#A01 INTEGERS 10 (2010), 1-11

CONGRUENCES FOR HYPER M-ARY OVERPARTITION FUNCTIONS

Lin Ma
School of Mathematical Science, Xuzhou Normal University, Xuzhou , P.R. China
malin6292@163.com

Qing-Lin Lu[1]
School of Mathematical Science, Xuzhou Normal University, Xuzhou, P.R. China
qllu@xznu.edu.cn

Received: 12/2/08, Revised: 10/27/09, Accepted: 10/30/09, Published: 3/5/10

Abstract

We discuss a new restricted m-ary overpartition function $\overline{h}_m(n)$, which is the number of hyper m-ary overpartitions of n, such that each power of m is allowed to be used at most m times as a non-overlined part. In this note we use generating function dissections to prove the following family of congruences for all $n \geq 0$, $m \geq 4$, $j \geq 0$, $3 \leq k \leq m-1$, and $t \geq 1$:

$$\overline{h}_m(m^{j+t}n + m^{j+t-1}k + \cdots + m^j k) \equiv 0 (\bmod\ 2^t(2^{j+1}-1)).$$

1. Introduction

Numerous functions which enumerate partitions into powers of a fixed number m (Here m is assumed to be bigger than 1) have been studied by Churchhouse [2], Rødseth [10], Andrews [1], Gupta [8] in the late 1960s and early 1970s, and Dirdal [5, 6] in the mid-1970s. For more recent work see [7, 11, 9].

Presently there are a lot of activities in the study of the objects named overpartitions by Corteel and Lovejoy [3]. Rødseth [12] discussed divisibility properties of the number of m-ary overpartitions of a natural number. Courtright and Sellers [4] gave arithmetic properties for hyper m-ary partition functions. In this note, we define $\overline{h}_m(n)$ to be the number of hyper m-ary overpartitions of n. A hyper m-ary overpartion of n is a non-increasing sequence of non-negative integral powers of m whose sum is n, and where the first occurrence (equivalently, the final occurrence) of a power of m may be overlined, such that each power of m is allowed to be used at most m times as a non-overlined part. We denote the number of hyper m-ary overpartitions of n by $\overline{h}_m(n)$ ($\overline{h}_m(n) = 0$ for all negative integers n). The overlined parts form an m-ary partition into distinct parts, and the non-overlined parts

[1]Research is jointly supported by the National Natural Science Foundation of China (10771100) and the Natural Science Foundation for Colleges and Universities in Jiangsu province (06KJD110179).

form a hyper m-ary partition. Thus, putting $\overline{h}_m(0) = 1$, we have the generating function

$$\overline{H}_m(q) := \sum_{n\geq 0} \overline{h}_m(n)q^n = \prod_{i\geq 0}(1+q^{m^i})\sum_{k=0}^{m} q^{k\cdot m^i}.$$

For example, for $m = 2$ we find

$$\sum_{n\geq 0} \overline{h}_2(n)q^n = 1 + 2q + 4q^2 + 5q^3 + 8q^4 + 10q^5 + 13q^6 + \cdots,$$

where the 10 hyper binary overpartitions of 5 are

$$4+1, \bar{4}+1, 4+\bar{1}, \bar{4}+\bar{1}, 2+2+1, \bar{2}+2+1,$$
$$2+2+\bar{1}, \bar{2}+2+\bar{1}, 2+\bar{1}+1+1, \bar{2}+\bar{1}+1+1.$$

From the generating function of $\overline{h}_m(n)$, we have

$$\overline{H}_m(q) = (1+q)(1+q+\cdots+q^m)\overline{H}_m(q^m), \tag{1}$$

from which we obtain the following recurrences:

$$\overline{h}_m(mn) = \overline{h}_m(n) + 2\overline{h}_m(n-1), \tag{2}$$

$$\overline{h}_m(mn+1) = 2\overline{h}_m(n) + \overline{h}_m(n-1), \tag{3}$$

$$\overline{h}_m(mn+k) = 2\overline{h}_m(n) \quad \text{for} \quad 2 \leq k \leq m-1\ . \tag{4}$$

The main object of this note is to prove the following family of congruences for all $n \geq 0,\ m \geq 4,\ j \geq 0,\ t \geq 1$, and k satisfying $3 \leq k \leq m-1$,

$$\overline{h}_m(m^{j+t}n + m^{j+t-1}k + \cdots + m^j k) \equiv 0 (\text{mod } 2^t(2^{j+1}-1)).$$

2. Congruences for Hyper Binary and Trinary Overpartitions

We now focus our attention on the function $\overline{h}_2(n)$.

Lemma 1 *For all $n \geq 0$, we have*

$$\overline{h}_2(3n+1) \equiv 0 (\text{mod } 2), \quad \overline{h}_2(3n+2) \equiv 0 (\text{mod } 2).$$

Proof. We prove this lemma via induction on n. First, the lemma holds for the case $n = 0$ since $\overline{h}_2(1) = 2 \equiv 0 \pmod 2$, $\overline{h}_2(2) = 4 \equiv 0 \pmod 2$. Now, we assume the lemma is true for all $n \leq k$. Then we consider the case $n = k+1$.

