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NONCONVEX SWEEPING PROCESS

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Abstract. We discuss the existence of BV and Lipschitzean solutions for the sweeping process associated to a nonconvex closed moving set and its applications to a new class of evolution problem governed by a subdifferential of a Lipschitzean function. Convex and nonconvex perturbations of the preceding evolution problem are also studied.

1. Introduction. Convex sweeping process was introduced by J.J. Moreau in 1971 (*in French, rafle*, see [25], [26], [27]). We refer to [6] and [22] for a complete bibliography on the subject. However not much study has been done for the sweeping process without convexity even in case when the closed moving set $C(t)$ has the form $C_0 + v(t)$ where C_0 is a fixed closed nonconvex subset in $\mathbb{R}^d$ and v is a $\mathbb{R}^d$ valued Lipschitzean mapping defined on an interval $[0, T]$. In ([29], [30]) Valadier studied some cases of the sweeping process without convexity, mainly when the closed moving set is the complementary of the interior of a closed convex moving set, and also when the closed moving set is of the form $epi f + (0, \varphi(t))$ where f belongs to a special class of Lipschitzean functions defined on $\mathbb{R}^d$ and φ is an increasing Lipschitzean real function defined on $[0, T]$. In the present paper

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we will deal with convex and nonconvex perturbations in $\mathbb{R}^d$ of the sweeping process associated with a closed moving set $C(t)$ in $\mathbb{R}^d$:

$$u'(t) \in -N_{C(t)}(u(t)) + F(t, u(t))$$

where $N_{C(t)}(u(t))$ denotes the Clarke normal cone to $C(t)$ at $u(t)$ and F is a bounded closed–valued multifunction defined on $[0, T] \times \mathbb{R}^d$.

In Section 2 we recall for completeness some useful results and the relationship between Clarke normal cone and Mordukhovich normal cone (alias *limiting proximal cone*).

In Section 3 we present an abstract existence result of BV solutions for the sweeping process by nonconvex closed moving sets.

Section 4 is devoted to the closure properties of the Mordukhovich normal cone to a closed moving set $C(t)$ of the form $C(t) = A_t(C_0)$ where A_t is a linear isomorphism from $\mathbb{R}^d$ to $\mathbb{R}^d$ depending continuously on $t \in [0, T]$.

In Section 5 we provide several applications to convex and nonconvex perturbations of the above differential inclusion and also the existence of Lipschitzean solutions (y, θ) with values in $\mathbb{R}^{d+1}$ for the evolution problem of the form

$$\begin{cases} \theta(t) = f(y(t)) + \varphi(t) \\ y'(t) \in -\theta'(t)\partial f(y(t)) \end{cases}$$

where φ is an increasing Lipschitzean function defined on $[0, T]$, f is a real valued Lipschitzean function defined on $\mathbb{R}^d$ and ∂f is the Clarke subdifferential of f.

2. Notations and Preliminaries. We will use the following notions and notations.

– $\mathbb{R}^d$ is endowed with its canonical Euclidean structure. The scalar product of x and y is denoted by $\langle x, y\rangle$.

– If I denotes the interval $[0, T]$ $(T > 0)$ of $\mathbb{R}$, dt is the Lebesgue measure λ on I, $\tau_\lambda(I)$ is the σ-algebra of all Lebesgue measurable subsets of I and I_r is the interval I equipped with the right topology.

– A subdivision of $[0, T]$ $(T > 0)$ is a finite sequence $(t_0, ..., t_n)$ such that

$$0 = t_0 < t_1 < ... < t_n = T.$$

– *The variation of a mapping* $u : [0, T] \to \mathbb{R}^d$ is the supremum over the set of all subdivisions of $[0, T]$ of the sums $\Sigma_{i=1}^n ||u(t_i) - u(t_{i-1})||$. It is denoted by $var(u; 0, T)$. The mapping u has bounded variation (BV) if $var(u; 0, T) < +\infty$.

– If u is BV, its left–hand side limit $u^-(t)$ exists at any $t > 0$. By convention $u^-(0) = u(0)$. The righ–hand side limit is denoted by $u^+(t)$. Then there exists a vector measure denoted by Du such that

$$\forall a < b, Du([a, b]) = u^+(b) - u^-(a).$$

The measure Du is the *differential measure* of u.

– If μ is a positive Radon measure on I and if $f \in L^1_{\mathbb{R}^d}(I, \mu)$, then the mapping $u : [0, T] \to \mathbb{R}^d$ defined by $u(t) = u(0) + \int_{]0,t]} f(s)\mu(ds)$ is a BV and right continuous mapping.

– If C is a nonempty closed subset of $\mathbb{R}^d$ and $x \in \mathbb{R}^d$, we denote by $\delta^*(\cdot, C)$ the support function of C and we set

$$\text{proj}_C(x) = \{y \in C : ||x - y|| = d(x, C)\}$$

where $d(x, C)$ is the distance from x to C. A vector $y \in \mathbb{R}^d$ is *normal proximal to C at $x \in C$* iff there exists a positive scalar σ such that:

$$\langle y, a - x\rangle \leq \sigma||a - x||^2$$

for all $a \in C$. We denote by $\Pi_C(x)$ the set of all normal proximal vectors to C at $x \in C$ and the Mordukhovich normal cone of C at x (alias *limiting proximal normal cone*) is defined by:

$$M_C(x) := \{\lim_{i\to\infty} y_i : \forall i, y_i \in \Pi_C(x_i),\ x_i \in C \text{ and } \lim_{i\to\infty} x_i = x\}.$$

– If A and B are subsets of $\mathbb{R}^d$, the excess of A over B is

$$e(A, B) = \sup\{d(a, B) : a \in A\}.$$

(where $d(a, B)$ is the distance from a to B) and their Hausdorff distance is

$$h(A, B) = \max(e(A, B), e(B, A)).$$

The excess $e(A, 0)$ is denoted by $|A|$.

– $c(\mathbb{R}^d)$ (resp. $k(\mathbb{R}^d)$ (resp. $cc(\mathbb{R}^d)$ (resp. $ck(\mathbb{R}^d)$) is the collection of all nonempty closed (resp. compact) (resp. closed convex) (resp. convex compact) subsets of $\mathbb{R}^d$.

– $Isom(\mathbb{R}^d, \mathbb{R}^d)$ is the set of all linear isomorphisms from $\mathbb{R}^d$ to $\mathbb{R}^d$.

For the sake of completeness we summarize the characterizations of proximal points and normal proximal vectors to a nonempty closed set C at $x \in C$ and also the relationship between the Mordukhovich normal cone and the Clarke normal cone to C at x.

The following easy result is standard.

Lemma 2.1. *Let C be a nonempty closed subset of $\mathbb{R}^d$. Then the following assertions hold:*

(a) *Let $x \in \mathbb{R}^d$. Then $y \in proj_C(x)$ iff:*

$$\forall a \in C, \langle x - y, a - y\rangle \leq \frac{1}{2}||a - y||^2.$$

(b) *Let $u \in \mathbb{R}^d$. If $v \in proj_C(u)$, then for every $w \in [v, u]$, $v \in proj_C(w)$.*

Proof. Indeed, to prove the first assertion, observe that, for every $a \in C$, we have

$$\begin{aligned} ||a - x||^2 - ||y - x||^2 &= \langle (a - x) - (y - x), (a - x) + (y - x)\rangle \\ &= \langle a - y, a + y - 2x\rangle \\ &= ||a - y||^2 + \langle a - y, a + y - 2x - a + y\rangle \\ &= ||a - y||^2 - 2\langle a - y, x - y\rangle. \end{aligned}$$

Let us prove the second assertion. Assume that $v \in \mathrm{proj}_C(u)$. Then we have

$$\forall c \in C, \langle u - v, c - v\rangle \leq \frac{1}{2}||c - v||^2.$$

Let $w \in [v, u]$. Then w has the form $w = v + t(u - v)$ with $0 \leq t \leq 1$ so that

$$\begin{aligned} \langle w - v, c - v\rangle &= t\langle u - v, c - v\rangle \\ &\leq t\frac{1}{2}||c - v||^2 \\ &\leq \frac{1}{2}||c - v||^2 \end{aligned}$$

for all $c \in C$. Hence $v \in \mathrm{proj}_C(w)$. □

Remarks 2.2. (a) $\Pi_C(x)$ is a convex cone and in view of Lemma 2.1 we have the following characterization of $\Pi_C(x)$: $y \in \Pi_C(x) \iff$ there exists $\delta > 0$ such that $x \in \mathrm{proj}_C(x + \delta y)$. It is enough to take $\delta = \frac{1}{2\sigma}$ in the definition of $\Pi_C(x)$.

