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ON THE LEAST COMMON MULTIPLE OF Q-BINOMIAL COEFFICIENTS

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Abstract

We first prove the following identity

$$\operatorname{lcm}\left(\begin{bmatrix}n\\0\end{bmatrix}_q, \begin{bmatrix}n\\1\end{bmatrix}_q, \dots, \begin{bmatrix}n\\n\end{bmatrix}_q\right) = \frac{\operatorname{lcm}([1]_q, [2]_q, \dots, [n+1]_q)}{[n+1]_q},$$

where ${n \brack k}_q$ denotes the q-binomial coefficient and $[n]_q = \frac{1-q^n}{1-q}$. Then we show that this identity is indeed a q-analogue of that of B. Farhi.

1. Introduction

An equivalent form of the prime number theorem states that $\log \operatorname{lcm}(1, 2, \dots, n) \sim n$ as $n \to \infty$ (see, for example, [4]). Nair [7] gave a nice proof for the well-known estimate $\operatorname{lcm}\{1, 2, \dots, n\} \geq 2^{n-1}$, while Hanson [3] already obtained $\operatorname{lcm}\{1, 2, \dots, n\} \leq 3^n$. Recently, Farhi [1] established the following interesting result.

Theorem 1 (Farhi) *For any positive integer n, there holds*

$$\operatorname{lcm}\left(\binom{n}{0}, \binom{n}{1}, \dots, \binom{n}{n}\right) = \frac{\operatorname{lcm}(1, 2, \dots, n+1)}{n+1}. \tag{1}$$

As an application, Farhi shows that the inequality $\operatorname{lcm}\{1, 2, \dots, n\} \geq 2^{n-1}$ follows immediately from (1).

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The purpose of this note is to give a q-analogue of (1) by using cyclotomic polynomials. Recall that a natural q-analogue of the nonnegative integer n is given by $[n]_q = \frac{1-q^n}{1-q}$. The corresponding q-factorial is $[n]_q! = \prod_{k=1}^n [k]_q$ and the q-binomial coefficient ${M \brack N}_q$ is defined as

$$ {M \brack N}_q = \begin{cases} \dfrac{[M]_q!}{[N]_q![M-N]_q!}, & \text{if } 0 \le N \le M, \\ 0, & \text{otherwise.} \end{cases} $$

Let lcm also denote the least common multiple of a sequence of polynomials in $\mathbb{Z}[q]$. Our main results can be stated as follows:

Theorem 2 *For any positive integer n, there holds*

$$ \operatorname{lcm}\left({n \brack 0}_q, {n \brack 1}_q, \dots, {n \brack n}_q \right) = \frac{\operatorname{lcm}([1]_q, [2]_q, \dots, [n+1]_q)}{[n+1]_q}. \tag{2} $$

Theorem 3 *The identity* (2) *is a q-analogue of Farhi's identity* (1)*, i.e.,*

$$ \lim_{q\to 1} \operatorname{lcm}\left({n \brack 0}_q, {n \brack 1}_q, \dots, {n \brack n}_q \right) = \operatorname{lcm}\left(\binom{n}{0}, \binom{n}{1}, \dots, \binom{n}{n} \right), \tag{3} $$

and

$$ \lim_{q\to 1} \frac{\operatorname{lcm}([1]_q, [2]_q, \dots, [n+1]_q)}{[n+1]_q} = \frac{\operatorname{lcm}(1, 2, \dots, n+1)}{n+1}. \tag{4} $$

Although it is clear that

$$ \lim_{q\to 1} {n \brack k}_q = \binom{n}{k}, $$

the identities (3) and (4) are not trivial. For example, we have

$$ 4 = \lim_{q\to 1} \operatorname{lcm}\left(1+q, 1+q^2\right) \neq \operatorname{lcm}\left(\lim_{q\to 1}(1+q), \lim_{q\to 1}(1+q^2) \right) = 2. $$

2. Proof of Theorem 2

Let $\Phi_n(x)$ be the n-th *cyclotomic polynomial*. The following easily proved result can be found in [5, (10)] and [2].

Lemma 4 *The q-binomial coefficient $\begin{bmatrix} n \\ k \end{bmatrix}_q$ can be factorized into*

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \prod_d \Phi_d(q),$$

where the product is over all positive integers $d \le n$ such that $\lfloor k/d \rfloor + \lfloor (n-k)/d \rfloor < \lfloor n/d \rfloor$.

Lemma 5 *Let n and d be two positive integers with $n \ge d$. Then there exists at least one positive integer k such that*

$$\lfloor k/d \rfloor + \lfloor (n-k)/d \rfloor < \lfloor n/d \rfloor \tag{5}$$

if and only if d does not divide $n+1$.

Proof. Suppose that (5) holds for some positive integer k. Let

$$k \equiv a \pmod{d}, \qquad (n-k) \equiv b \pmod{d}$$

for some $1 \le a, b \le d-1$. Then $n \equiv a+b \pmod{d}$ and $d \le a+b \le 2d-2$. Namely, $n+1 \equiv a+b+1 \not\equiv 0 \pmod{d}$. Conversely, suppose that $n+1 \equiv c \pmod{d}$ for some $1 \le c \le d-1$. Then $k = c$ satisfies (5). This completes the proof. □

