VOLUME AS A MEASURE OF APPROXIMATION FOR THE JACOBI-PERRON ALGORITHM

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Abstract

We consider the values of the consecutive minima of the quantities $F_j(x;g) = (A_0^{(g+d+1)} + \sum_{j=1}^d A_0^{(g+j)} y_j)(A_0^{(g+j)})^{-1}$, $1 \leq j \leq d$. W. Schmidt, in 1958, calculated the first and second minimum for $j = 1$ and $d = 2$. Schweiger, in 1975, considered the case $j = 1$ for any $d \geq 2$. This note is a continuation of these investigations.

1. Introduction

W. Schmidt opened a new route on Diophantine approximation by the Jacobi-Perron algorithm when he introduced volume as a measure of approximation. For $g \geq d+1$, let $p^{(g)} = \left(\frac{A_1^{(g)}}{A_0^{(g)}}, \ldots, \frac{A_d^{(g)}}{A_0^{(g)}}\right)$ be the rational approximation to the point $x = (x_1, \ldots, x_d)$ provided by the Jacobi-Perron algorithm. Then d consecutive points $p^{(g+1)}, ..., p^{(g+d)}$, and x form a simplex with volume ($y = T^g x$)

$$V(x;g) = \frac{1}{d! A_0^{(g+1)} ... A_0^{(g+d)} (A_0^{(g+d+1)} + \sum\limits_{j=1}^{d} A_0^{(g+j)} y_j)}.$$

The Jacobi-Perron algorithm can be described by iteration of the map T on the d-dimensional unit cube as follows (see [4]):

$$\begin{aligned} T(x_1, \ldots, x_d) &= \left(\frac{x_2}{x_1} - k_1(x), \ldots, \frac{1}{x_1} - k_d(x)\right) \\ k_j(x) &= \left[\frac{x_{j+1}}{x_1}\right], 1 \leq j \leq d-1, k_d(x) = \left[\frac{1}{x_1}\right], \\ k(x) &= (k_1(x), \ldots, k_d(x)). \end{aligned}$$

The points x and z are called *equivalent* if there are $n \geq 0, m \geq 0$ such that $T^n x = T^m z$.

We further introduce the sequence $k^{(g)}(x) = k(T^{g-1}x)$ and the matrices

$$\beta^{(g)}(x) = \begin{pmatrix} k_d^{(g)} & 0 & \dots & 0 & 1 \\ 1 & 0 & \dots & 0 & 0 \\ k_1^{(g)} & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ k_{d-1}^{(g)} & 0 & \dots & 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} A_0^{(g+d+1)} & A_0^{(g+1)} & \dots & A_0^{(g+d)} \\ A_1^{(g+d+1)} & A_1^{(g+1)} & \dots & A_1^{(g+d)} \\ \vdots & \vdots & \ddots & \vdots \\ A_d^{(g+d+1)} & A_d^{(g+1)} & \dots & A_d^{(g+d)} \end{pmatrix} = \beta^{(1)}(x) \circ \dots \circ \beta^{(g)}(x).$$

In this note we consider the quantities

$$F_j(x;g) = \frac{A_0^{(g+d+1)} + \sum_{k=1}^{d} A_0^{(g+k)} y_k}{A_0^{(g+j)}}, \; 1 \le j \le d.$$

Let $\xi > 1$ be the largest root of $X^{d+1} - X^d - 1 = 0$. In [2] the following conjecture was stated. For all x with non-terminating expansion there are infinitely many values of g such that the inequality

$$F_j(x,g) > \xi^{d+1-j} + d\xi^{-j}$$

is satisfied.

Since for $x^* = (\frac{1}{\xi}, \frac{1}{\xi^2}, ..., \frac{1}{\xi^d})$ it is easy to see that $\lim_{g\to\infty} F_j(x^*, g) = \xi^{d+1-j} + d\xi^{-j}$. This result was thought to be best possible.

For $d = 1$ this conjecture is true by Hurwitz' famous result on continued fraction. (Note that for $d = 1$ we have $\xi + \xi^{-1} = \sqrt{5}$.)