(b) $y \in M_C(x)$ iff : for every $\varepsilon > 0$, there exists $(u, v) \in C \times \mathbb{R}^d$ such that $v \in \Pi_C(u)$, $||x - u|| < \varepsilon$, $||y - v|| < \varepsilon$.

The following result is well known (see, for example, [12] and [17]) and we provide a proof for the convenience of the reader.

Proposition 2.3. *Let C be a nonempty closed subset of $\mathbb{R}^d$ with $x \in C$ and let $N_C(x)$ be the Clarke normal cone to C at the point x. Then the following hold:*

(a) $N_C(x) = \overline{co}\, M_C(x)$.
(b) *The multifunction* $x \mapsto M_C(x)$ *has closed graph.*
(c) *Let* V *be a closed neighbourhood of* x. *Then the following equalities hold:*

$$\Pi_C(x) = \Pi_{C\cap V}(x),\ M_C(x) = M_{C\cap V}(x),\ N_C(x) = N_{C\cap V}(x).$$

(d) *If* C *is convex, then*

$$N_C(x) = M_C(x) = \Pi_C(x).$$

Proof. (a) Let us recall that $N_C(x) = \overline{\bigcup_\lambda \lambda \partial d_C(x)}$ where $d_C(\cdot) := d(\cdot, C)$. But $\partial d_C(x) = co\, D_C(x)$ where

$$D_C(x) = \{0\} \cup \{v = \lim_{i\to\infty} \frac{v_i}{||v_i||} : v_i \text{ perpendicular to } C \text{ at } x_i \text{ with } x_i \to x \text{ and } v_i \to 0\}$$

where v_i perpendicular to C at x_i means that: $v_i \neq 0$ and $\mathrm{proj}_C(x_i + v_i) = \{x_i\}$. It is clear that $\forall \lambda \geq 0$, we have $\lambda D_C(x) \subset M_C(x)$. Hence it follows that

$$\lambda \partial d_C(x) = co[\lambda D_C(x)] \subset co\, M_C(x).$$

Therefore $N_C(x) \subset \overline{co}\, M_C(x)$. Let us check the converse inclusion. Let any nonzero $y \in M_C(x)$ with $y = \lim_{i\to\infty} y_i$, $y_i \in \Pi_C(x_i)$ and $x_i \to x \in C$. Choose $t_i \to 0_+$ such that $x_i \in \mathrm{proj}_C(x_i + t_i y_i)$. Now fix $v \in T_C(x)$ where $T_C(x)$ is the Clarke tangent cone to C. Then there exist $v_i \to v$ such that $x_i + t_i^2 v_i \in C$. By Lemma 2.1 one has

$$\langle x_i + t_i y_i - x_i \,, x_i + t_i^2 v_i - x_i \rangle \leq \frac{1}{2} t_i^4\, ||v_i||^2$$

i.e. $\langle y_i, v_i \rangle \leq \frac{1}{2} t_i\, ||v_i||^2$ and hence, taking the limit, one gets $\langle y, v \rangle \leq 0$. So $y \in N_C(x)$ and hence $\overline{co}\, M_C(x) \subset N_C(x)$. (*)[1]

(b) follows from the definition of $M_C(x)$.

(c) First we prove equality $\Pi_C(x) = \Pi_{C\cap V}(x)$. Let $y \in \Pi_C(x)$. Then there exists $\delta > 0$ such that $x \in \mathrm{proj}_C(x + \delta y)$. Whence a fortiori $x \in \mathrm{proj}_{C\cap V}(x + \delta y)$. So $y \in \Pi_{C\cap V}(x)$. Conversely assume that $y \in \Pi_{C\cap V}(x)$. Then there exists $\eta > 0$ such that $x \in \mathrm{proj}_{C\cap V}(x + \eta y)$. Pick $r > 0$ such that $\overline{B}(x, r) \subset V$ and choose $\delta > 0$ such that $0 < \delta \leq \eta$ and that $||\delta y|| < \frac{1}{2} r$. Since $x + \delta y \in [x, x + \eta y]$, by Lemma 2.1 $x \in \mathrm{proj}_{C\cap V}(x + \delta y)$. Now we claim that

$$(**) \qquad x \in \mathrm{proj}_C(x + \delta y).$$

[1](*) The preceding arguments are kindly communicated to us by Lionel Thibault.

Note that $(**)$ is equivalent to the following

$$(***) \qquad \forall c \in C,\ ||\delta y|| \leq ||x + \delta y - c||.$$

Let $c \in C$. If $c \in C \cap V$, then $(***)$ holds because $x \in \mathrm{proj}_{C\cap V}(x + \delta y)$. So assume that $c \notin C \cap V$, whence $c \notin \overline{B}(x, r)$. Then we have

$$||x + \delta y - c|| \geq ||x - c|| - ||\delta y|| > r - ||\delta y|| > ||\delta y||$$

thus proving $(***)$ and consequently $y \in \Pi_C(x)$. Hence we conclude that $\Pi_C(x) = \Pi_{C\cap V}(x)$.

Equality $M_C(x) = M_{C\cap V}(x)$ follows easily from the preceding equality and the definition of M_C. Indeed the relations $y = \lim_{i\to\infty} y_i$ *with* $\forall i,\ y_i \in \Pi_C(x_i), x_i \in C$ *and* $x = \lim_{i\to\infty} x_i$ are equivalent to : $y = \lim_{i\to\infty} y_i\ (i \geq i_0)$ *with* $\forall i \geq i_0,\ y_i \in \Pi_C(x_i) = \Pi_{C\cap V}(x_i),\ x_i \in C \cap \mathit{int}\, V$ *and* $\lim_{i\to\infty} x_i = x\ (i \geq i_0)$.

Finally we have

$$N_C(x) = \overline{co}\, M_C(x) = \overline{co}\, M_{C\cap V}(x) = N_{C\cap V}(x).$$

(d) If C is convex and closed, $N_C(x) \subset \Pi_C(x)$ since

$$y \in N_C(x) \Longleftrightarrow x = \mathrm{proj}_C(x + y).$$

□

To end this section we would like to mention several extensions to infinite dimensional spaces of the notions of normal cone in some series of papers by Borwein–Strowjas [4], Ioffe [18], Jofre–Thibault [19], Loewen ([20], [21]), Mordukhovich–Shao ([23], [24]).

3. Existence of BV solutions: An abstract formulation. We will provide in this section an existence theorem for BV and right continuous solutions for the sweeping process by nonconvex closed moving sets in $\mathbb{R}^d$ which is analoguous to a result due to Moreau ([27], Proposition 3b) ensuring the existence of BV and right continuous solutions for sweeping process by convex closed moving sets in a Hilbert space. We refer to Valadier [29], Castaing–Duc Ha–Valadier [6], Benabdellah–Castaing–Salvadori ([2], [3]) and Castaing–Marques ([7], [9]) for the case when the closed moving set is the complementary of the interior of a convex closed moving set. The use of the limiting proximal normal cone will be decisive in our proof. We would like to thank Lionel Thibault for recommending this fact.

Theorem 3.1. *Let μ be a positive Radon measure on $[0, T]$ and let $C : [0, T] \to c(\mathbb{R}^d)$ satisfying the following conditions:*

(i) $h(C(t), C(\tau)) \leq \mu(]\tau, t])$ *whenever* $0 \leq \tau < t \leq T$.
(ii) *The graph of the multifunction*

$$G : (t, x) \to \begin{cases} M_{C(t)}(x) & \text{if } x \in C(t) \\ \emptyset & \text{if } x \notin C(t) \end{cases}$$

is closed in $[0, T]_r \times \mathbb{R}^d \times \mathbb{R}^d$ *where* $M_{C(t)}(x)$ *is the limiting proximal normal cone to* $C(t)$ *at* $x \in C(t)$.

