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ON RELATIVELY PRIME SUBSETS AND SUPERSETS

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Abstract

A nonempty finite set of positive integers A is relatively prime if $\gcd(A) = 1$ and it is relatively prime to n if $\gcd(A \cup \{n\}) = 1$. The number of nonempty subsets of A which are relatively prime to n is $\Phi(A, n)$ and the number of such subsets of cardinality k is $\Phi_k(A, n)$. Given positive integers l_1, l_2, m_2, and n such that $l_1 \leq l_2 \leq m_2$ we give $\Phi([1, m_1] \cup [l_2, m_2], n)$ along with $\Phi_k([1, m_1] \cup [l_2, m_2], n)$. Given positive integers l, m, and n such that $l \leq m$ we count for any subset A of $\{l, l+1, \dots, m\}$ the number of its supersets in $[l, m]$ which are relatively prime and we count the number of such supersets which are relatively prime to n. Formulas are also obtained for corresponding supersets having fixed cardinalities. Intermediate consequences include a formula for the number of relatively prime sets with a nonempty intersection with some fixed set of positive integers.

1. Introduction

Throughout let k, l, m, n be positive integers such that $l \leq m$, let $[l, m] = \{l, l+1, \dots, m\}$, let μ be the Möbius function, and let $\lfloor x \rfloor$ be the floor of x. If A is a set of integers and $d \neq 0$, then $\frac{A}{d} = \{a/d : \ a \in A\}$. A nonempty set of positive integers A is called *relatively prime* if $\gcd(A) = 1$ and it is called *relatively prime to n* if $\gcd(A \cup \{n\}) = \gcd(A, n) = 1$. Unless otherwise specified A and B will denote nonempty sets of positive integers. We will need the following basic identity on binomial coefficients stating that for nonnegative integers $L \leq M \leq N$

$$\sum_{j=M}^{N} \binom{j}{L} = \binom{N+1}{L+1} - \binom{M}{L+1}. \tag{1}$$

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Definition 1. Let

$$\begin{aligned}\Phi(A,n) &= \#\{X \subseteq A: \ X \neq \emptyset \text{ and } \gcd(X,n)=1\},\\ \Phi_k(A,n) &= \#\{X \subseteq A: \ \#X = k \text{ and } \gcd(X,n)=1\},\\ f(A) &= \#\{X \subseteq A: \ X \neq \emptyset \text{ and } \gcd(X)=1\},\\ f_k(A) &= \#\{X \subseteq A: \ \#X = k \text{ and } \gcd(X)=1\}.\end{aligned}$$

Nathanson in [5] introduced $f(n)$, $f_k(n)$, $\Phi(n)$, and $\Phi_k(n)$ (in our terminology $f([1,n])$, $f_k([1,n])$, $\Phi([1,n],n)$, and $\Phi_k([1,n],n)$ respectively) and gave their formulas along with asymptotic estimates. Formulas for $f([m,n])$, $f_k([m,n])$, $\Phi([m,n],n)$, and $\Phi_k([m,n],n)$ are found in [3, 6] and formulas for $\Phi([1,m],n)$ and $\Phi_k([1,m],n)$ for $m \leq n$ are obtained in [4]. Recently Ayad and Kihel in [2] considered phi functions for sets which are in arithmetic progression and obtained the following more general formulas for $\Phi([l,m],n)$ and $\Phi_k([l,m],n)$.

Theorem 2. *We have*

$$\text{(a)} \quad \Phi([l,m],n) = \sum_{d|n} \mu(d) 2^{\lfloor m/d \rfloor - \lfloor (l-1)/d \rfloor},$$

$$\text{(b)} \quad \Phi_k([l,m],n) = \sum_{d|n} \mu(d) \binom{\lfloor m/d \rfloor - \lfloor (l-1)/d \rfloor}{k}.$$

2. Relatively Prime Subsets for $[1, m_1] \cup [l_2, m_2]$

If $[1,m_1] \cap [l_2,m_2] \neq \emptyset$, then phi functions for $[1,m_1] \cup [l_2,m_2] = [1,m_2]$ are obtained by Theorem 2. So we may assume that $1 \leq m_1 < l_2 \leq m_2$.

