$

□

The following second lemma will also be needed.

Lemma 3.2. *Let p be given by the lemma above. Then for any $y^* \in \partial p(\overline{y} - b)$ (the convex subdifferential) one has $y^* \neq 0$ and $\langle y^*, y \rangle \geq 0$ for all $y \in K$.*

Proof. If we fix $y^* \in \partial p(\overline{y} - b)$, then we have for all $y \in Y$

$$\langle y^*, y - \overline{y} + b \rangle \leq p(y) - p(\overline{y} - b) \tag{3.1}$$

and hence it is easy to see (taking $y = 0$ in (3.1)) that $y^* \neq 0$. Now fix any $y \in K$. Since $2\overline{y} - 2b \in \mathrm{int}_Y K$, there exists some $t > 0$ such that $2\overline{y} - 2b - ty \in \mathrm{int}_Y K$ and hence $ty \in K \cap (2\overline{y} - 2b - \mathrm{int}_Y K)$. Then it follows from (3.1) and Lemma 3.1 that

$$\langle y^*, -ty \rangle \leq p(\overline{y} - b - ty) - p(\overline{y} - b) \leq 0$$

and hence $\langle y^*, y \rangle \geq 0$, which completes the proof. □

We can now prove necessary optimality conditions for unconstrained problems.

Theorem 3.3. *Let $\overline{x}$ be a local solution in $\overline{y}$ of the problem*

$$\text{Minimize } F(x), x \in X.$$

Then there exists a nonzero $y^* \in Y^*$ such that $\langle y^*, y\rangle \geq 0$ for all $y \in K$ and

$$0 \in D_A^* F(\overline{x}, \overline{y})(y^*) \quad (\text{resp. } 0 \in D_C^* F(\overline{x}, \overline{y})(y^*)).$$

Proof. Lemma 3.1 and the Clarke penalization (Proposition 2.4.3 in [11]) ensure that (for some $k > 0$) $(\overline{x}, \overline{y})$ is a local minimizer of $(x, y) \longmapsto p(y - b) + k\, d(x, y; Gr\, F)$. By subdifferential calculus rules (see [18]) one has

$$(0, 0) \in \{0\} \times \partial p(\overline{y} - b) + k\, \partial_A d(\overline{x}, \overline{y}; Gr\, F)$$

and hence there exists $y^* \in \partial p(\overline{y} - b)$ with $(0, -y^*) \in k\, \partial_A d(\overline{x}, \overline{y}; Gr\, F)$. This implies $0 \in D_A^*\, F(\overline{x}, \overline{y})(y^*)$ and completes the proof since one always has $D_A^* \subset D_C^*$. □

Remark. Note that the scalarization method by Ciligot-Travain [8] (via the signed distance function) could be also used.

Before proving optimality conditions in terms of F and G separately, we are going to establish optimality conditions in terms of the coderivative of (F, G). Recall that $(F, G)(x) = F(x) \times G(x)$.

Proposition 3.4. *Assume that $\overline{x}$ is a local solution in $\overline{y}$ of the problem*

$$\text{Minimize } F(x) \text{ subject to } G(x) \cap D \neq \emptyset$$

where D is a convex subset of Z with nonempty interior. Then for any $\overline{z} \in G(\overline{x}) \cap D$ there exists a nonzero pair $(y^*, z^*) \in Y^* \times Z^*$ such that

$$\langle y^*, y\rangle \geq 0 \quad \text{for all } y \in K \quad \text{and} \quad \langle z^*, z\rangle \leq \langle z^*, \overline{z}\rangle \quad \text{for all } z \in D$$

and

$$0 \in D_A^*(F, G)(\overline{x}, \overline{y}, \overline{z})(y^*, z^*) \quad (\text{resp. } 0 \in D_C^*(F, G)(\overline{x}, \overline{y}, \overline{z})(y^*, z^*)).$$

Proof. For $Q := -(D - \overline{z})$, the set $K \times Q$ is a convex subset with nonempty interior and $(0, 0) \in (K \times Q) \setminus \text{int}\,(K \times Q)$. Denote by V a neighborhood of $\overline{x}$ over which the problem is solved by $\overline{x}$. Then for any $x \in V$ satisfying $(G(x) - \overline{z}) \cap (-\text{int}_Z Q) \neq \emptyset$, we have

$$F(x) - \overline{y}) \cap (-\text{int}_Y K) \neq \emptyset.$$

Therefore for any $x \in V$ we have

$$(F(x) \times G(x) - (\overline{y}, \overline{z})) \cap (-\mathrm{int}\,(K \times Q)) \neq \emptyset$$

and hence $\overline{x}$ solves in $\overline{y}$ (with respect to $K \times Q$) the unconstrained problem