Then given $x_0 \in C(0)$ *there exist a right continuous mapping of bounded variation* $u : [0, T] \to \mathbb{R}^d$ *and* $u' \in L^1_{\mathbb{R}^d}([0, T], \mu)$ *such that*

(a) $\forall t \in [0, T],\ u(t) = x_0 + \int_{]0,t]} u'(s)\mu(ds)$.
(b) $-u'(t) \in N_{C(t)}(u(t))$ *μ-a.e.*

Proof. First step : Algorithm. We will adopt an algorithm developed by Moreau ([27], p. 368–369) (see also Castaing–Marques ([7], Theorem 4.2) providing the existence of approximate solutions for the sweeping process by $C(t)$. Let $(\varepsilon_n)_{n\in\mathbb{N}}$ be an infinite sequence of positive real numbers, converging to zero. Set $v(t) = \mu(]0, t])$ for all $t \in [0, T]$, then v is increasing and right continuous. Hence by taking the inverse images under v of intervals of the form $[p^n_i, p^n_{i+1}[$ with $p^n_{i+1} - p^n_i < \varepsilon_n$, one can construct an infinite sequence $(S_n)_{n\in\mathbb{N}}$ of finite partition of I into subintervals of the form

$$S_n : [0, t^n_1[, [t^n_1, t^n_2[, ..[t^n_{\nu(n)-1}, T[, \{T\}$$

with the following properties :

(j) the increment of v on every interval constituting S_n is $\leq \epsilon_n$,
(jj) $\lim_{n\to\infty} \max\{t^n_i - t^n_{i-1} : i = 1, ..., \nu_n\} = 0$.

For every $n \in \mathbb{N}$, let us define inductively a finite sequence $(x^n_i)_{0\leq i\leq \nu_n}$ of points of $\mathbb{R}^d$ by

$$x^n_0 = a,\ x^n_i \in \text{proj}(x^n_{i-1}, C(t^n_i)). \tag{3.1.1}$$

This implies

$$||x^n_i - x^n_{i-1}|| \leq h(C(t^n_{i-1}), C(t^n_i)) \leq v(t^n_i) - v(t^n_{i-1}).$$

For every $n \in \mathbb{N}$, let us contruct a mapping $u_n : [0, T] \to \mathbb{R}^d$ as follows:

(1) if $v(t^n_{i-1}) = v(t^n_i)$, set

$$\forall t \in [t^n_{i-1}, t^n_i[,\ u_n(t) = x^n_{i-1} = x^n_i$$

(2) if $v(t^n_{i-1}) > v(t^n_i)$, set

$$\forall t \in [t^n_{i-1}, t^n_i[,\ u_n(t) = \frac{(v(t^n_i) - v(t))x^n_{i-1} + (v(t) - v(t^n_{i-1}))x^n_i}{v(t^n_i) - v(t^n_{i-1})}$$

(3) for $t = T, u_n(T) = x^n_{t^n_{\nu_n}}$.

Then u_n is right continuous with bounded variation, namely we have that

$$||u_n(t) - u_n(s)|| \leq v(t) - v(s) \tag{3.1.2}$$

whenever $0 \leq s \leq t \leq T$. Moreover the differential measure du_n of u_n is equal to $u'_n d\mu$ where $u'_n : [0,T] \to \mathbb{R}^d$ is a step function with values in the closed unit ball of $\mathbb{R}^d$. We refer to Moreau for details ([27], Lemma 1, p. 369–370). Now let $\theta_n : [0,T] \to [0,T]$ be defined as follows : $\theta_n(0) = 0, \theta_n(t) = t^n_i$ if $t \in]t^n_{i-1}, t^n_i]$ for $1 \leq i \leq \nu_n$. Then by (3.1.1), the construction of u_n and the properties of the limiting proximal normal cone to a closed set (see Remark 2.2 (a)), we have

$$u'_n(t) \in -M_{C(\theta_n(t))}(u_n(\theta_n(t)) \tag{3.1.3}$$

for all $t \in [0,T]$.

Second step : Convergence of the approximate functions. Since $||u'_n(t)|| \leq 1$ for all n and all $t \in [0,T]$, by extracting a subsequence if necessary we may assume that $u'_n \to u'$ for $\sigma(L^1, L^\infty)$-topology with $||u'(t)|| \leq 1$ *μ-a.e.* By (3.1.2) we have $||u_n(\theta_n(t))) - u_n(t)|| \leq v(\theta_n(t)) - v(t)$, for all $n \in \mathbb{N}$ and all $t \in [0,T]$, so that

$$\lim_{n\to\infty} u_n(\theta_n(t)) = \lim_{n\to\infty} u_n(t) := u(t) \tag{3.1.4}$$

where $u(t) = x_0 + \int_{]0,t]} u'(s)\mu(ds)$, with $u(t) \in C(t)$ for all $t \in [0,T]$. Moreover (3.1.3) yields

$$u'_n(t) \in -M_{C(\theta_n(t))}(u_n(\theta_n(t)) \cap \overline{B}(0,1)$$

where $\overline{B}(0,1)$ is the closed unit ball of $\mathbb{R}^d$. Now by assumption (ii), one sees that the multifunction $\Gamma : [0,T]_r \times \mathbb{R}^d \to c(\mathbb{R}^d) \cup \{\emptyset\}$ defined by

$$\Gamma(t,x) := M_{C(t)}(x) \cap \overline{B}(0,1)$$

has closed graph in $[0,T]_r \times \mathbb{R}^d \times \mathbb{R}^d$, hence is upper semicontinuous. Since by (3.1.4)

$$\lim_{n\to\infty} \theta_n(t) = t,\ \lim_{n\to\infty} u_n(\theta_n(t)) = u(t),\ \forall t \in [0,T],$$

then by using Mazur's lemma, the upper semicontinuity of Γ and property (b) in Proposition 2.2, it is easy to conclude that

$$-u'(t) \in \bigcap_n \overline{co} \bigcup_{k\geq n} \Gamma(\theta_k(t), u_k(\theta_k(t))) \subset \overline{co}\,\Gamma(t,u(t)) \subset \overline{co}\, M_{C(t)}(u(t)) = N_{C(t)}(u(t))$$

μ-a.e. □

Assumption (ii) is crucial in Theorem 3.1. Also we will give in the next section sufficient conditions for which $M_C(\cdot,\cdot)$ has closed graph.

4. On the limiting proximal normal cone to a closed moving set. In this section we present two closure theorems concerning the graph of limiting proximal normal cone to a closed moving set in $\mathbb{R}^d$.

Proposition 4.1. *Let $I_r = [0,T]_r$ $(T > 0)$ and $C : I_r \to cc(\mathbb{R}^d)$. Assume that the function $(t,x) \mapsto \delta^*(x, C(t))$ is lower semicontinuous on $[0,T]_r \times \mathbb{R}^d$ and the graph of C is closed in $[0,T]_r \times \mathbb{R}^d$. Then the multifunction*

$$F(t,x) = \begin{cases} N_{C(t)}(x) & \text{if } x \in C(t) \\ \emptyset & \text{if} x \notin C(t) \end{cases}$$

has closed graph in $I_r \times \mathbb{R}^d \times \mathbb{R}^d$.

Proof. We have

$$y \in N_{C(t)}(x) \iff x \in C(t) \text{ and } \langle y, x\rangle = \delta^*(y, C(t)).$$

Let $x_n \in C(t_n)$ and let $y_n \in N_{C(t_n)}(x_n)$ with $t_n \to t$ in I_r, $x_n \to x$, $y_n \to y$. We need to check that $y \in N_{C(t)}(x)$. Since the graph of C is closed in $I_r \times \mathbb{R}^d$ by hypothesis, we have $x \in C(t)$. Since $(t,x) \mapsto \delta^*(x, C(t))$ is lower semicontinuous on $[0,T]_r \times \mathbb{R}^d$ by hypothesis, we have

$$\delta^*(x, C(t)) \le \liminf_{n\to\infty} \delta^*(x_n, C(t_n)) \le \lim_{n\to\infty} \langle x_n, y_n\rangle = \langle x, y\rangle.$$

Hence $y \in N_{C(t)}(x)$. □

Remarks. (a) If $C : I_r \to cc(\mathbb{R}^d)$ is lower semicontinuous, then $(t,x) \mapsto \delta^*(x, C(t))$ is lower semicontinuous on $[0,T]_r \times \mathbb{R}^d$ by Michael selection theorem.