Lemma 3. *Let*

$$\Psi(m_1,l_2,m_2,n) = \#\{X \subseteq [1,m_1] \cup [l_2,m_2]: \ l_2 \in X \ \textit{and} \ \gcd(X,n)=1\}$$

and

$$\Psi_k(m_1,l_2,m_2,n) = \#\{X \subseteq [1,m_1] \cup [l_2,m_2]: \ l_2 \in X, \ |X|=k, \ \textit{and} \ \gcd(X,n)=1\}.$$

Then

$$\text{(a)} \quad \Psi(m_1,l_2,m_2,n) = \sum_{d|(l_2,n)} \mu(d) 2^{\lfloor m_1/d \rfloor + \lfloor m_2/d \rfloor - l_2/d},$$

$$\text{(b)} \quad \Psi_k(m_1,l_2,m_2,n) = \sum_{d|(l_2,n)} \mu(d) \binom{\lfloor m_1/d \rfloor + \lfloor m_2/d \rfloor - l_2/d}{k-1}.$$

Proof. (a) Assume first that $m_2 \leq n$. Let $\mathcal{P}(m_1, l_2, m_2)$ denote the set of subsets of $[1, m_1] \cup [l_2, m_2]$ containing l_2 and let $\mathcal{P}(m_1, l_2, m_2, d)$ be the set of subsets X of $[1, m_1] \cup [l_2, m_2]$ such that $l_2 \in X$ and $\gcd(X, n) = d$. It is clear that the set $\mathcal{P}(m_1, l_2, m_2)$ of cardinality $2^{m_1+m_2-l_2}$ can be partitioned using the equivalence relation of having the same gcd (dividing l_2 and n). Moreover, the mapping $A \mapsto \frac{1}{d}X$ is a one-to-one correspondence between $\mathcal{P}(m_1, l_2, m_2, d)$ and the set of subsets Y of $[1, \lfloor m_1/d \rfloor] \cup [l_2/d, \lfloor m_2/d \rfloor]$ such that $l_2/d \in Y$ and $\gcd(Y, n/d) = 1$. Then

$$\#\mathcal{P}(m_1, l_2, m_2, d) = \Psi(\lfloor m_1/d \rfloor, l_2/d, \lfloor m_2/d \rfloor, n/d).$$

Thus,

$$2^{m_1+m_2-l_2} = \sum_{d|(l_2,n)} \#\mathcal{P}(m_1, l_2, m_2, d) = \sum_{d|(l_2,n)} \Psi(\lfloor m_1/d \rfloor, l_2/d, \lfloor m_2/d \rfloor, n/d),$$

which by the Möbius inversion formula extended to multivariable functions [3, Theorem 2] is equivalent to

$$\Psi(m_1, l_2, m_2, n) = \sum_{d|(l_2,n)} \mu(d) 2^{\lfloor m_1/d \rfloor + \lfloor m_2/d \rfloor - l_2/d}.$$

Assume now that $m_2 > n$ and let a be a positive integer such that $m_2 \leq n^a$. As $\gcd(X, n) = 1$ if and only if $\gcd(X, n^a) = 1$ and $\mu(d) = 0$ whenever d has a nontrivial square factor, we have

$$\begin{aligned}\Psi(m_1, l_2, m_2, n) &= \Psi(m_1, l_2, m_2, n^a) \\ &= \sum_{d|(l_2,n^a)} \mu(d) 2^{\lfloor m_1/d \rfloor + \lfloor m_2/d \rfloor - l_2/d} \\ &= \sum_{d|(l_2,n)} \mu(d) 2^{\lfloor m_1/d \rfloor + \lfloor m_2/d \rfloor - l_2/d}.\end{aligned}$$

