Journal of Applied Analysis
Vol. 1, No. 2 (1995), pp. 125–134

NONLINEAR CONTRACTIONS ON SEMIMETRIC SPACES

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Abstract. Let (X, d) be a Hausdorff semimetric (d need not satisfy the triangle inequality) and d–Cauchy complete space. Let f be a selfmap on X, for which $d(fx, fy) \leq \phi(d(x, y))$, $(x, y \in X)$, where ϕ is a non– decreasing function from $\mathbf{R}_+$, the nonnegative reals, into $\mathbf{R}_+$ such that $\phi^n(t) \to 0$, for all $t \in \mathbf{R}_+$. We prove that f has a unique fixed point if there exists an $r > 0$, for which the diameters of all balls in X with radius r are equibounded. Such a class of semimetric spaces includes the Frechet spaces with a regular ecart, for which the Contraction Principle was established earlier by M. Cicchese [5], however, with some further restrictions on a space and a map involved. We also demonstrate that for maps f satisfying the condition $d(fx, fy) \leq \phi(\max\{d(x, fx), d(y, fy)\})$, $(x, y \in X)$ (the Bianchini [2] type condition), a fixed point theorem holds under substantially weaker assumptions on a distance function d.

1991 *Mathematics Subject Classification.* 47H10, 54H25.
Key words and phrases. Fixed point, nonlinear contraction, semimetric, symmetric, space with a regular ecart, E–space, d–Cauchy completeness.

*The author passed away in October 30, 1994.

1. Introduction. A *distance function* for a set X is a function d from $X \times X$ into $\mathbf{R}_+$, the nonnegative reals, such that $d(x,y) = 0$ iff $x = y$, and $d(x,y) = d(y,x)$ for all $x, y \in X$. A distance function is also called a *symmetric*. The space (X,d) in which limitting points are defined in the usual way is called an *E–space*. The idea of E–spaces goes back to Frechet and Menger. The pioneer works in this setting were the papers of E. W. Chittenden [4] and W. A. Wilson [17].

In every symmetric space (X,d) one may introduce a topology τ_d by defining the family of closed sets as follows: a set $A \subseteq X$ is closed iff for any $x \in X$, $d(x,A) = 0$ implies $x \in A$, where

$$d(x,A) := \inf\{d(x,a) : a \in A\}.$$

A topological space (X,τ) is *symmetrizable* iff there exists a symmetric d for which τ_d coincides with τ. A space (X,τ) is *semimetrizable* iff there is a distance function d such that for any $A \subseteq X$, $\bar{A} = \{x \in X : d(x,A) = 0\}$. In this case d is said to be a semimetric. In other words, without involving a topology, d is a semimetric if the operator

$$cl(A) := \{x \in X : d(x,A) = 0\}, \text{ for } A \subseteq X,$$

is the closure operator (it suffices here that cl is idempotent, i.e., $cl(cl(A)) = cl(A)$ for all $A \subseteq X$). For a discussion of the differences between a semimetric space and a symmetric space, see [1] and the references in [3].

Further, a symmetric or semimetric space (X,d) is *d–Cauchy complete* if every d–Cauchy sequence is *τ–convergent* (a sequence $\{x_n\}_{n=1}^{\infty}$ is d–Cauchy if given $\epsilon > 0$, there is a $k \in \mathbf{N}$ such that $d(x_n, x_m) < \epsilon$ for all $n, m \geq k$). We emphasize here that there are several concepts of completeness in semimetric spaces (see [15], [8]), but for our purposes we shall employ only the above concept. Similarly, an E–space (X,d) is *complete* if every d–Cauchy sequence $\{x_n\}_{n=1}^{\infty}$ is *d–convergent*, i.e., $d(x_n, x_0) \to 0$, for some $x_0 \in X$. Since in semimetrizable spaces d–convergence coincides with τ–convergence (see, e.g., [8]), we may conclude that a semimetric space (X,d) is d–Cauchy complete iff the E-space (X,d) is complete.