(b) Proposition 4.1 is valid in Hilbert spaces.

In order to state the second closure theorem we need the two following lemmas.

Lemma 4.2. *Let E be a Banach space and $\mathcal{L}(E,E)$ the set of all linear continuous mappings from E into E. Let $A : [0,T]_r \to \mathcal{L}(E,E)$ such that, $\forall x \in E$, $t \mapsto A_t(x)$ is continuous on $[0,T]_r$. Then the mapping*

$$\zeta : (t,x) \mapsto A_t(x)$$

from $[0,T]_r \times E$ into E is also continuous.

Proof. By standard properties of the right topology, we need only to show that ζ is sequentially continuous. Let (t_n, x_n) be a sequence in $[0,T]_r \times E$ such that $(t_n, x_n) \to (t, x)$ in $[0,T]_r \times E$. Clearly the set $D := \{t\} \cup \{t_n : n \in \mathbb{N}\}$ is compact in $[0,T]_r$. So, $\forall x \in E, \bigcup_{s\in D}\{A_s(x)\}$ is compact in E by hypothesis. Hence, $\forall x \in E, \sup_n ||A_{t_n}(x)|| < \infty$. By virtue of the Banach–Steinhaus theorem, this implies that $M := \sup_n ||A_{t_n}|| < +\infty$. To conclude the proof, it is enough to observe that

$$\begin{aligned} ||A_{t_n}(x_n) - A_t(x)|| &= ||A_{t_n}(x_n - x) + A_{t_n}(x) - A_t(x)|| \\ &\leq M\,||x_n - x|| + ||A_{t_n}(x) - A_t(x)||. \end{aligned}$$

□

Lemma 4.3. *Let $C_0 \in c(\mathbb{R}^d)$ and let $A \in Isom(\mathbb{R}^d, \mathbb{R}^d)$. Then we have*

$$z \in \Pi_{A(C_0)}(y) \Longleftrightarrow A^{-1}(y) \in C_0 \text{ and } A^*(z) \in \Pi_{C_0}(A^{-1}(y)) \tag{4.3.1}$$

where A^* denotes the adjoint operator of A.

Proof. Indeed $z \in \Pi_{A(C_0)}(y)$ iff $y \in A(C_0)$ and there exist $\sigma > 0$ such that

$$\forall c \in C_0, \langle z, A(c) - y\rangle \leq \sigma ||A(c) - y||^2.$$

But

$$\langle z, A(c) - y\rangle = \langle z, A(c - A^{-1}(y))\rangle = \langle A^*(z), c - A^{-1}(y)\rangle \tag{4.3.2}$$

and

$$||A(c) - y||^2 = ||A(c - A^{-1}(y))||^2 \leq ||A||^2 ||c - A^{-1}(y)||^2. \tag{4.3.3}$$

So if $z \in \Pi_{A(C_0)}(y)$ then $y \in A(C_0)$ and there exists $\sigma > 0$ such that

$$\forall c \in C_0, \langle A^*(z), c - A^{-1}(y)\rangle \leq \sigma ||A||^2 ||c - A^{-1}(y)||^2 \tag{4.3.4}$$

using (4.3.2) and (4.3.3) and hence $A^*(z) \in \Pi_C(A^{-1}(y))$. The other implication follows by symmetry. □

Theorem 4.4. *Let $I = [0,T]$. Let $C_0 \in c(\mathbb{R}^d)$ and $t \mapsto A_t \in Isom(\mathbb{R}^d, \mathbb{R}^d)$ such that for every $x \in \mathbb{R}^d$, $t \mapsto A_t(x)$ and $t \mapsto A_t^{-1}(x)$ are continuous on I_r. Then the multifunction*

$$G : (t,x) \to \begin{cases} M_{A_t(C_0)}(x) & \text{if } x \in A_t(C_0) \\ \emptyset & \text{if } x \notin A_t(C_0) \end{cases}$$

has closed graph in $I_r \times \mathbb{R}^d \times \mathbb{R}^d$.

Proof. For each $t \in I$, we denote by A_t^* the adjoint operator of A_t and we set $C(t) = A_t(C_0)$. Obviously we have $(A_t^*)^{-1} = (A_t^{-1})^*$. Let $y_n \in M_{C(t_n)}(x_n)$ with $t_n \to t$ in I_r, $x_n \to x$, $y_n \to y$. For each n pick $x_n^{k_n}$ and $y_n^{k_n}$ with $y_n^{k_n} \in \Pi_{C(t_n)}(x_n^{k_n})$ such that $||x_n^{k_n} - x_n|| \leq 2^{-n}$ and $||y_n^{k_n} - y_n|| \leq 2^{-n}$ (see Remarks 2.2). By the characterization (4.3.1) we have

$$A_{t_n}^{-1}(x_n^{k_n}) \in C_0 \text{ and } A_{t_n}^*(y_n^{k_n}) \in \Pi_{C_0}(A_{t_n}^{-1}(x_n^{k_n})). \tag{4.4.1}$$

By Lemma 4.2 the mappings $(t, x) \mapsto A_t(x)$ and $(t, x) \mapsto A_t^{-1}(x)$ are continuous on $I_r \times \mathbb{R}^d$ and so is $(t, x) \mapsto A_t^*(x)$. Consequently we get

$$\lim_{n\to\infty} A_{t_n}^{-1}(x_n^{k_n}) = A_t^{-1}(x) \text{ and } \lim_{n\to\infty} A_{t_n}^*(y_n^{k_n}) = A_t^*(y). \tag{4.4.2}$$

Since C_0 is closed, we have $A_t^{-1}(x) \in C_0$ and so $x \in C(t)$. Using (4.4.1) and (4.3.1) we deduce that

$$v_n := A_t(A_{t_n}^{-1}(x_n^{k_n})) \in C(t) \text{ and } w_n := (A_t^*)^{-1}(A_{t_n}^*(y_n^{k_n})) \in \Pi_{C(t)}(w_n).$$

By (4.4.2) and the continuity of the mappings A_t and $(A_t^*)^{-1}$ we obtain

$$\lim_{n\to\infty} v_n = A_t(A_t^{-1}(x)) = x \text{ and } \lim_{n\to\infty} w_n = (A_t^*)^{-1}(A_t^*(y)) = y.$$

This proves that $y \in M_{C(t)}(x)$. Hence the graph of G is closed. □

5. Applications: Convex and nonconvex sweeping process and evolution equation. We will present in this section existence theorems for sweeping process by nonconvex closed moving sets and applications to a new class of evolution equation.

Proposition 5.1. *Let μ be a positive Radon measure on $[0,T]$ and let $v : [0,T] \to \mathbb{R}^d$ satisfying : for all $s < t$ in $[0,T]$, $||v(t) - v(s)|| \leq \mu(]s,t])$. Let $C_0 \in c(\mathbb{R}^d)$. Then given $a \in C_0 + v(0)$ there are a BV right continuous mapping $u : [0,T] \to \mathbb{R}^d$ and $u' \in L^\infty_{\mathbb{R}^d}([0,T], \mu)$ satisfying the following properties:*

(1) *$\forall t,\ u(t) = a + \int_{]0,t]} u'(s)\mu(ds)$ with $||u'(s)|| \leq 1$ μ-a.e.*
(2) *$-u'(t) \in N_{C_0+v(t)}(u(t))$ μ-a.e.*

Proof. Set $C(t) = C_0 + v(t)$ for all $t \in [0,T]$. Then it is obvious that the multifunction C satisfies

$$\forall s < t \in [0,T],\ h(C(t), C(s)) \leq \mu(]s,t]). \tag{5.1.1}$$

In view of Theorem 4.4, the multifunction G associated to the limiting proximal normal cone to C given by

$$G(t,x) = \begin{cases} M_{C(t)}(x) & \text{if } x \in C(t) \\ \emptyset & \text{if } x \notin C(t) \end{cases}$$

has closed graph in $I_r \times \mathbb{R}^d \times \mathbb{R}^d$ so that by (5.1.1) we may apply Theorem 3.1 to get the result. □

Here is a particular case of Theorem 5.1.