$$\text{Minimize } (F, G)(x), x \in X.$$

By Theorem 3.3 there exists a nonzero pair $(y^*, z^*) \in Y^* \times Z^*$ such that

$$\langle y^*, y\rangle + \langle z^*, z\rangle \geq 0 \quad \text{for all} \quad (y, z) \in K \times Q \tag{3.2}$$

and

$$0 \in D_A^*(F, G)(\overline{x}, \overline{y})(y^*, z^*).$$

Moreover it is obvious that (3.2) ensures that

$$\langle y^*, y\rangle \geq 0 \quad \text{for all } y \in K \quad \text{and} \quad \langle z^*, z\rangle \leq \langle z^*, \overline{z}\rangle \quad \text{for all } z \in D$$

and hence the proof is complete. □

Recall that $F_S(x) = F(x)$ if $x \in S$ and $F_S(x) = \emptyset$ otherwise.

Theorem 3.5. *Assume that D is a convex subset with non empty interior and that $\overline{x}$ solves locally (P) in $\overline{y}$. Then for any $\overline{z} \in G(\overline{x}) \cap D$, there exists a nonzero pair $(y^*, z^*) \in Y^* \times Z^*$ such that*

$$\langle y^*, y\rangle \geq 0 \quad \textit{for all } y \in K \quad \textit{and} \quad \langle z^*, z\rangle \geq \langle z^*, \overline{z}\rangle \quad \textit{for all } z \in D$$

and

$$0 \in D_A^*(F_S, G_S)(\overline{x}, \overline{y}, \overline{z})(y^*, z^*) \quad (\textit{resp. } 0 \in D_C^*(F_S, G_S)(\overline{x}, \overline{y}, \overline{z})(y^*, z^*)).$$

Proof. It is enough to see that $\overline{x}$ is a local solution of the problem

$$\text{Minimize } F_S(x) \quad \text{subject to } G_S(x) \cap D \neq \emptyset$$

and to apply Theorem 3.4. □

At this stage, we can already deduce the main result (Theorem 5.1) in Corley [12]. It is a direct consequence of Theorem 3.5 and the definition of the Clarke coderivative. Recall that for a set-valued mapping M from X into Z with $\overline{z} \in M(\overline{x})$, the Clarke tangent derivative $D_C M(\overline{x}, \overline{y})$ of M at $(\overline{x}, \overline{y})$ is the set-valued mapping from X into Z whose graph is the Clarke tangent cone to Gr M at $(\overline{x}, \overline{z})$.

Corollary 3.6. *Assume that D is a convex cone with nonemtpy interior and that $\overline{x}$ is a local solution of (P) in $\overline{y}$. Then for any $\overline{z} \in G(\overline{x}) \cap D$, there exists a nonzero pair $(y^*, z^*) \in Y^* \times Z^*$ such that $\langle y^*, y\rangle \geq 0$ for all $y \in K$,*

$$\langle z^*, z\rangle \geq 0 \quad \textit{for all } z \in D \textit{ with } \langle z^*, \overline{z}\rangle = 0$$

and for all $x \in$ dom $D_C(F_S, G_S)(\overline{x}, \overline{y}, \overline{z})$ and $(y, z) \in D_C(F_S, G_S)(\overline{x}, \overline{y}, \overline{z})(x)$.

Now we are going to establish necessary optimality conditions for the problem (P) in terms of the coderivatives of F and G separately.

Considering the particular case in definition 2.1 with $G_1 = G$ and $Gr\, G_2 = S \times D$, we will say that G is metrically regular around $(\overline{x}, \overline{y})$ relatively to $S \times D$ if there exist $\gamma \geq 0$ and $r > 0$ such that

$$d(x, z; (S \times D) \cap Gr\, G) \leq \gamma\, d(z; G(x)) \tag{3.3}$$

for all $(x, z) \in [(\overline{x} + r\,\mathbb{B}_{\mathbb{X}}) \times (\overline{F} + \setminus \mathbb{B}_{\mathbb{Z}})] \cap (\mathbb{C} \times \mathbb{D})$.

In the remainder of this section we will suppose that S and D are closed *subsets* of X and Z, that $\overline{x}$ solves locally (P) in $\overline{y}$ and that $\overline{z} \in G(\overline{x}) \cap D$. We will also suppose that F and G are pseudo-Lipschitzian around $(\overline{x}, \overline{y})$ and $(\overline{x}, \overline{z})$ respectively.