Our main purpose is to extend some fundamental metric fixed point theorems to a non–metric setting. Namely, we generalize Theorem 1.2 [13] of the second named author (see also [6], Theorem 3.2, or [14], Theorem 2) by considering selfmaps on some d–Cauchy complete semimetric spaces. This class of spaces is large enough to include the spaces (X,d) studied in [4] (called by Frechet spaces with a *regular ecart*), for which d is assumed to satisfy the following condition, a relaxation of the triangle inequality.

$$d(x,y) \leq \epsilon(\max\{d(x,z), d(z,y)\}), \quad \text{for } x, y, z \in X, \tag{1}$$

where a function $\epsilon : \mathbf{R}_+ \mapsto \mathbf{R}_+$ is such that $\lim_{t\to 0^+} \epsilon(t) = 0$. (Recently, a comprehensive study of such spaces with ϵ being linear has been made by the third named author [16] in connexion with studying the so–called small system convergence [12].). Our Theorem 1 generalizes an earlier result of M. Cicchese [5], who has considered the Banach contractions on a semimetric space with d satisfying a strengthened form of (1). Moreover, a restriction on a contractive constant was made in [5].

We also give an example of a fixed point free Banach contraction on a d–Cauchy complete semimetric space in order to demonstrate that an additional condition imposed on d in Theorem 1 cannot be omitted (see Example 2). On the other hand, our Theorem 2 shows that this condition is unnecessary if one considers a map f satisfying the inequality introduced by R. M. Bianchini [2].

$$d(fx, fy) \leq h \max\{d(x, fx), d(y, fy)\}, \quad \text{for an } h \in (0,1) \text{ and all } x, y \in X. \tag{2}$$

Then, however, a continuity argument must be used to ensure the existence of a fixed point (see Example 3 and Remark 3).

Finally, we would like to call the reader's attention to the recent papers [9], [10] and [11] of T. Hicks and B. E. Rhoades, in which the authors have obtained several fixed point theorems for maps on so–called d–complete topological spaces. Here a distance function d need not be even symmetric. However, they use a different concept of completeness: a space (X, d) is said to be (Σ) *d–complete* iff for any sequence $\{x_n\}_{n=1}^{\infty}$, $\sum_{n=1}^{\infty} d(x_n, x_{n+1}) < \infty$ implies that $\{x_n\}_{n=1}^{\infty}$ is τ-convergent. This notion let to obtain almost immediately an extension of the Contraction Principle and many other theorems to such spaces. However, in a semimetric setting, this concept of completeness is rather strong (see Proposition 2).

2. Preliminary results. We begin with the following simple extension of the Contraction Principle. The letter f^n denotes the nth iterate of a map f.

Proposition 1. *Let (X, d) be a Hausdorff semimetric and d-Cauchy complete space and let f be a selfmap on X satisfying the Banach contractive condition:*

$$d(fx, fy) \leq hd(x, y), \quad \text{for an } h \in (0,1) \text{ and } x, y \in X. \tag{3}$$

If (X, d) is bounded, i.e., $M := \sup\{d(x, y) : x, y \in X\} < \infty$, then f has a unique fixed point p, and for any $x \in X$, $\{f^n x\}_{n=1}^{\infty}$ converges to p.

Proof. Fix an $x \in X$. That $\{f^n x\}_{n=1}^{\infty}$ is d-Cauchy follows easily from the inequality

$$d(f^n x, f^{n+m} x) \le h^n d(x, f^m x) \le h^n M, \text{ for all } n, m \in \mathbf{N}$$

because of the convergence $h^n M \to 0$. By the completeness, there is a $p \in X$ such that $\{f^n x\}_{n=1}^{\infty}$ τ-converges to p. Since d is a semimetric, (3) implies that f is τ-continuous. Therefore, $\{f^{n+1} x\}_{n=1}^{\infty}$ τ-converges to fp. Since (X, d) is Hausdorff, we may infer that $p = fp$. Clearly, (3) guarantees the uniqueness of a fixed point. □

The following example shows that under assumptions of Proposition 1 a space (X, d) need not be (Σ) d-complete.