(b) For the same reason as before, we may assume that $m_2 \leq n$. Noting that the correspondence $X \mapsto \frac{1}{d}X$ defined above preserves the cardinality and using an argument similar to the one in part (a), we obtain the following identity

$$\binom{m_1 + m_2 - l_2}{k-1} = \sum_{d|(l_2,n)} \Psi_k(\lfloor m_1/d \rfloor, l_2/d, \lfloor m_2/d \rfloor, n/d)$$

which by the Möbius inversion formula [3, Theorem 2] is equivalent to

$$\Psi_k(m_1, l_2, m_2, n) = \sum_{d|(l_2,n)} \mu(d) \binom{\lfloor m_1/d \rfloor + \lfloor m_2/d \rfloor - l_2/d}{k-1},$$

as desired. □

Theorem 4. *We have*

$$\text{(a)} \quad \Phi([1, m_1] \cup [l_2, m_2], n) = \sum_{d|n} \mu(d) 2^{\lfloor \frac{m_1}{d} \rfloor + \lfloor \frac{m_2}{d} \rfloor - \lfloor \frac{l_2-1}{d} \rfloor},$$

$$\text{(b)} \quad \Phi_k([1, m_1] \cup [l_2, m_2], n) = \sum_{d|n} \mu(d) \binom{\lfloor \frac{m_1}{d} \rfloor + \lfloor \frac{m_2}{d} \rfloor - \lfloor \frac{l_2-1}{d} \rfloor}{k}.$$

Proof. (a) Clearly

$$\begin{aligned}
\Phi([1, m_1] \cup [l_2, m_2], n) &= \Phi([1, m_1] \cup [l_2 - 1, m_2], n) - \Psi(m_1, l_2 - 1, m_2, n) \\
&= \Phi([1, m_1] \cup [m_1 + 1, m_2], n) - \sum_{i=m_1+1}^{l_2-1} \Psi(m_1, i, m_2, n) \\
&= \Phi([1, m_2] - \sum_{i=m_1+1}^{l_2-1} \Psi(m_1, i, m_2, n) \qquad (2) \\
&= \sum_{d|n} \mu(d) 2^{\lfloor m_2/d \rfloor} - \sum_{i=m_1+1}^{l_2-1} \sum_{d|(n,i)} \mu(d) 2^{\lfloor \frac{m_1}{d} \rfloor + \lfloor \frac{m_2}{d} \rfloor - \frac{i}{d}},
\end{aligned}$$

where the last identity follows by Theorem 2 for $l = 1$ and Lemma 3. Rearranging the last summation in (2) gives

$$\begin{aligned}
\sum_{i=m_1+1}^{l_2-1} \sum_{d|(n,i)} \mu(d) 2^{\lfloor \frac{m_1}{d} \rfloor + \lfloor \frac{m_2}{d} \rfloor - \frac{i}{d}} &= \sum_{d|n} \sum_{\substack{i=m_1+1 \\ d|i}}^{l_2-1} \mu(d) 2^{\lfloor \frac{m_1}{d} \rfloor + \lfloor \frac{m_2}{d} \rfloor - \frac{i}{d}} \\
&= \sum_{d|n} \mu(d) 2^{\lfloor \frac{m_1}{d} \rfloor + \lfloor \frac{m_2}{d} \rfloor} \sum_{j=\lfloor \frac{m_1}{d} \rfloor + 1}^{\lfloor \frac{l_2-1}{d} \rfloor} 2^{-j} \qquad (3) \\
&= \sum_{d|n} \mu(d) 2^{\lfloor \frac{m_2}{d} \rfloor} \left(1 - 2^{-\lfloor \frac{l_2-1}{d} \rfloor + \lfloor \frac{m_1}{d} \rfloor}\right).
\end{aligned}$$

Now combining identities (2) and (3) yields the result.

(b) Proceeding as in part (a) we find

$$\Phi_k([1,m_1]\cup[l_2,m_2],n)=\sum_{d|n}\mu(d)\binom{\lfloor\frac{m_2}{d}\rfloor}{k}-\sum_{i=m_1+1}^{l_2-1}\sum_{d|(n,i)}\mu(d)\binom{\lfloor\frac{m_1}{d}\rfloor+\lfloor\frac{m_2}{d}\rfloor-\frac{i}{d}}{k-1}. \tag{4}$$