Theorem 3.7. *Under the assumptions above, if G is metrically regular around $(\overline{x}, \overline{z})$ relatively to $S \times D$, then for $k > 0$ large enough, there exists a pair $(y^*, z^*) \in Y^* \times Z^*$ such that*

$$y^* \neq 0,\ \langle y^*, y\rangle \geq 0 \quad \textit{for all } y \in K, z^* \in k\,\partial_A d(\overline{z}; D)$$
$$(\textit{resp. } z^* \in k\,\partial_C\, d(\overline{z}; D))$$

and

$$0 \in D_A^* F(\overline{x}, \overline{y})(y^*) + D_A^* G(\overline{x}, \overline{z})(z^*) + k\,\partial_A d(\overline{x}; S)$$
$$(\textit{resp. } 0 \in D_C^*\, F(\overline{x}, \overline{y})(y^*) + D_C^*\, G(\overline{x}, \overline{z})(z^*) + k\,\partial_C d(\overline{x}; S)).$$

Proof. If we put $q(x, y, z) := p(y - b)$ (where p is given by Lemma 3.1), it is not difficult to see that $(\overline{x}, \overline{y}, \overline{z})$ is a local minimizer of the problem

$$\text{Minimize } q(x, y, z) \text{ subject to } (x, y) \in Gr\, F \text{ and } (x, z) \in (S \times D) \cap Gr\, G.$$

Then by the Clarke penalization (see Proposition 2.4.3 in [9]), the metric regularity assumption and Proposition 2.6, for $k > 0$ large enough, $(\overline{x}, \overline{y}, \overline{z})$ is an unconstrained local minimizer of the function

$$(x, y, z) \longmapsto q(x, y, z) + kd(x, y; Gr\, F) + kd(x, z; Gr\, G) + kd(x, z; S \times D).$$

Therefore 0 is in the sum of the subdifferentials, that is there exist

$$y_1^* \in \partial p(\overline{y} - b),\ (x_2^*, y_2^*) \in k\partial_A d(\overline{x}, \overline{y}; Gr\, F)$$

$$(x_3^*, z_3^*) \in k\partial_A d(\overline{x}, \overline{z}; Gr\, G) \quad \text{and} \quad (x_4^*, z_4^*) \in k\partial_A d(\overline{x}, \overline{z}; S \times D)$$

such that

$$0 = x_2^* + x_3^* + x_4^*,\ 0 = y_1^* + y_2^* \text{ and } 0 = z_3^* + z_4^*.$$

Putting $y^* := y_1^* = -y_2^*$ and $z^* := z_4^* = -z_3^*$, we obtain

$$0 \in D_A^* F(\overline{x}, \overline{y})(y^*) + D_A^* G(\overline{x}, \overline{z})(z^*) + k\partial_A d(\overline{x}; S).$$

To conclude, it remains to apply Lemma 3.2 to get $y^* \neq 0$ and $\langle y^*, y\rangle \geq 0$ for all $y \in K$. □

The corollaries below are direct consequences of Theorem 3.7, 2.4, 2.5 and Proposition 2.6.

Corollary 3.8. *Suppose that, in place of the metric regularity of G in Theorem 3.7, both assumptions below are fulfilled*

i) *for each nonzero $z^* \in \mathbb{R}_+\ \partial_{\mathbb{A}}(\overline{F}; \mathbb{D})$ one has (see 2.10)*

$$0 \notin D_A^* G(\overline{x}, \overline{z})(z^*) + \mathbb{R}_+ \partial_{\mathbb{A}}(\overline{\frown}; \mathbb{S});$$

ii) *G is partially normally stable at $(\overline{x}, \overline{z})$ and (see 2.9)*

$$D_A^* G(\overline{x}, \overline{z})(0) \cap (-\partial_A d(\overline{x}; S)) = \{0\}.$$

Then the conclusion of Theorem 3.7 holds.

Corollary 3.9. *Suppose that in corollary 3.8 the assumption ii) is replaced by one of the following assumptions*

iii) G is partially uniformly normally stable at $(\overline{x}, \overline{y})$;

iv) D is normally stable at $(\overline{x}, \overline{y})$.

Then the conclusion of Theorem 3.7 holds.

According to Corolary 2.7 we also have the following corollary.

Corollary 3.10. *Suppose that assumption i) in Corollary 3.8 is fulfilled and that Z is finite dimensional. Then the conclusion of Theorem 3.7 holds.*

Now we are going to show that optimality conditions with Lagrange–Fritz John multipliers can be derived from the results above.