Example 1. Let $X := \mathbf{N}$. Define the function d by putting

$$d(n, n+1) := \frac{1}{2^n} =: d(n+1, n), \ d(n, n) := 0, \quad \text{for } n \in \mathbf{N},$$

and $d(n, m) := 1$, for all $n, m \in \mathbf{N}$ with $|n - m| > 1$. Then d is the semimetric. Clearly, $\sum_{n=1}^{\infty} d(n, n+1) < \infty$ and the sequence $\{n\}_{n=1}^{\infty}$ is not convergent. Thus, (X, d) is not (Σ) $d-$ complete. On the other hand, every d-Cauchy sequence is constant for sufficiently large n, hence convergent, so (X, d) is d-Cauchy complete.

Moreover, for a large class of semimetric spaces, (Σ) d-completeness implies d-Cauchy completeness.

Proposition 2. *Let (X, d) be a semimetric space satisfying Wilson's Axiom IV* [17], *i.e., given $\{x_n\}_{n=1}^{\infty}$, $\{y_n\}_{n=1}^{\infty}$ and an x in X,*

$$d(x_n, x) \to 0 \ \text{ and } \ d(x_n, y_n) \to 0 \ \text{ imply that } \ d(y_n, x) \to 0.$$

If (X, d) is (Σ) d-complete, then (X, d) is d-Cauchy complete.

Proof. Let $\{x_n\}_{n=1}^{\infty}$ be a d-Cauchy sequence. Then there exists a subsequence $\{x_{k_n}\}_{n=1}^{\infty}$ such that $d(x_{k_n}, x_{k_{n+1}}) < \frac{1}{2^n}$. By hypothesis, $\{x_{k_n}\}_{n=1}^{\infty}$ is τ-convergent to an $x \in X$. Since d is a semimetric, we get $d(x_{k_n}, x) \to 0$. Simultaneously, $d(x_n, x_{k_n}) \to 0$ because of the Cauchy condition. So, by Axiom IV, we may infer that $d(x_n, x) \to 0$, which implies that $\{x_n\}_{n=1}^{\infty}$ is τ- convergent to x (this implication also holds if d is a symmetric). □

The following example shows that Proposition 1 cannot be extended to unbounded semimetric spaces.

Example 2. Let $X := \mathbf{N}$, $fn := n + 1$ for $n \in \mathbf{N}$, and

$$d(n, m) := \frac{|n - m|}{2^{\min\{n, m\}}}, \quad \text{for } n, m \in \mathbf{N}.$$

Then d is the semimetric. Let $\{x_n\}_{n=1}^{\infty}$ be a d-Cauchy sequence. Then $\{x_n\}_{n=1}^{\infty}$ is bounded; for otherwise, there is a subsequence $\{x_{k_n}\}_{n=1}^{\infty}$, $x_{k_n} \to \infty$, and then, for any $n \in \mathbf{N}$,

$$\lim_{m\to\infty} d(x_{k_n}, x_{k_m}) = \lim_{m\to\infty} \frac{|x_{k_n} - x_{k_m}|}{2^{x_{k_n}}} = \infty,$$

violating the Cauchy condition. Therefore, we may infer that $\{x_n\}_{n=1}^{\infty}$ is constant for sufficiently large n, since it is d-Cauchy. Thus (X, d) is d-Cauchy complete, but f has no a fixed point though it satisfies (3) with $h = \frac{1}{2}$.

Now, we give some equivalent formulations of a condition imposed on a function d in our Theorem 1 (see Section 3).

Proposition 3. *Let (X, d) be an E-space. The following conditions are equivalent.*

(i): *There exists an $r > 0$ such that*

$$R := \sup\{diam\ K(x, r) : x \in X\} < \infty,$$

i.e., the diameters of open balls with radius r are equibounded.