W. Schmidt [1] proved the conjecture for $d = 2$ and $j = 1$. For infinitely many $g \ge 1$, the inequality

$$F_2(x^*, g) > 3\xi - 2 = \lim_{s\to\infty} F_2((\frac{1}{\xi}, \frac{1}{\xi^2}), s) \sim 2,39671\dots$$

is true. Moreover, he showed that if x is not equivalent to $x^* = (\frac{1}{\xi}, \frac{1}{\xi^2})$, then the constant $\xi^2 + 2\xi^{-1}$ could be replaced by the greater value $\gamma \sim 4.26459\dots$ which is related to $z^* = (\frac{1}{\eta} + \frac{1}{\eta^2}, \frac{1}{\eta})$ where $\eta^3 - 2\eta^2 - 3\eta = 1,\ \eta > 3$. Again, this result is best possible in an obvious sense. If x is not equivalent to x^* or z^* then we obtain

$$F_1(x, g) > \frac{13}{3}$$

for infinitely many values of g.

Schweiger [3]proved that the conjecture is true for any $d \geq 1$ and $j = 1$. Schweiger [2] additionally proved that the conjecture is true for $d = 2$ and $j = 2$. In this paper this matter is further explored. For $d = 2$ and $j = 2$ the second minimum is calculated. Surprisingly the second minimum of $F_2(x, g)$ is given by $y^* = (\frac{1}{\lambda}, \frac{1}{\lambda} + \frac{1}{\lambda^2})$, $\lambda^3 = 2\lambda^2 + 1$, and not by $z^* = (\frac{1}{\eta} + \frac{1}{\eta^2}, \frac{1}{\eta})$ as for $j = 1$. Furthermore, it is shown that the conjecture is not true for $d = 3$ and $j = 3$.

2. The Second Minimum

Theorem. *Let $\lambda > 1$ be the greatest root of $\lambda^3 - 2\lambda^2 - 1 = 0$. Then for all x which are not equivalent to $(\frac{1}{\xi}, \frac{1}{\xi^2})$ for infinitely many $g \geq 1$ we have*

$$F_2(x, g) > 3\lambda - 4 = \lim_{s \to \infty} F_2\left(\left(\frac{1}{\lambda}, \frac{1}{\lambda} + \frac{1}{\lambda^2}\right), s\right) \sim 2.61671\ldots.$$

Proof. Here and in the sequel, overlines refer to a periodic expansion. We first consider two special cases:

`Case 1.` $\left(\frac{1}{\alpha}, \frac{1}{\alpha} + \frac{1}{\alpha^2}\right) = \left(\begin{array}{c} \overline{1} \\ 1 \end{array}\right), \alpha^3 - \alpha^2 - \alpha - 1 = 0$. Then

$$\lim_{s \to \infty} F_2\left(\left(\frac{1}{\alpha}, \frac{1}{\alpha} + \frac{1}{\alpha^2}\right), s\right) = -\alpha^2 + 4\alpha - 1 \sim 2.97417\ldots.$$

`Case 2.` $\left(\frac{1}{\beta}, \frac{1}{\beta} + \frac{1}{\beta^2}\right) = \left(\begin{array}{c} \overline{1} \\ 2 \end{array}\right), \beta^3 - 2\beta^2 - \beta - 1 = 0$. Then

$$\lim_{s \to \infty} F_2\left(\left(\frac{1}{\beta}, \frac{1}{\beta} + \frac{1}{\beta^2}\right), s\right) = -\beta^2 + 5\beta - 3 \sim 3.24781\ldots.$$

Now consider $F_2(x, g+2) = k_2^{(g+2)} + x_2^{(g+2)} + \frac{A_0^{(g+3)}}{A_0^{(g+4)}}(k_1^{(g+2)} + x_1^{(g+2)}) + \frac{A_0^{(g+2)}}{A_0^{(g+4)}}$. If $F_2(x, g) < 2.62$ for all $g \geq g_0$, then clearly $k_2^{(t)} \leq 2$. Clearly, we may assume that $g_0 = 1$, so that we have $x_2^{(g+2)} = \frac{k_1^{(g+3)} + x_1^{(g+3)}}{k_2^{(g+3)} + x_2^{(g+3)}} \geq \frac{1}{9}$ and $x_1^{(g+2)} = \frac{1}{k_2^{(g+3)} + x_2^{(g+3)}} \geq \frac{1}{3}$.