Rearranging the last summation on the right of (4) gives

$$\begin{aligned}\sum_{i=m_1+1}^{l_2-1}\sum_{d|(n,i)}\binom{\lfloor\frac{m_1}{d}\rfloor+\lfloor\frac{m_2}{d}\rfloor-\frac{i}{d}}{k-1}&=\sum_{d|n}\mu(d)\sum_{j=\lfloor\frac{m_1}{d}\rfloor+1}^{\lfloor\frac{l_2-1}{d}\rfloor}\binom{\lfloor\frac{m_1}{d}\rfloor+\lfloor\frac{m_2}{d}\rfloor-j}{k-1}\\&=\sum_{d|n}\mu(d)\sum_{i=\lfloor\frac{m_1}{d}\rfloor+\lfloor\frac{m_2}{d}\rfloor-\lfloor\frac{l_2-1}{d}\rfloor}^{\lfloor\frac{m_2}{d}\rfloor-1}\binom{i}{k-1}\\&=\sum_{d|n}\mu(d)\left(\binom{\lfloor\frac{m_2}{d}\rfloor}{k}-\binom{\lfloor\frac{m_1}{d}\rfloor+\lfloor\frac{m_2}{d}\rfloor-\lfloor\frac{l_2-1}{d}\rfloor}{k}\right),\end{aligned} \tag{5}$$

where the last identity follows by formula (1). Then identities (4) and (5) yield the desired result. □

Definition 5. Let

$$\varepsilon(A,B,n)=\#\{X\subseteq B:\ X\neq\emptyset,\ X\cap A=\emptyset,\ \text{and}\ \gcd(X,n)=1\},$$
$$\varepsilon_k(A,B,n)=\#\{X\subseteq B:\ \#X=k,\ X\cap A=\emptyset,\ \text{and}\ \gcd(X,n)=1\}.$$

If $B=[1,n]$ we will simply write $\varepsilon(A,n)$ and $\varepsilon_k(A,n)$ rather than $\varepsilon(A,[1,n],n)$ and $\varepsilon_k(A,[1,n],n)$ respectively.

Theorem 6. *If $l\le m<n$, then*

$$\text{(a) } \varepsilon([l,m],n)=\sum_{d|n}\mu(d)2^{\lfloor(l-1)/d\rfloor+n/d-\lfloor m/d\rfloor},$$

$$\text{(b) } \varepsilon_k([l,m],n)=\sum_{d|n}\mu(d)\binom{\lfloor(l-1)/d\rfloor+n/d-\lfloor m/d\rfloor}{k}.$$

Proof. Immediate from Theorem 4 since $\varepsilon([l,m],n)=\Phi([1,l-1]\cup[m+1,n],n)$ and $\varepsilon_k([l,m],n)=\Phi_k([1,l-1]\cup[m+1,n],n)$. □

3. Relatively Prime Supersets

In this section the sets A and B are not necessary nonempty.

Definition 7. If $A \subseteq B$ let

$$\overline{\Phi}(A, B, n) = \#\{X \subseteq B : \ X \neq \emptyset, \ A \subseteq X, \text{ and } \gcd(X, n) = 1\},$$

$$\overline{\Phi}_k(A, B, n) = \#\{X \subseteq B : \ A \subseteq X, \ \#X = k, \text{ and } \gcd(X, n) = 1\},$$

$$\overline{f}(A, B) = \#\{X \subseteq B : \ X \neq \emptyset, \ A \subseteq X, \text{ and } \gcd(X) = 1\},$$

$$\overline{f}_k(A, B) = \#\{X \subseteq B : \ \#X = k, \ A \subseteq X, \text{ and } \gcd(X) = 1\}.$$

The purpose of this section is to give formulas for $\overline{f}(A, [l, m])$, $\overline{f}_k(A, [l, m])$, $\overline{\Phi}(A, [l, m], n)$, and $\overline{\Phi}_k(A, [l, m], n)$ for any subset A of $[l, m]$. We need a lemma.