(ii): *There exist $\delta, \eta > 0$ such that, given $x, y, z \in X$,*

$$d(x, z) + d(z, y) \le \delta \ \text{ implies that } \ d(x, y) \le \eta.$$

(iii): *There do not exist sequences $\{x_n\}_{n=1}^{\infty}$, $\{y_n\}_{n=1}^{\infty}$, $\{z_n\}_{n=1}^{\infty}$ such that*

$$d(x_n, z_n) \to 0,\ d(z_n, y_n) \to 0 \ \text{ and } \ d(x_n, y_n) \to \infty.$$

Proof. To prove (i) implies (ii) it suffices to put $\delta := \frac{r}{2}$ and $\eta := R$. To prove (ii) implies (iii) suppose, on the contrary, there exist sequences as in (iii). Then $d(x_n, z_n) + d(z_n, y_n) \le \delta$ for n large enough so, by (ii), $d(x_n, y_n) \le \eta$, which contradicts the convergence $d(x_n, y_n) \to \infty$. Further, it is easy to verify the implication $\neg(i) \Rightarrow \neg(iii)$. □

3. Nonlinear contractions on a semimetric space. Obviously, condition (iii) of Proposition 3 is satisfied if there do not exist sequences $\{x_n\}_{n=1}^{\infty}$, $\{y_n\}_{n=1}^{\infty}$, $\{z_n\}_{n=1}^{\infty}$ such that

$$d(x_n, z_n) \to 0, \ \ d(z_n, y_n) \to 0 \ \text{ and } \ d(x_n, y_n) \not\to 0.$$

By Theorem 1 of Wilson [17], the last condition holds iff (X, d) has a regular ecart. So, in particular, the following fixed point theorem may be applied to selfmaps on such a space.

Theorem 1. *Let (X,d) be a Hausdorff semimetric and d-Cauchy complete space satisfying one (hence all) of conditions (i)–(iii) of Proposition 3. Let f be a selfmap on X, for which*

$$d(fx,fy) \le \phi(d(x,y)), \quad \text{for all } x,y \in X, \tag{4}$$

where $\phi : \mathbf{R}_+ \mapsto \mathbf{R}_+$ is a non-decreasing function such that $\lim_{n\to\infty} \phi^n(t) = 0$, $(t \in \mathbf{R}_+)$. Then f has a unique fixed point p and $f^n x \to p$, for all $x \in X$.

Proof. Assume that condition (ii) of Proposition 3 holds. Fix an $x \in X$. By (4) and the monotonicity of ϕ, we get that

$$d(f^n x, f^{n+m}x) \le \phi^n(d(x,f^m x)), \quad \text{for all } n,m \in \mathbf{N}. \tag{5}$$

In particular, $d(f^n x, f^{n+1}x) \le \phi^n(d(x,fx))$, which implies, by hypothesis, that $d(f^n x, f^{n+1}x) \to 0$. Therefore, there is a $k \in \mathbf{N}$ such that $d(f^k x, f^{k+1}x) \le \min\{\frac{\delta}{2}, \eta\}$. Assume that $\phi(\eta) \le \frac{\delta}{2}$. We shall apply induction with respect to n to show that, for all $n \in \mathbf{N}$,

$$d(f^k x, f^{k+n}x) \le \eta. \tag{6}$$

By the definition of k, (6) holds for $n = 1$. Assume that (6) is satisfied for some $n \in \mathbf{N}$. Since $d(f^k x, f^{k+1}x) \le \frac{\delta}{2}$ and

$$d(f^{k+1}x, f^{k+n+1}x) \le \phi(d(f^k x, f^{k+n}x)) \le \phi(\eta) \le \frac{\delta}{2},$$

we get that $d(f^k x, f^{k+1}x) + d(f^{k+1}x, f^{k+n+1}x) \le \delta$, which, by (ii), implies that $d(f^k x, f^{k+n+1}x) \le \eta$, completing the induction. Hence and by (5), we may infer that

$$d(f^{k+n}x, f^{k+n+m}x) \le \phi^n(\eta), \quad \text{for all } n,m \in \mathbf{N},$$