Proof of Theorem 2. By Lemma 4, we have

$$\operatorname{lcm}\left(\begin{bmatrix} n \\ 0 \end{bmatrix}_q, \begin{bmatrix} n \\ 1 \end{bmatrix}_q, \dots, \begin{bmatrix} n \\ n \end{bmatrix}_q \right) = \prod_d \Phi_d(q), \tag{6}$$

where the product is over all positive integers $d \le n$ such that for some k $(1 \le k \le n)$ there holds $\lfloor k/d \rfloor + \lfloor (n-k)/d \rfloor < \lfloor n/d \rfloor$. On the other hand, since

$$[k]_q = \frac{q^k - 1}{q - 1} = \prod_{d \mid k,\ d>1} \Phi_d(q),$$

we have

$$\frac{\operatorname{lcm}([1]_q, [2]_q, \dots, [n+1]_q)}{[n+1]_q} = \prod_{d \le n,\ d \nmid (n+1)} \Phi_d(q). \tag{7}$$

By Lemma 5, one sees that the right-hand sides of (6) and (7) are equal. This proves the theorem. □

3. Proof of Theorem 3

We need the following property.

Lemma 6 *For any positive integer n, there holds*

$$\Phi_n(1) = \begin{cases} p, & \text{if } n = p^r \text{ is a prime power,} \\ 1, & \text{otherwise.} \end{cases}$$

Proof. See for example [6, p. 160]. □

In view of (6), we have

$$\lim_{q\to 1} \operatorname{lcm}\left(\begin{bmatrix} n \\ 0 \end{bmatrix}_q, \begin{bmatrix} n \\ 1 \end{bmatrix}_q, \dots, \begin{bmatrix} n \\ n \end{bmatrix}_q \right) = \prod_d \Phi_d(1), \tag{8}$$

where the product is over all positive integers $d \le n$ such that for some k $(1 \le k \le n)$ there holds $\lfloor k/d \rfloor + \lfloor (n-k)/d \rfloor < \lfloor n/d \rfloor$. By Lemma 6, the right-hand side of (8) can be written as

$$\prod_{\text{primes } p \le n} p^{\sum_{r=1}^{\infty} \max_{0\le k\le n} \{\lfloor n/p^r \rfloor - \lfloor k/p^r \rfloor - \lfloor (n-k)/p^r \rfloor\}}. \tag{9}$$

We now claim that

$$\sum_{r=1}^{\infty} \max_{0\le k\le n} \{\lfloor n/p^r \rfloor - \lfloor k/p^r \rfloor - \lfloor (n-k)/p^r \rfloor\}$$
$$= \max_{0\le k\le n} \sum_{r=1}^{\infty} (\lfloor n/p^r \rfloor - \lfloor k/p^r \rfloor - \lfloor (n-k)/p^r \rfloor). \tag{10}$$

Let $n = \sum_{i=0}^{M} a_i p^i$, where $0 \le a_0, a_1, \dots, a_M \le p-1$ and $a_M \ne 0$. By Lemma 5, the left-hand side of (10) (denoted LHS(10)) is equal to the number of r's such that $p^r \le n$ and $p^r \nmid n+1$. It follows that

$$LHS(10) = \begin{cases} 0, & \text{if } n = p^{M+1} - 1, \\ M - \min\{i\colon a_i \ne p-1\}, & \text{otherwise.} \end{cases}$$

It is clear that the right-hand side of (10) is less than or equal to LHS(10). If $n = p^{M+1} - 1$, then both sides of (10) are equal to 0. Assume that $n \ne p^{M+1} - 1$ and $i_0 = \min\{i\colon a_i \ne p-1\}$. Taking $k = p^M - 1$, we have

$$\lfloor n/p^r \rfloor - \lfloor k/p^r \rfloor - \lfloor (n-k)/p^r \rfloor = \begin{cases} 0, & \text{if } r = 1, \dots, i_0, \\ 1, & \text{if } r = i_0 + 1, \dots, M, \end{cases}$$

and so
$$\sum_{r=1}^{\infty} (\lfloor n/p^r \rfloor - \lfloor k/p^r \rfloor - \lfloor (n-k)/p^r \rfloor) = M - i_0.$$

Thus (10) holds. Namely, the expression (9) is equal to

$$\prod_{\text{primes } p \le n} p^{\max_{0\le k\le n} \sum_{r=1}^{\infty} (\lfloor n/p^r \rfloor - \lfloor k/p^r \rfloor - \lfloor (n-k)/p^r \rfloor)}$$
$$= \operatorname{lcm}\left(\binom{n}{0}, \binom{n}{1}, \dots, \binom{n}{n}\right).$$

This proves (3). To prove (4), we apply (7) to get

$$\lim_{q\to 1} \frac{\operatorname{lcm}([1]_q, [2]_q, \dots, [n+1]_q)}{[n+1]_q} = \prod_{d\le n,\ d\nmid (n+1)} \Phi_d(1),$$

which, by Lemma 6, is clearly equal to

$$\frac{\operatorname{lcm}(1, 2, \dots, n+1)}{n+1}.$$

Finally, we mention that (10) has the following interesting conclusion.

Corollary 7 *Let p be a prime number and let $k_1, k_2, \dots, k_m \le n$, $r_1 < r_2 < \cdots < r_m$ be positive integers such that*

$$\lfloor n/p^{r_i} \rfloor - \lfloor k_i/p^{r_i} \rfloor - \lfloor (n-k_i)/p^{r_i} \rfloor = 1 \quad \text{for } i = 1, 2, \dots, m.$$

Then there exists a positive integer $k \le n$ such that

$$\lfloor n/p^{r_i} \rfloor - \lfloor k/p^{r_i} \rfloor - \lfloor (n-k)/p^{r_i} \rfloor = 1 \quad \text{for } i = 1, 2, \dots, m.$$

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