Now let $k_2^{(t)} = 2$ infinitely often. Assume that $k_2^{(g+2)} = 2$.
If

$$k_1^{(g+2)} = 2, \tag{1}$$

then clearly $A_0^{(g+4)} \le 2A_0^{(g+3)} + 3A_0^{(g+2)}$. Hence,

$$\begin{aligned} F_2(x, g+2) &\ge 2 + \frac{1}{9} + \frac{A_0^{(g+3)}}{A_0^{(g+4)}}(2 + \frac{1}{3}) + \frac{A_0^{(g+2)}}{A_0^{(g+4)}} \\ &= \frac{19}{9} + \frac{7A_0^{(g+3)} + 3A_0^{(g+2)}}{3A_0^{(g+4)}} \\ &\ge \frac{19}{9} + \frac{7A_0^{(g+3)} + 3A_0^{(g+2)}}{6A_0^{(g+3)} + 9A_0^{(g+2)}} \\ &\ge \frac{19}{9} + \frac{2}{3} = \frac{25}{9} > \frac{26}{10}. \end{aligned}$$

Next, assume that $k_1^{(g+2)} = 1$. Looking at Case 2 and Equation 1 we may assume

$$k_2^{(g+1)} = 2, k_1^{(g+1)} = 0, k_2^{(g+1)} = k_1^{(g+1)} = 1,$$

or $k_2^{(g+1)} = 1, k_1^{(g+1)} = 0$. In any case we obtain

$$A_0^{(g+4)} \le 2A_0^{(g+3)} + A_0^{(g+1)}.$$

Then again

$$\begin{aligned} F_2(x, g+2) &\ge \frac{19}{9} + \frac{A_0^{(g+3)}}{A_0^{(g+4)}}(1 + \frac{1}{3}) + \frac{A_0^{(g+2)}}{A_0^{(g+4)}} \\ &= \frac{19}{9} + \frac{4A_0^{(g+3)} + 3A_0^{(g+2)}}{3A_0^{(g+4)}} \\ &\ge \frac{19}{9} + \frac{4A_0^{(g+3)} + 3A_0^{(g+2)}}{6A_0^{(g+4)} + 3A_0^{(g+1)}} \ge \frac{25}{9}. \end{aligned}$$

Now assume that $k_1^{(g+2)} = 0$ and note that only the digits $\begin{pmatrix} 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ must be considered. Hence, the remaining case is

$$k_2^{(g+1)} = 1, k_1^{(g+1)} = 0$$

(the case $k_1^{(g+1)} = 1$ is not allowed by Perron's condition for the digits).

We have $A_0^{(g+4)} = A_0^{(g+3)} + A_0^{(g+1)}$, and we estimate

$$F_2(x,g+2) \geq \frac{19}{9} + \frac{A_0^{(g+3)}}{A_0^{(g+4)}}\frac{1}{3} + \frac{A_0^{(g+2)}}{A_0^{(g+4)}} = \frac{19}{9} + \frac{A_0^{(g+3)} + 3A_0^{(g+2)}}{3A_0^{(g+3)} + 3A_0^{(g+1)}}.$$

Since $k_2^{(g)} = k_1^{(g)}$ is not allowed, the cases $k_2^{(g)} = 1, k_1^{(g)} = 0$ and $k_2^{(g)} = 2, k_1^{(g)} = 0$ remain.

If $k_2^{(g)} = 1, k_1^{(g)} = 0$, then $A_0^{(g+3)} = A_0^{(g+2)} + A_0^{(g)}$, so that

$$\frac{A_0^{(g+3)} + 3A_0^{(g+2)}}{3A_0^{(g+3)} + 3A_0^{(g+1)}} = \frac{4A_0^{(g+2)} + A_0^{(g)}}{3A_0^{(g+2)} + 3A_0^{(g+1)} + 3A_0^{(g)}} \geq \frac{5}{9}.$$

If $k_2^{(g)} = 2, k_1^{(g)} = 0$, then we may assume that $k_2^{(g-1)} = 2, k_1^{(g-1)} = 0$. Calculation gives $A_0^{(g+2)} = 2A_0^{(g+1)} + A_0^{(g-1)}$ and $A_0^{(g+3)} = 4A_0^{(g+1)} + A_0^{(g)} + 2A_0^{(g-1)}$, so that

$$\frac{A_0^{(g+3)} + 3A_0^{(g+2)}}{3A_0^{(g+3)} + 3A_0^{(g+1)}} = \frac{10A_0^{(g+1)} + A_0^{(g)} + 5A_0^{(g-1)}}{15A_0^{(g+1)} + 3A_0^{(g)} + 6A_0^{(g-1)}} \geq \frac{5}{9}.$$

However $\frac{19}{9} + \frac{5}{9} = \frac{24}{9} = \frac{8}{3} > 2.61671\ldots$.