Lemma 8. *If $A \subseteq [1, m]$, then*

$$\text{(a)} \quad \overline{\Phi}(A, [1, m], n) = \sum_{d|(A,n)} \mu(d) 2^{\lfloor m/d \rfloor - \#A},$$

$$\text{(b)} \ \overline{\Phi}_k(A, [1, m], n) = \sum_{d|(A,n)} \mu(d) \binom{\lfloor m/d \rfloor - \#A}{k - \#A} \text{ whenever } \#A \leq k \leq m.$$

Proof. If $A = \emptyset$, then clearly

$$\overline{\Phi}(A, [1, m], n) = \Phi([1, m], n) \text{ and } \overline{\Phi}_k(A, [1, m], n) = \Phi_k([1, m], n)$$

and the identities in (a) and (b) follow by Theorem 2 for $l = 1$. Assume now that $A \neq \emptyset$. If $m \leq n$, then

$$2^{m - \#A} = \sum_{d|(A,n)} \overline{\Phi}(\frac{A}{d}, [1, \lfloor m/d \rfloor], n/d)$$

and

$$\binom{m - \#A}{k - \#A} = \sum_{d|(A,n)} \mu(d) \overline{\Phi}_k(\frac{A}{d}, [1, \lfloor m/d \rfloor], n/d),$$

which by Möbius inversion [3, Theorem 2] are equivalent to the identities in (a) and in (b) respectively. If $m > n$, let a be a positive integer such that $m \leq n^a$.

As $\gcd(X, n) = 1$ if and only if $\gcd(X, n^a) = 1$ and $\mu(d) = 0$ whenever d has a nontrivial square factor we have

$$\begin{aligned}\overline{\Phi}(A,[1,m],n) &= \overline{\Phi}(A,[1,m],n^a)\\ &= \sum_{d|(A,n^a)} \mu(d)2^{\lfloor m/d\rfloor - \#A}\\ &= \sum_{d|(A,n)} \mu(d)2^{\lfloor m/d\rfloor - \#A}.\end{aligned}$$

The same argument gives the formula for $\overline{\Phi}_k(A,[1,m],n)$. □

Theorem 9. *If $A \subseteq [l,m]$, then*

$$\text{(a)} \quad \overline{\Phi}(A,[l,m],n) = \sum_{d|(A,n)} \mu(d)2^{\lfloor m/d\rfloor - \lfloor (l-1)/d\rfloor - \#A},$$

$$\text{(b)} \quad \overline{\Phi}_k(A,[l,m],n) = \sum_{d|(A,n)} \mu(d)\binom{\lfloor m/d\rfloor - \lfloor (l-1)/d\rfloor - \#A}{k-\#A}$$

whenever $\#A \le k \le m-l+1$.

Proof. If $A = \emptyset$, then clearly

$$\overline{\Phi}(A,[l,m],n) = \Phi([l,m],n)$$

and

$$\overline{\Phi}_k(A,[l,m],n) = \Phi_k([l,m],n)$$

and the identities in (a) and (b) follow by Theorem 2.
Assume now that $A \neq \emptyset$. Let

$$\Psi(A,l,m,n) = \#\{X \subseteq [l,m] : \ A \cup \{l\} \subseteq X, \text{and } \gcd(X,n) = 1\}.$$

Then

$$2^{m-l-\#A} = \sum_{d|(A,l,n)} \Psi(\frac{A}{d}, l/d, \lfloor m/d\rfloor, n/d),$$

which by Möbius inversion [3, Theorem 2] means that

$$\Psi(A,l,m,n) = \sum_{d|(A,l,n)} \mu(d)2^{\lfloor m/d\rfloor - l/d - \#A}. \tag{6}$$

Then combining identity (6) with Lemma 8 gives

$$\begin{aligned}\overline{\Phi}(A,[l,m],n) &= \overline{\Phi}([A,[1,m],n) - \sum_{i=1}^{l-1}\Psi(i,m,A,n)\\ &= \sum_{d\mid(A,n)}\mu(d)2^{\lfloor m/d\rfloor-\#A} - \sum_{i=1}^{l-1}\sum_{d\mid(A,i,n)}\mu(d)2^{\lfloor m/d\rfloor-i/d-\#A}\\ &= \sum_{d\mid(A,n)}\mu(d)2^{\lfloor m/d\rfloor-\#A} - \sum_{d\mid(A,n)}\mu(d)2^{\lfloor m/d\rfloor-\#A}\sum_{j=1}^{\lfloor (l-1)/d\rfloor}2^{-j}\\ &= \sum_{d\mid(A,n)}\mu(d)2^{\lfloor m/d\rfloor-\#A} - \sum_{d\mid(A,n)}\mu(d)2^{\lfloor m/d\rfloor-\#A}(1-2^{-\lfloor (l-1)/d\rfloor})\\ &= \sum_{d\mid(A,n)}\mu(d)2^{\lfloor m/d\rfloor-\lfloor (l-1)/d\rfloor-\#A}.\end{aligned} \tag{7}$$