which easily yields the Cauchy condition for $\{f^n x\}_{n=1}^{\infty}$. Further, use the same argument as in the proof of Proposition 1 to obtain that $f^n x \to p = fp$. Thus the proof is completed if $\phi(\eta) \le \frac{\delta}{2}$. If not, then, however, there exists a $j \in \mathbf{N}$ such that $\phi^j(\eta) \le \frac{\delta}{2}$. Since the iterate f^j satisfies (4) with ϕ replaced by ϕ^j, we may conclude by the preceding part of the proof, that f^j has a unique fixed point p and for all $x \in X$, $f^{jn}x \to p$ as $n \to \infty$. It is well-known that this implies that p is a unique fixed point of f and $f^n x \to p$, for all $x \in X$ (clearly, the proof of this fact in a metric setting remains valid for semimetrics). □

Remark 1. Theorem 1 generalizes Theorem 2 of Cicchese [5], who has imposed on d the condition

$$d(x,y) \le \epsilon(d(x,z)) + kd(y,z), \quad \text{for all } x,y,z \in \mathbf{R}_+,$$

where $k \geq 1$, $\epsilon : [0, a) \mapsto \mathbf{R}_+$, $(a > 0)$ and $\lim_{t \to 0^+} \epsilon(t) = 0$. By Theorem 1 [17], this condition is stronger than (1). Furthermore, Cicchese has assumed that f satisfies (3) with $h < \frac{1}{k}$.

Remark 2. Theorem 1 can be carried over to a complete E-space (X, d) satisfying (i) of Proposition 3 and Wilson's Axiom III [17] given in our Theorem 2.

4. Bianchini's maps on an E–space. The following example shows that Proposition 1 cannot be extended to maps satisfying condition (2).

Example 3. Let $X := \{0\} \cup \{\frac{1}{n} : n \in \mathbf{N}\}$ and $fx := \frac{x}{4}$ for $x \neq 0$, $f0 := 1$. Further, let

$$d(0,1) := 1 =: d(1,0), \quad d(1, \frac{1}{n}) := \frac{2}{3} =: d(\frac{1}{n}, 1) \qquad \text{for } n \geq 2,$$
$$d(1,1) := 0 \quad \text{and} \quad d(x,y) := |x-y|, \qquad \text{for } x, y \in X - \{1\}. \tag{7}$$

Then d is the semimetric. Let $\{x_n\}_{n=1}^{\infty}$ be a d–Cauchy sequence. Since $(X - \{1\}, d)$ is the complete metric space, it suffices to consider the case, in which there is a subsequence $\{x_{k_n}\}_{n=1}^{\infty}$ such that $x_{k_n} = 1$, for all $n \in \mathbf{N}$. Then $x_n = 1$ for sufficiently large n; for otherwise, there is a subsequence $\{x_{m_n}\}_{n=1}^{\infty}$ such that $x_{m_n} \neq 1$ for all $n \in \mathbf{N}$ so, by (7), $d(x_{k_n}, x_{k_m}) \geq \frac{2}{3}$, $(n \in \mathbf{N})$, violating the Cauchy condition. Thus (X, d) is d–Cauchy complete.

Now, we verify condition (2). Let $X_0 := X - \{0, 1\}$. Then (X_0, d) is the metric space and $f|_{X_0}$ is the Banach contraction with the constant $h = \frac{1}{4}$. Hence, by the triangle inequality, $f|_{X_0}$ satisfies (2) with the constant $\frac{2h}{1-h}$ $(= \frac{2}{3})$. Further, for all $x \in X$, $d(f0, fx) \leq \frac{2}{3} = \frac{2}{3} d(0, f0)$, and $d(f1, fx) = \frac{1}{4} - \frac{1}{4}x < \frac{4}{9} = \frac{2}{3} d(1, f1)$, for $x \neq 0$. So f satisfies (2) with $h = \frac{2}{3}$, but there is no fixed point for f.

Unexpectedly, Proposition 1 does extend to *continuous* maps satisfying (2) even if (X, d) is unbounded and d is not symmetric. Such a space (X, d) endowed with the right convergence operator we also call an E-space.