Finally, the case $k_2^{(t)} = 1$ for all $t \geq t_0$ leads to the periodic cases $\begin{pmatrix} \overline{1} \\ 1 \end{pmatrix}$ and $\begin{pmatrix} \overline{0} \\ 1 \end{pmatrix}$. □

Remark. The point

$$x = (\lambda, \frac{1}{\lambda}) = \lim_{g\to\infty} \left(\frac{A_0^{(g+3)}}{A_0^{(g+2)}}, \frac{A_0^{(g+1)}}{A_0^{(g+2)}} \right)$$

lies in every triangle spanned by three successive points $\left(\frac{A_0^{(j+2)}}{A_0^{(j+1)}}, \frac{A_0^{(j)}}{A_0^{(j+1)}} \right)$, $j = g+1, g+2, g+3$. Furthermore, this point lies on the straight line with the equation

$$x_1 + x_2\frac{1}{\lambda} + \frac{1}{\lambda^2} = 3\lambda - 4.$$

Therefore there are infinitely many values g such that

$$\frac{A_0^{(g+3)}}{A_0^{(g+2)}} + \frac{A_0^{(g+1)}}{A_0^{(g+2)}}\frac{1}{\lambda} + \frac{1}{\lambda^2} > 3\lambda - 4.$$

Remark. For $F_1(x,g)$ the second minimum is given by the point

$$(\frac{1}{\eta}+\frac{1}{\eta^2},\frac{1}{\eta}) = \begin{pmatrix} \overline{0 \quad 0} \\ 2 \quad 1 \end{pmatrix}$$

where $\eta^3 - 2\eta^2 - 3\eta = 1, \eta > 3$.

For $F_2(x,g)$ this expansion gives two points of accumulation:

$$\lim_{s\to\infty} F_2((\frac{1}{\eta}+\frac{1}{\eta^2},\frac{1}{\eta}),2s+1) = \frac{\eta+1}{\eta^2}(-3\eta^2+9\eta+5) \sim 1.83445\ldots$$

and

$$\lim_{s\to\infty} F_2((\frac{1}{\eta}+\frac{1}{\eta^2},\frac{1}{\eta}),2s) = \frac{2\eta+1}{\eta^2}(-3\eta^2+9\eta+5) \sim 3.21924\ldots.$$

Therefore this expansion is not related to the second minimum.

3. A Counterexample

The general conjecture about the first minimum of the quantities $F_j(x,g)$ [2,4] is not true.

Letting $j = d = 3$, we have

$$F_3(x,g) = \frac{A_0^{(g+4)} + x_1^{(g)} A_0^{(g+1)} + x_2^{(g)} A_0^{(g+2)} + x_3^{(g)} A_0^{(g+3)}}{A_0^{(g+3)}}.$$

Let $\xi^4 = \xi^3 + 1$ and consider again

$$z = (\frac{1}{\xi},\frac{1}{\xi^2},\frac{1}{\xi^3}) = \begin{pmatrix} \overline{0} \\ 0 \\ 1 \end{pmatrix}.$$

Then

$$\lim_{s\to\infty} F_3(z,s) = \xi + \frac{3}{\xi^3} = 4\xi - 3 \sim 2.52112\ldots.$$

But if we consider $\lambda > 1$, the greatest root of $\lambda^4 = 2\lambda^3 + 1$, then the expansion of

$$w = (\frac{1}{\lambda},\frac{1}{\lambda^2},\frac{1}{\lambda^3}) = \begin{pmatrix} \overline{0} \\ 0 \\ 2 \end{pmatrix}$$

gives the smaller value

$$\lim_{s\to\infty} F_3(w,s) = \lambda + \frac{3}{\lambda^3} = 4\lambda - 6 \sim 2.42768\ldots.$$

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