Proposition 5.2. *Let $v : [0,T] \to \mathbb{R}^d$ be a 1–Lipschitzean mapping and let $C_0 \in c(\mathbb{R}^d)$. Then given $a \in C_0 + v(0)$, there there is a 1–Lipschitzean mapping function $u : [0,T] \to \mathbb{R}^d$ satisfying the following properties:*

(1) $\forall t, u(t) = a + \int_{]0,t]} u'(s)ds$ *with* $||u'(s)|| \leq 1$ *dt-a.e.*

(2) $-u'(t) \in N_{C_0+v(t)}(u(t))$ *dt-a.e.*

Proof. It is enough to see that the multifunction $C(\cdot) := C_0 + v(\cdot)$ is 1–Lipschitzean for the Hausdorff distance so that Proposition 5.2 follows from Proposition 5.1 by taking $\mu = dt$. □

From Proposition 5.2 we derive an existence result for a new class of evolution equation. We are inspired by ([30], Theorem 7).

Proposition 5.3. *Let f be a real–valued Lipschitzean function defined on $\mathbb{R}^d$ with $f(0) = 0$. Let $\varphi : I \to [0, \infty[$ be a positive increasing Lipschitzean function with $\varphi(0) = 0$. Then there exists a Lipschitzean mapping $x := (y, \theta)$ from I into $\mathbb{R}^{d+1}$ with $x(0) = (0,0)$ which satisfies the following properties:*

(a) $\forall t, \theta(t) = f(y(t)) + \varphi(t)$.

(b) $y'(t) \in -\theta'(t)\partial f(y(t))$ *dt-a.e., where ∂f denotes the Clarke subdifferential of f.*

Proof. Set $C(t) = epi f + (0, \varphi(t))$ for all $t \in I$. Then Proposition 5.2 applied to the sweeping process

$$\begin{cases} (y'(t), \theta'(t)) \in -N_{C(t)}(y(t), \theta(t)) \\ (y(0), \theta(0)) = (0,0) \end{cases} \tag{5.3.1}$$

yields a Lipschitzean solution $x := (y, \theta)$ to (5.3.1). It is also obvious that $\forall t, \theta(t) \geq f(y(t)) + \varphi(t)$. Now using the continuity of f and applying Valadier's arguments in ([30], Lemma 6) we get (a) as follows. Assume by contradiction that $\Omega := \{t : \theta(t) > f(y(t)) + \varphi(t)\}$ is nonempty. Then there is $]t_0, t_1[\subset \Omega$ with $\theta(t_0) = f(y(t_0)) + \varphi(t_0)$. On $]t_0, t_1[$, $x'(t) = 0$ since $-N_{C(t)}(c) = 0$ whenever $c \in \operatorname{int} C(t)$. But

$$\theta(t_0) = \theta(t) > f(y(t)) + \varphi(t) = f(y(t_0)) + \varphi(t)$$

yields a contradiction because φ is increasing.

(b) follows easily from (a) and the inclusions

$$\begin{aligned}(y'(t), \theta'(t)) \in -N_{epif+(0,\varphi(t))}&(y(t), f(y(t)) + \varphi(t))\\ &= -N_{epif}(y(t), f(y(t)))\\ &= \bigcup_{\lambda \in [0,\infty[} -\lambda[\partial f(y(t)) \times \{-1\}].\end{aligned}$$

□

REMARKS. Property (a) allows to obtain (b) and relies on the arguments of Lemma 6 in Valadier [30] using the continuity of f. When both f and φ belong to a special class (S) of Lipschitzean functions [30], Valadier obtains more information about properties of solutions of the evolution equation $y'(t) \in -\theta'(t)\partial f(y(t))$. Note that Proposition 5.3 completes Valadier's work since it provides existence of Lipschitzean solutions $x(\cdot) = (y(\cdot), \theta(\cdot))$ for the sweeping process by $C(t) = epif + (0, \varphi(t))$. Unfortunately we are unable to formulate an analogue for property (b) when f is lower semicontinuous although solutions x do exist for the sweeping process by $C(t) = epif + (0, \varphi(t))$ with f lower semicontinuous. This is an open problem. There is another difficult problem concerning the case when f is a Carathéodory continuous integrand.

Now we establish some results on convex and nonconvex perturbations for the sweeping process associated to a nonconvex closed moving set.

Proposition 5.4. *Let $C : [0, T] \to c(\mathbb{R}^d)$ be a k–Lipschitzean multifunction such that the multifunction*

$$G(t, x) = \begin{cases} M_{C(t)}(x) & \text{if } x \in C(t) \\ \emptyset & \text{if } x \notin C(t) \end{cases}$$

has closed graph in $[0, T] \times \mathbb{R}^d \times \mathbb{R}^d$. Let $F : [0, T] \times \mathbb{R}^d \to ck(\mathbb{R}^d)$ be a multifunction such that, $\forall (t, x) \in [0, T] \times \mathbb{R}^d, |F(t, x)| \le m$ for some positive constant m and that : for every fixed $t \in [0, T], F(t, \cdot)$ is upper semicontinuous on $\mathbb{R}^d$, for every fixed $x \in \mathbb{R}^d, F(\cdot, x)$ has a Lebesgue measurable *selection. Then, given $a \in C(0)$, there is a Lipschitzean solution x to the differential inclusion*

$$\begin{cases} -x'(t) \in N_{C(t)}(x(t)) + F(t, x(t)) \ a.e. \\ x(t) \in C(t), \forall t \in [0, T] \\ x(0) = a \end{cases}$$

Proof. For every $n \in \mathbb{N}$, set $t_i^n := i\frac{T}{2^n}$, $0 \le i \le \nu_n = 2^n$, and $x_0^n = a$. Pick $x_1^n \in \text{proj}_{C(t_1^n)}(x_0^n)$ and a measurable (hence integrable) selection σ_1^n of the multifunction $F(\cdot, a)$. For every $t \in [t_0^n, t_1^n]$, we set

$$x_n(t) = a + \frac{x_1^n - x_0^n}{t_1^n - t_0^n}(t - t_0^n) - \int_0^t \sigma_1^n(s)ds.$$

Pick $x_2^n \in \text{proj}_{C(t_2^n)}(x_n(t_1^n))$ and an integrable selection σ_2^n of the multifunction $F(\cdot, x_n(t_1^n))$. For every $t \in [t_1^n, t_2^n]$, we set

$$x_n(t) = x(t_1^n) + \frac{x_2^n - x_n(t_1^n)}{t_2^n - t_1^n}(t - t_1^n) - \int_{t_1^n}^t \sigma_2^n(s)ds.$$

Then, by induction, we obtain a mapping $x_n : [0, T] \to \mathbb{R}^d$ such that

$$x_n(t) = x(t_{i-1}^n) + \frac{x_i^n - x_n(t_{i-1}^n)}{t_i^n - t_{i-1}^n}(t - t_{i-1}^n) - \int_{t_{i-1}^n}^t \sigma_i^n(s)ds$$

for every $t \in [t_{i-1}^n, t_i^n]$ with $i = 1, .., \nu_n$ where $x_i^n \in \text{proj}_{C(t_i^n)}(x_n(t_{i-1}^n))$ and σ_i^n is an integrable selection of the multifunction $F(\cdot, x_n(t_{i-1}^n))$. We consider the two following mappings δ_n and θ_n from $[0, T]$ to $[0, T]$ defined by:

$$\delta_n(t_i^n) = \theta_n(t_i^n) = t_i^n, \text{ and } \delta_n(t) = t_{i-1}^n, \ \theta_n(t) = t_i^n$$

for $t \in]t_{i-1}^n, t_i^n[$ and for $1 \le i \le \nu_n$. Let us denote by

$$J_i^n = [t_{i-1}^n, t_i^n[\text{ for } 1 \le i < \nu_n \text{ and } J_{\nu_n}^n = [t_{\nu_n - 1}^n, T].$$

Let us consider the mappings $\widetilde{x}_n$, y_n, z_n defined on $[0, T]$ as follows. For all $i = 1, ..., \nu_n$ and for all $t \in J_i^n$, we set

$$\widetilde{x}_n(t) = x_i^n, \ \ y_n(t) = \frac{x_i^n - x_n(t_{i-1}^n)}{t_i^n - t_{i-1}^n} \text{ and } z_n(t) = -\sigma_i^n(t).$$

Then one can easily check that the following properties hold:

(5.4.1) $\quad \forall t \in [0, T], \ t - T/2^n \le \delta_n(t) \le t \le \theta_n(t) \le t + T/2^n.$