Theorem 3.11. *Suppose that either assumption ii) in Corollary 3.8 is fulfilled or Z is finite dimensional. Then there exist some $k > 0$ and a nonzero pair $(y^*, z^*) \in Y^* \times Z^*$ such that*

$$\langle y^*, y\rangle \geq 0 \text{ for all } \ y \in K, z^* \in k\partial_A d(\overline{z}; D) \quad (\text{resp. } z^* \in k\partial_C d(\overline{z}; D))$$

and

$$0 \in D_A^* F(\overline{x}, \overline{y})(y^*) + D_A^* G(\overline{x}, \overline{z})(z^*) + k\partial_A d(\overline{x}; S)$$

(resp. $0 \in D_C^* F(\overline{x}, \overline{y})(y^*) + D_C^* G(\overline{x}, \overline{z})(z^*) + k\partial_C d(\overline{x}; S)$*).*

Proof. If the assumption i) in Corollary 3.8 is satisfied, then the result follows from this corollary. Otherwise there exists a nonzero $z^* \in \mathbb{R}_+ \partial_{\mathbb{A}}(\overline{F}; \mathbb{D})$ such that

$$0 \in D_A^* G(\overline{x}, \overline{z})(z^*) + \mathbb{R}_+ \partial_{\mathbb{A}}(\overline{\curvearrowleft}; \mathbb{S})$$

and hence it is enough to choose $y^* = 0$. □

One can also easily derive from the results above the necessary optimality conditions established in El Abdouni and Thibault [14], Thibault [38] and Jourani [21] for Pareto optimization problems with single-valued objective mappings which are compactly Lipschitzian in the sense introduced by the second author (see [35, 36]).

4. The convex case. In this section we are going to consider the convex case. We will show in this case that all the preceding necessary optimality conditions are sufficient too.

Recall that the set-valued mapping F is convex if its graph is a convex subset of $X \times Y$.

Theorem 4.1. *Assume that F and G are convex and that S and D are convex subsets of X and Z. Let $\overline{x} \in S \cap G^-(D)$ and $\overline{y} \in F(\overline{x})$. If there exist $\overline{z} \in G(\overline{x}) \cap D$, a nonzero y^* in Y^* with $\langle y^*, y \rangle \geq 0$ for all $y \in K$ and $z^* \in \mathbb{R}_+ \partial_{\mathbb{A}}(\overline{F}; \mathbb{D})$ such that*

$$0 \in D_A^* F(\overline{x}, \overline{y})(y^*) + D_A^* G(\overline{x}, \overline{z})(z^*) + \mathbb{R}_+ \partial_{\mathbb{A}}(\overline{\curvearrowleft}; \mathbb{S}),$$

then $\overline{x}$ solves the problem (P) in $\overline{y}$.

Proof. Suppose that $\overline{x}$ does not solve (P) in $\overline{y}$. Then, there exist $x \in S \cap G^-(D)$ and $y \in F(x)$ such that

$$y_1 := y - \overline{y} \in -\text{int}_Y K.$$

As $y^* \neq 0$, there exists $y_0 \in Y$ with $\langle y^*, y_0 \rangle > 0$, and since $y_1 \in -\text{int}_Y K$, we may choose $t > 0$ such that $-y_1 - ty_0 \in \text{int}_Y K$ which ensures

$$\langle y^*, y_1 \rangle \leq -t\langle y^*, y_0 \rangle < 0. \tag{4.1}$$

Moreover, we may write (because of the assumptions)

$$0 = x_1^* + x_2^* + x_3^* \tag{4.2}$$

with $(x_1^*, -y^*) \in \mathbb{R}_+\partial_{\mathbb{A}}(\overline{\curvearrowleft}, \overline{\curvearrowright}; \mathbb{G}\setminus \mathbb{F})$, $(\curvearrowleft^*_{\not\vDash}, -\digamma^*) \in \mathbb{R}_+\partial_{\mathbb{A}}(\overline{\curvearrowleft}, \overline{\digamma}; \mathbb{G}\setminus \mathbb{G})$ and $x_3^* \in \mathbb{R}_+\partial_{\mathbb{A}}(\overline{\curvearrowleft}; \mathbb{S})$. Since $x \in S \cap G^-(D)$, there exists some $z \in G(x) \cap D$ and by subdifferential calculus rules in convex analysis we have

$$\langle x_1^*, x - \overline{x}\rangle - \langle y^*, y - \overline{y}\rangle \leq 0 \text{ and } \langle x_2^*, x - \overline{x}\rangle - \langle z^*, z - \overline{z}\rangle \leq 0.$$

Therefore

$$\langle x_1^* + x_2^*, x - \overline{x}\rangle - \langle y^*, y_1\rangle - \langle z^*, z - \overline{z}\rangle \leq 0$$

and hence, since $x_3^* \in \mathbb{R}_+\partial_{\mathbb{A}}(\overline{\curvearrowleft}; \mathbb{S})$ and $z^* \in \mathbb{R}_+\partial_{\mathbb{A}}(\overline{\digamma}; \mathbb{D})$, we obtain by (4.2)

$$\langle y^*, y_1\rangle \geq -\langle x_3^*, x - \overline{x}\rangle - \langle z^*, z - \overline{z}\rangle \geq 0,$$

which is in contradiction with (4.1). So the proof is complete. □

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