Theorem 2. *Let X be a nonempty set and $d : X \times X \mapsto \mathbf{R}_+$ be a function such that, given $x, y \in X$,*

$$d(x,y) = 0 \quad iff \quad x = y.$$

Let f be a selfmap on X such that condition (2) holds and f is d-continuous, i.e., given $\{x_n\}_{n=1}^{\infty}$ and x in X, $d(x_n, x) \to 0$ implies $d(fx_n, fx) \to 0$. If the E-space (X, d) is complete and d satisfies Wilson's Axiom III [17], *i.e.,* *given* $\{x_n\}_{n=1}^{\infty}$, x *and* y *in* X,

$$d(x_n, x) \to 0 \quad \textit{and} \quad d(x_n, y) \to 0 \quad \textit{imply that} \quad x = y,$$

then f *has a unique fixed point* p, *and* $d(f^n x, p) \to 0$, *for all* $x \in X$.

Proof. Define $\alpha(x) := d(x, fx)$, for $x \in X$. Then (2) easily implies that $\alpha^{-1}(0)$ is at most a singleton and $\alpha(fx) \le h\alpha(x)$, for $x \in X$. Hence if

$$\bar{d}(x, y) := \max\{\alpha(x), \alpha(y)\} \quad \text{for } x \ne y, \quad \text{and } \bar{d}(x, x) := 0 \quad \text{for } x \in X,$$

then one can verify that $\bar{d}$ is the metric; in particular, $\bar{d}(x, y) \le \max\{\bar{d}(x, z), \bar{d}(z, y)\}$, for $x, y, z \in X$, so $\bar{d}$ is the ultrametric (see, e.g., [7], p.504). Moreover, f is the Banach contraction with respect to $\bar{d}$ with the same constant h as in (2). By the proof of the Contraction Principle, for any $x \in X$ the sequence $\{f^n x\}_{n=1}^{\infty}$ is $\bar{d}$-Cauchy. By (2), for $n, m \in \mathbf{N}$, if $f^n x \ne f^m x$ then

$$d(f^{n+1}x, f^{m+1}x) \le h \max\{\alpha(f^n x), \alpha(f^m x)\} = h\bar{d}(f^n x, f^m x).$$

Hence, $d(f^{n+1}x, f^{m+1}x) \le h\bar{d}(f^n x, f^m x)$, which holds also in case, in which $f^n x = f^m x$. Therefore, we may conclude that $\{f^n x\}_{n=1}^{\infty}$ is d–Cauchy. By the completeness, there is a $p \in X$ such that $d(f^n x, p) \to 0$. Then $d(f^{n+1}x, fp) \to 0$ because of the continuity of f. Hence, $p = fp$ since d satisfies Axiom III. Moreover, p does not depend on x, since the fixed point is unique. □

Remark 3. If d is continuous with respect to the first variable, i.e., given $\{x_n\}_{n=1}^{\infty}$, x, y in X, $d(x_n, x) \to 0$ implies $d(x_n, y) \to d(x, y)$ (this forces Axiom III for such a d), then the assumption in Theorem 2 that f be continuous can be dropped. To see that, observe that, by the proof of Theorem 2, given $x \in X$, there is a $p \in X$ such that $d(f^n x, p) \to 0$. By the continuity of d and the inequality

$$d(f^{n+1}x, fp) \le h \max\{d(f^n x, f^{n+1}x), d(p, fp)\}, \quad \text{for } x \in X,$$

we get letting $n \to \infty$ that $d(p, fp) \le hd(p, fp)$, and hence $p = fp$.

Remark 4. Theorem 2 can be extended to maps satisfying more general contractive condition:

$$d(fx, fy) \le \phi(\max\{d(x, fx), d(y, fy)\}), \quad \text{for } x, y \in X,$$

where ϕ is a function as in Theorem 1. Then the above given proof needs a slight modification only; that $\{f^n x\}_{n=1}^{\infty}$ is $\bar{d}$-Cauchy follows this time from the proof of Theorem 1.2 [13] and the fact that f satisfies (4) in a metric space $(X, \bar{d})$.

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