(5.4.2) $\quad \forall t \in [0, T], \ \widetilde{x}_n(t) \in C(\theta_n(t))$ and $||x_n(\theta_n(t)) - \widetilde{x}_n(t)|| \le m\,T/2^n.$

(5.4.3) $\quad \forall t \in [0, T], \ ||y_n(t)|| \le k + m$ and $||z_n(t)|| \le m.$

(5.4.4) $\forall t \in [0,T],\ y_n(t) \in -\Pi_{C(\theta_n(t))}(\widetilde{x}_n(t))$ and $z_n(t) \in -F(t, x_n(\delta_n(t)))$.

$$(5.4.5) \qquad \forall t \in [0,T],\ x_n(t) = a + \int_0^t x'_n(s)ds \text{ where } x'_n := y_n + z_n.$$

By standard arguments, it is not difficult to see that (y_n) and (z_n) are relatively weakly compact in $L^1_{\mathbb{R}^d}([0,T], dt)$, and (x_n) is relatively compact in the Banach space $\mathcal{C}_{\mathbb{R}^d}([0,T])$. By extracting subsequences, we may ensure that $y_n \to y, z_n \to z$ weakly in $L^1_{\mathbb{R}^d}([0,T], dt)$ and $x_n \to x$ in $\mathcal{C}_{\mathbb{R}^d}([0,T])$ so that

$$(5.4.6) \qquad x(t) = a + \int_0^t x'(s)ds,\ \forall t \in [0,T],\ \text{where } x' := y + z.$$

On the other hand, since

$$(5.4.7) \qquad \forall t \in [0,T],\ \lim_{n\to\infty} x_n(\theta_n(t)) = \lim_{n\to\infty} x_n(t) = x(t)$$

and the graph of C is closed, we have

$$(5.4.8) \qquad \forall t \in [0,T],\ x(t) \in C(t).$$

Now let us consider the multifunction $\Gamma : [0,T] \times \mathbb{R}^d \to ck(\mathbb{R}^d) \cup \{\emptyset\}$:

$$\forall (t,x) \in [0,T] \times \mathbb{R}^d, \Gamma(t,x) := -G(t,x) \cap \overline{B}(0, k+m).$$

Then by hypothesis, Γ has closed graph, hence Γ is upper semicontinuous on $[0,T] \times \mathbb{R}^d$. Moreover, by (5.4.3) and (5.4.4), we have

$$(5.4.9) \qquad \forall t \in [0,T],\ y_n(t) \in \Gamma(\theta_n(t), \widetilde{x}_n(t)).$$

Since $\theta_n(t) \to t$ and $\widetilde{x}_n(t) \to x(t)$ for every $t \in [0,T]$, by upper semicontinuity property of Γ, we conclude that

$$(5.4.10) \qquad \forall t \in [0,T],\ \bigcap_n \overline{co} \bigcup_{p=n}^{\infty} \Gamma(\theta_p(t), \widetilde{x}_p(t)) \subset \overline{co}\, \Gamma(t, x(t)).$$

By Mazur's lemma, we have

$$(5.4.11) \qquad y(t) \in \bigcap_n \overline{co} \bigcup_{p=n}^{\infty} \{y_p(t)\}\ a.e.$$

So (5.4.9), (5.4.10), (5.4.11) imply $y(t) \in \overline{co}\, \Gamma(t, x(t))$ a.e. Hence

$$(5.4.12) \qquad y(t) \in -N_{C(t)}(x(t))\ a.e.$$

because

$$\overline{co}\, \Gamma(t, x(t)) \subset \overline{co}\, [-M_{C(t)}(x(t))] = -N_{C(t)}(x(t))$$

by Proposition 2.3 (a). Since $\forall t$, $x_n(\delta_n(t)) \to x(t)$, then by using similar arguments and (5.4.4), we get

$$z(t) \in -F(t, x(t)) \ a.e. \tag{5.4.13}$$

This completes the proof. □

Now we consider nonconvex perturbation of the sweeping process associated to a nonconvex closed moving set.

Proposition 5.5. *Let C_0 and v as in Proposition 5.2. Let $F : [0,T] \times \mathbb{R}^d \to k(\mathbb{R}^d)$ be a multifunction such that, $\forall (t,x) \in [0,T] \times (C_0 + v([0,T]))$, $|F(t,x)| \leq k$ for some positive constant k. Assume that F is uniformly continuous on $[0,T] \times (C_0 + v([0,T]))$. Then given $a \in C(0)$ there is a Lipschitzean solution to the differential inclusion*

$$\begin{cases} -u'(t) \in N_{C_0+v(t)}(u(t)) + F(u(t)) \\ u(0) = a \end{cases}$$

Proof. The details of the proof is very long since both C_0 and F are not convex valued. Also we only sketch a proof which relies on Proposition 5.2 and a special algorithm due to Gamal ([15], [16]) which has been already exploited by ([1], [14], [28]) in similar problems. Since this algorithm needs several steps of computation, we do not produce the details because of the lack of place. However the idea of the proof is quite simple. Using Gamal's algorithm ([15], [16]) in accord with the properties of limiting proximal normal cone, we construct as in the proof of Proposition 5.4 two sequences (θ_n) and (δ_n) of simple mappings from $[0,T]$ to $[0,T]$, a sequence (u_n) of equi–Lipschitzean mappings from $[0,T]$ to $\mathbb{R}^d$, a sequence of measurable mappings (h_n) from $[0,T]$ to $\mathbb{R}^d$ with the following properties:

$$\forall t, \ \lim_{n\to\infty} \theta_n(t) = \lim_{n\to\infty} \delta_n(t) = t. \tag{5.5.1}$$

$$\forall t, \ ||u_n'(t)|| \leq 1 + 2k. \tag{5.5.2}$$

$$u_n'(t) - h_n(t) \in M_{C(\theta_n(t))}(u_n(\theta_n(t)) \ dt\text{-}a.e. \tag{5.5.3}$$

$$h_n(t) \in F(\delta_n(t), u_n(\delta_n(t)) \ dt\text{-}a.e. \tag{5.5.4}$$

(5.5.5) (h_n) is relatively compact for the norm of $L^1_{\mathbb{R}^d}([0,T], dt)$.

Using (5.5.2), (5.5.4), weak compactness of (u_n') in $L^1_{\mathbb{R}^d}([0,T], dt)$ and (5.5.5), we may suppose that $u_n' \to u'$ weakly in $L^1_{\mathbb{R}^d}([0,T], dt)$ so that by (5.5.1)

$$\lim_{n\to\infty} u_n(\theta_n(t)) = \lim_{n\to\infty} u_n(\delta_n(t)) = u(t)$$

with $\forall t,\ u(t) = a + \int_0^t u'(s)ds$. By (5.5.5) we may suppose that $h_n \to h$ *strongly* in $L^1_{\mathbb{R}^d}([0,T], dt)$ and almost everywhere. Now by Theorem 4.4 the graph of $(t,x) \to M_{C(t)}(x)$ is closed so that the multifunction Γ defined by

$$\Gamma(t,x) = M_{C(t)}(x) \cap \overline{B}(0, 1+2k)$$

has closed graph and compact values, hence upper semicontinuous on $I \times \mathbb{R}^d$. So using Mazur lemma, the upper semicontinuity of Γ, the fact that $N_{C(t)}(x) = \overline{co}\, M_{C(t)}(x)$ for $x \in C(t)$ (see Proposition 2.3) and (5.5.3) we get $u'(t) - h(t) \in N_{C(t)}(u(t))$ dt-*a.e.*, with $h(t) \in F(t, u(t))$ dt-*a.e.* by using (5.5.4). □

To end this paper we present in $\mathbb{R}^2$ an example characterizing the limiting proximal normal cone to a closed nonconvex moving set and providing also an interesting variant of Theorem 3.1.

Let us consider the space $\mathbb{R}^d$ with $d = 2$, $e_1 = (1,0)$ and $e_2 = (0,1)$. Given ρ and r_0 in $\mathbb{R}^+$, we set

$$\forall t \in [0,1], r(t) = \rho t + r_0, \text{ and } S(t) = \{x \in \mathbb{R}^2 : ||x|| = r(t)\}.$$

and we consider the following closed moving set

$$C(t) = S(t) \cup [r(t)e_1, (r_0 + \rho)e_1].$$

Let $0 \le t < 1$. The closed set $C(t) = S(t) \cup [r(t)e_1, r(1)e_1]$ is geometrically represented as follows.