This completes the proof of (a). Part (b) follows similarly. □

As to $\overline{f}(A,[l,m])$ and $\overline{f}_k(A,[l,m])$ we similarly have:

Theorem 10. *If $A\subseteq[l,m]$, then*

$$\text{(a)}\quad \overline{f}(A,[l,m]) = \sum_{d\mid \gcd(A)}\mu(d)2^{\lfloor \frac{m}{d}\rfloor-\lfloor\frac{l-1}{d}\rfloor-\#A},$$

$$\text{(b)}\quad \overline{f}_k(A,[l,m]) = \sum_{d\mid \gcd(A)}\mu(d)\binom{\lfloor \frac{m}{d}\rfloor-\lfloor\frac{l-1}{d}\rfloor-\#A}{k-\#A},$$

whenever $\#A\le k\le m-l+1$.

We close this section by formulas for relatively prime sets which have a nonempty intersection with A.

Definition 11. Let

$$\overline{\varepsilon}(A,B,n) = \#\{X\subseteq B:\ X\cap A\neq\emptyset \text{ and } \gcd(X,n)=1\},$$

$$\overline{\varepsilon}_k(A,B,n) = \#\{X\subseteq B:\ \#X=k,\ X\cap A\neq\emptyset, \text{ and } \gcd(X,n)=1\},$$

$$\overline{\varepsilon}(A,B) = \#\{X\subseteq B:\ X\cap A\neq\emptyset \text{ and } \gcd(X)=1\},$$

$$\overline{\varepsilon}_k(A,B) = \#\{X\subseteq B:\ \#X=k,\ X\cap A\neq\emptyset, \text{ and } \gcd(X)=1\}.$$

Theorem 12. *We have*

$$\text{(a)}\quad \overline{\varepsilon}(A,[l,m],n)=\sum_{\emptyset\neq X\subseteq A}\sum_{d|(X,n)}\mu(d)2^{\lfloor\frac{m}{d}\rfloor-\lfloor\frac{l-1}{d}\rfloor-\#X},$$

$$\text{(b)}\quad \overline{\varepsilon}_k(A,[l,m],n)=\sum_{\substack{\emptyset\neq X\subseteq A\\ \#X\leq k}}\sum_{d|(X,n)}\mu(d)\binom{\lfloor\frac{m}{d}\rfloor-\lfloor\frac{l-1}{d}\rfloor-\#X}{k-\#X},$$

$$\text{(c)}\ \overline{\varepsilon}(A,B)=\sum_{\emptyset\neq X\subseteq A}\sum_{d|\,\gcd(X)}\mu(d)2^{\lfloor\frac{m}{d}\rfloor-\lfloor\frac{l-1}{d}\rfloor-\#X},$$

$$\text{(d)}\ \overline{\varepsilon}_k(A,B)=\sum_{\substack{\emptyset\neq X\subseteq A\\ \#X\leq k}}\sum_{d|\,\gcd(X)}\mu(d)\binom{\lfloor\frac{m}{d}\rfloor-\lfloor\frac{l-1}{d}\rfloor-\#X}{k-\#X}.$$

Proof. These formulas follow by Theorems 4 and 5 along with the facts:

$$\overline{\varepsilon}(A,[l,m],n)=\sum_{\emptyset\neq X\subseteq A}\overline{\Phi}(X,[l,m],n),$$

$$\overline{\varepsilon}_k(A,[l,m],n)=\sum_{\substack{\emptyset\neq X\subseteq A\\ \#X\leq k}}\overline{\Phi}_k(X,[l,m],n),$$

$$\overline{\varepsilon}(A,[l,m])=\sum_{\emptyset\neq X\subseteq A}\overline{f}(X,[l,m]),$$

$$\overline{\varepsilon}_k(A,[l,m])=\sum_{\substack{\emptyset\neq X\subseteq A\\ \#X\leq k}}\overline{f}_k(X,[l,m]).$$

□

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