It is easy to see that the multifunction C is ρ–Lipschitzean on $[0,1]$. Let $x \in C(t)$. In order to characterize $M_{C(t)}(x)$, it is enough to determine $\Pi_{C(t)}(x)$. We use the following property

$$\zeta \in \Pi_{C(t)}(x) \iff \exists y \in \mathbb{R}^d,\ \delta > 0 \text{ such that } x \in \text{proj}_{C(t)}(y) \text{ and } \zeta = \delta(y - x).$$

So it suffices to determine all the vectors $y - x$ with $y \in \mathbb{R}^d$ such that $x \in \text{proj}_{C(t)}(y)$ since all homothetic vectors $\delta(y-x)$ with $\delta \ge 0$ belong to $\Pi_{C(t)}(x)$.

Given $x \in C(t)$ there are only three positions of x on $C(t)$ which are represented by the vectors u, v and w (see Figure 1).

(1) *First case: Figure 2* : $x = u$.

Since u is situated on the circle $S(t)$ with radius $r(t)$ and center 0, we have

$$\Pi_{C(t)}(u) = M_{C(t)}(u) = N_{C(t)}(u) = \{\lambda u : \lambda \in \mathbb{R}\} = \mathbb{R}.u$$

Indeed only the points y in $\mathbb{R}^2$ such that

$$u \in \text{proj}_{S(t)}(y)$$

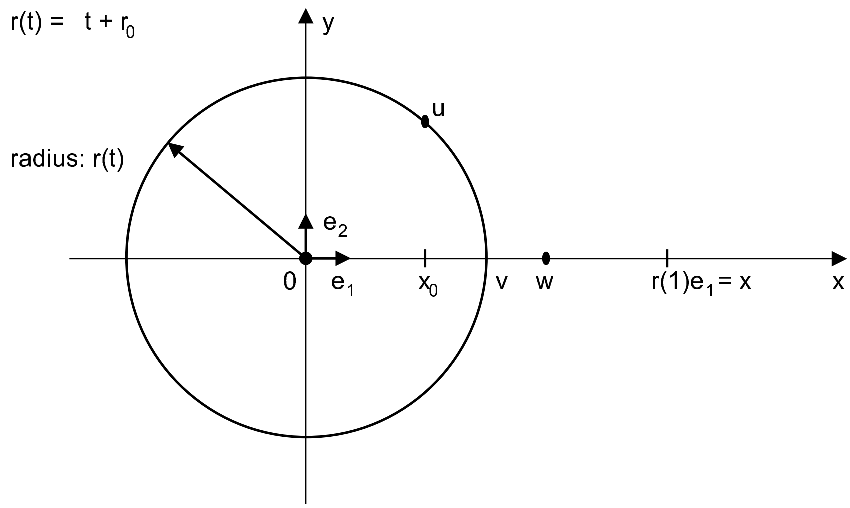

FIGURE 1

are those which are situated on the line $\Delta_u = \mathbb{R}.u$

(2) *Second case: Figure 3* : $x = v$.

Is is easy to see that only the points y in $\mathbb{R}^2$ such that

$$v \in \mathrm{proj}_{C(t)}(y)$$

are those which are situated on the segment $[0, v]$. Indeed, consider the domains D_1, D_2, D_3 and D_4 presented in the Figure 3.

If either $y \in D_1$ or $y \in D_2$, then the proximal points of y to $C(t)$ are situated on either $S(t) \setminus \{v\}$ or $]v, x_\infty]$.

If either $y \in D_3$ or $y \in D_4$, then the proximal points of y to $C(t)$ are situated on $S(t)$.

So only the points $y \in [0, v]$ satisfy $v \in \mathrm{proj}_{C(t)}(y)$ and if $y \in [0, v]$, we have

$$y - v \in \mathbb{R}^- e_1 = \{\lambda e_1 : \lambda \leq 0\} = \mathbb{R}^- \times \{0\}.$$

Consequently

$$\Pi_{C(t)}(v) = \mathbb{R}^- e_1 = \mathbb{R}^- \times \{0\}.$$

Third case : Figure 4 : $x = w$.

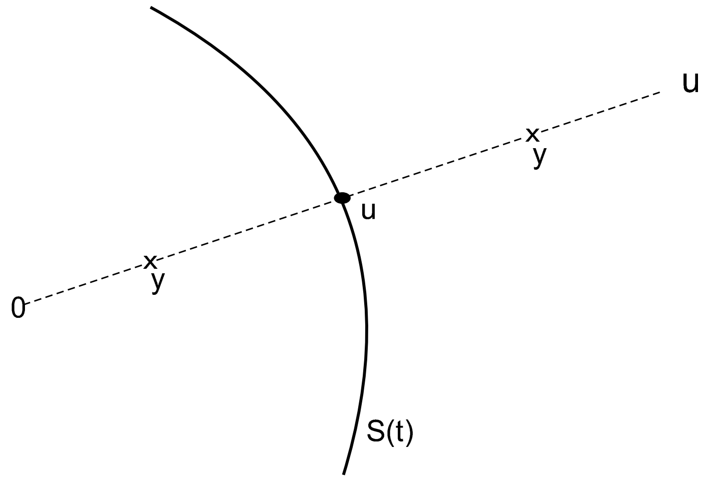

FIGURE 2

Since $w \in]r(t)e_1, r(1)e_1[$ and $]r(t)e_1, r(1)e_1[$ is a convex subset of the closed subset $C(t)$, we have

$$\Pi_{C(t)}(w) = M_{C(t)}(w) = N_{C(t)}(w) = \mathbb{R}.e_2 = \{0\} \times \mathbb{R}.$$

So only the points y in $\mathbb{R}^2$ for which $w \in \text{proj}_{C(t)}(y)$ are those situated on the line

$$\Delta_w = w + \mathbb{R}.e_2.$$

Now we will give a direct proof of the existence for the following sweeping process associated to the closed moving set $C(t)$. Given $a \in C(0)$ with $a \neq r_0 e_1$. (**)[2] Then the sweeping process $\mathcal{S}W^*$

$$\begin{cases} u'(t) \in -N_{C(t)}(u(t)) \ dt\text{-}a.e. \ t \in [0,1] \\ u(t) \in C(t) \\ u(0) = a \end{cases}$$

admits an absolutely continuous solution.

Proof. First case : $r_0 \neq 0$. In this case $C(0) = S(0) \cup [x_0, x_\infty]$ where $S(0)$ is the circle of center 0 with radius $r_0 > 0$. Since $a \in C(0)$, we have either $a \in S(0)$ or $a \in [x_0, x_\infty]$.

[2](**) It is also interesting to consider the case where $a = r_0 e_1$.

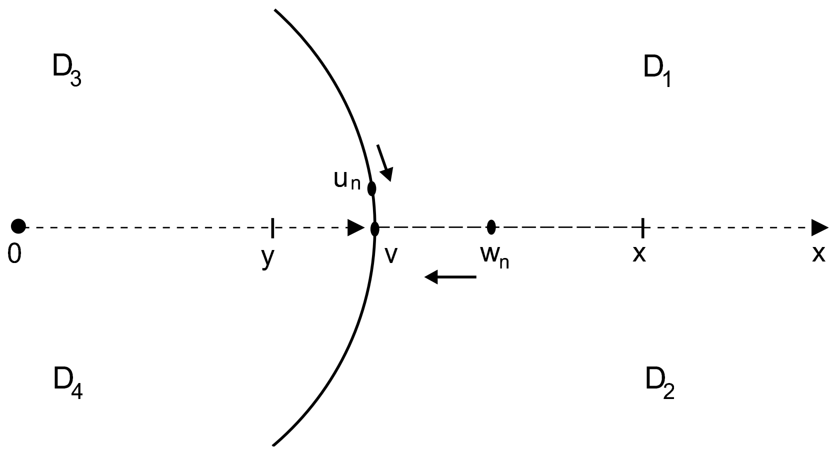

Figure 3

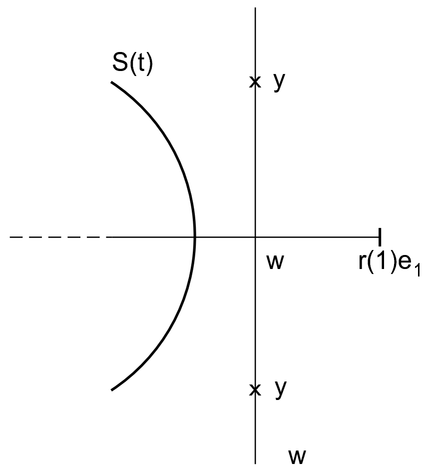

Figure 4

If $a \in S(0)$, $||a|| = r_0$. Set

$$\forall t \in [0,t],\ u(t) = r(t).\frac{a}{||a||} = (\rho t + r_0)\frac{a}{||a||}.$$

Then u is absolutely continuous on $[0,1]$ with $u(t) \in S(t)$ for all $t \in [0,1]$ and $u'(t) = \rho\, a/||a||$. By the case of Figure 2 we have

$$N_{C(t)}(u(t)) = \mathbb{R}.u(t) = \mathbb{R}.a.$$

Since $a \neq r_0 e_1$ we have $\rho\, a/||a|| \in -N_{C(t)}(u(t) = \mathbb{R}.a$ for all $t \in [0,1]$. Hence u is solution of $\mathcal{S}W^*$.

If $a \in]x_0, x_\infty]$, then $a = r(t_0)e_1$ with $0 < t_0 \leq 1$. Set

$$u(t) = \begin{cases} a & \text{if } t \in [0, t_0[\\ r(t)e_1 & \text{if } t \in [t_0, 1] \end{cases}$$

then u is absolutely continuous and

$$u'(t) = \begin{cases} 0 & \text{if } t \in [0, t_0[\\ \rho e_1 & \text{if } t \in [t_0, 1] \end{cases}$$

Then $\forall t \in [0, t_0[$, $u(t) = a \in [r(t)e_1, r(1)e_1] \subset C(t)$ and $u'(t) = 0 \in N_{C(t)}(u(t))$. Now $\forall t \in [t_0, 1[$, we have $u(t) = r(t)e_1 \in S(t) \cap [r(t)e_1, r(1)e_1]$. So $u(t)$ corresponds to the case of Figure 3. Since $-u'(t) = -\rho e_1 \in \Pi_{C(t)}(u(t)) = \mathbb{R}^- e_1$, we may conclude that u is a solution of the sweeping process $\mathcal{S}W^*$.

Second case : $r_0 = 0$. In this case $S(0) = \{0\}$ and

$$a \in C(0) \setminus \{0\} =]0, \rho e_1]$$

Hence $a = r(t_0)e_1$ with $0 < t_0 \leq 1$. Then as above it is easy to check that

$$u(t) = \begin{cases} a & \text{if } t \in [0, t_0[\\ r(t)e_1 & \text{if } t \in [t_0, 1] \end{cases}$$

is a solution of the sweeping process $\mathcal{S}W^*$. □

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References

[1] Bardaro, C., Pucci, P., *Some contributions to the theory of Multivalued Differential Equations*, Atti. Sem. Mat. Univ. Modena 32 (1983), 175–202.

[2] Benabdellah, H., Castaing, C., Salvadori, A., *Compactness and discretization methods for differential inclusions and evolution problems*, Rapporto tecnico n. 13 (1994), Del Dipartimento di Matematica, Perugia.

[3] Benabdellah, H., Castaing, C., Salvadori, A., *Compactness and discretization methods for differential inclusions and evolution problems*, C. R. Acad. Sci., t. 320, Ser. I (1995), 769–774.

[4] Borwein, J.M., Strojwas, H.M., *Proximal analysis and boundaries of closed sets in Banach space*, Part I, Theory, Canadian J. Math. 38 (1986), 431–452.

[5] Brezis, H., *Opérateurs maximaux monotones*, North–Holland, Amsterdam (1973).

[6] Castaing, C., Duc Ha, T.X. and Valadier, M., *Evolution equations governed by the sweeping process*, Set–Valued Analysis 1 (1993), 109–139.

[7] Castaing, C., Monteiro Marques, M.D.P, *Evolution problems associated with convex and nonconvex closed moving sets*, Preprint (1994)/15, Université Montpellier II, Département de Mathématiques.

[8] Castaing, C., Monteiro Marques, M.D.P, *Sweeping process by nonconvex closed moving sets with perturbation*, C. R. Acad. Sci., t. 319, Ser. I (1994), 127–132.

[9] Castaing, C., Monteiro Marques, M.D.P, *Evolution problems associated with nonconvex closed moving sets with bounded variation*, Portugalia Mathematica, Vol 53, Fas 1 (1996), 73–87.

[10] Castaing, C., Monteiro Marques, M.D.P, *A multivalued version of Scorza–Dragoni theorem with an application to normal integrands*, Bull. Pol. Acad. Math., Vol 42, No 2 (1994), 133–140.

[11] Castaing, C., Valadier, M., *Convex Analysis and Measurable Multifunctions*, Lecture Notes in Math. 580 (1977), Springer Verlag, Berlin.

[12] Clarke, F.H., *Optimization and Nonsmooth Analysis*, John Wiley, New-York (1983).

[13] Clarke, F.H., *Methods of Dynamic and Nonsmooth Optimization*, Society for Industrial and Applied Mathematics Philadelphia, Pennsylvania (1989).

[14] Duc Ha, T.X., Monteiro Marques, M.D.P., *Nonconvex Second–Order Differential Inclusions with Memory*, Set–Valued Analysis, Vol 3, No 1 (1995).

[15] Gamal, M.A., *Perturbation non convexe d'un problème d'évolution dans un espace hilbertien*, Sém. Anal. Convexe Montpellier (1981), Exposé No 16.

[16] Gamal, M.A., *Perturbation non convexe d'un problème d'évolution dans un espace de Banach*, Sém. Anal. Convexe Montpellier (1982), Exposé No 17.

[17] Ioffe, A.D., *Approximate subdifferentials and applications I: The finite dimensional theory*, Trans. Amer. Math. 281 (1984), 389–416.

[18] Ioffe, A.D., *Proximal analysis and approximate subdifferentials*, J. London Math. Soc. 41 (1990), 175–192.

[19] Jofre, A., Thibault., L. *Proximal and Fréchet normal formulae for some small normal cone in Hilbert spaces*, Nonlinear Anal. Th. Meth. Appl. 19 (1992), 599–612.

[20] Loewen, P.D., *The proximal formula in Hilbert space*, Nonlinear Anal. Th. Meth. Appl. 11 (1987), 979–995.

[21] Loewen, P.D., *Limits of Fréchet Normals in Nonsmooth Analysis, Optimization and Nonlinear Analysis* (A.D. Ioffe, L. Marcus, and S. Reich, eds.), Pitman Research Notes Math. Ser. 244 (1992), 178–188.

[22] Monteiro Marques, M.D.P., *Differential Inclusions in Nonsmooth Mechanical Problems–Shocks and Dry Friction*, Birkhäuser Verlag (1993).

[23] Mordukhovich, B.S., *Approximation Methods in Problems of Optimization and Control*, Nauka, Moscow (1988).

[24] Mordukhovich, B.S., Shao Y., *Nonsmooth sequential analysis in Asplund spaces*, Preprint, Wayne State University, Detroit, Michigan (June 1994) (to appear in Trans. A. M. S.).

[25] Moreau, J.J., *Rafle par un convexe variable* (Première partie), Sém. Anal. Convexe (1971), Exposé No 15 (43 pages).

[26] Moreau, J.J., *Rafle par un convexe variable* (Deuxième partie), Sém. Anal. Convexe (1972), Exposé No 3 (36 pages).

[27] Moreau, J.J., *Evolution problem associated with a moving convex set in a Hilbert space*, J. Differential Equations 26 (1977), 347–374.

[28] Syam, A., *Contribution aux inclusions différentielles*, Thèse, Université de Montpellier II, Montpellier (1993).

[29] Valadier, M., *Quelques problèmes d'entrainement unilatéral en dimension finie*, Sém. Anal. Convexe Montpellier (1988), Exposé No 8 and C. R. Acad. Sci. Paris Sér. 1, 308 (1989) 241–244.

[30] Valadier, M., *Lignes de descente de fonctions Lipschitziennes non pathologiques*, Sém. Anal. Convexe Montpellier (1989), Exposé No 9.

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