Case 1. $k = 2j$ for some integer $j < k$. Then from (2), (3) and induction hypothesis we have

$$\begin{aligned}
\overline{h}_2(3(k+1)+1) &= \overline{h}_2(2(3j+2)) \\
&= \overline{h}_2(3j+2) + 2\overline{h}_2(3j+1) \equiv 0 \pmod 2, \\
\overline{h}_2(3(k+1)+2) &= \overline{h}_2(2(3j+2)+1) \\
&= 2\overline{h}_2(3j+2) + \overline{h}_2(3j+1) \equiv 0 \pmod 2.
\end{aligned}$$

Case 2. $k = 2j+1$ for some integer $j < k$. We also have

$$\begin{aligned}
\overline{h}_2(3(k+1)+1) &= \overline{h}_2(2(3j+3)+1) \\
&= 2\overline{h}_2(3j+3) + \overline{h}_2(3j+2) \equiv 0 \pmod 2, \\
\overline{h}_2(3(k+1)+2) &= \overline{h}_2(2(3j+4)) \\
&= \overline{h}_2(3j+4) + 2\overline{h}_2(3j+3) \equiv 0 \pmod 2.
\end{aligned}$$

So the lemma is true for the case $n = k+1$ and the proof is completed. □

By the lemma and similar techniques we can prove the following theorem:

Theorem 2 *For all $n \geq 0$, we have*

$$\overline{h}_2(n) \equiv 0\,(mod\ 2) \text{ if and only if } n \equiv 1, 2 \pmod 3.$$

Proof. The sufficiency is handled in Lemma 2.1. We now prove the necessity. We need only to prove $\overline{h}_2(3n) \equiv 1 (\text{mod } 2)$ by induction on n. First, the case $n = 0$ is clear. Now, we assume the result is true for all $n \leq k$. Then we consider the case $n = k+1$.

Case 1. $k = 2j$ for some integer $j < k$. Then from (2), (3), and the induction hypothesis we have

$$\overline{h}_2(3(k+1)) = \overline{h}_2(2(3j+1)+1) = 2\overline{h}_2(3j+1) + \overline{h}_2(3j) \equiv 1 \pmod 2.$$

Case 2. $k = 2j+1$ for some integer $j < k$. We also have

$$\overline{h}_2(3(k+1)) = \overline{h}_2(2(3j+3)) = \overline{h}_2(3j+3) + 2\overline{h}_2(3j+2) \equiv 1 \pmod 2.$$

So the case $n = k+1$ is true. This completes the proof. □

From the proof of Theorem 2.2 and $\overline{h}_2(3) \equiv 1 (\text{mod } 4)$, we have

Corollary 3 *For all $n \geq 0$, we have $\overline{h}_2(3n) \equiv 1 (\text{mod } 4)$.*

Lemma 4 *For all $k \geq 0$, we have*

$$\begin{aligned} \overline{h}_2(2^k) - \overline{h}_2(2^k - 1) &= k + 1, \\ \overline{h}_2(2^k - 1) - \overline{h}_2(2^k - 2) &= 1, \\ \overline{h}_2(2^k - 2) - \overline{h}_2(2^k - 3) &= k. \end{aligned}$$

Proof. We prove this lemma by induction on k. First the case $k = 0$ is clear. Now, we assume the result is true for $k = n$ and we consider the case $k = n + 1$. By the recurrences (2) and (3), we have

$$\begin{aligned} &\overline{h}_2(2^{n+1}) - \overline{h}_2(2^{n+1} - 1) \\ &\quad = \overline{h}_2(2 \cdot 2^n) - \overline{h}_2(2(2^n - 1) + 1) \\ &\quad = \overline{h}_2(2^n) + 2\overline{h}_2(2^n - 1) - (2\overline{h}_2(2^n - 1) + \overline{h}_2(2^n - 2)) \\ &\quad = \overline{h}_2(2^n) - \overline{h}_2(2^n - 1) + (\overline{h}_2(2^n - 1) - \overline{h}_2(2^n - 2)) \\ &\quad = (n + 1) + 1 = n + 2. \end{aligned}$$

$$\begin{aligned} &\overline{h}_2(2^{n+1} - 1) - \overline{h}_2(2^{n+1} - 2) \\ &\quad = \overline{h}_2(2(2^n - 1) + 1) - \overline{h}_2(2(2^n - 1)) \\ &\quad = 2\overline{h}_2(2^n - 1) + \overline{h}_2(2^n - 2) - (\overline{h}_2(2^n - 1) + 2\overline{h}_2(2^n - 2)) \\ &\quad = \overline{h}_2(2^n - 1) - \overline{h}_2(2^n - 2) = 1. \end{aligned}$$

$$\begin{aligned} &\overline{h}_2(2^{n+1} - 2) - \overline{h}_2(2^{n+1} - 3) \\ &\quad = \overline{h}_2(2(2^n - 1)) - \overline{h}_2(2(2^n - 2) + 1) \\ &\quad = \overline{h}_2(2^n - 1) + 2\overline{h}_2(2^n - 2) - (2\overline{h}_2(2^n - 2) + \overline{h}_2(2^n - 3)) \\ &\quad = (\overline{h}_2(2^n - 1) - \overline{h}_2(2^n - 2)) + (\overline{h}_2(2^n - 2) - \overline{h}_2(2^n - 3)) \\ &\quad = 1 + n = n + 1. \end{aligned}$$

So the lemma is true for the case $k = n + 1$ and the proof is completed. □

By Lemma 4 and induction on n, we can prove

Theorem 5 *For all $n \geq 0$, we have*

$$\overline{h}_2(2^n) = \frac{1}{2}(3^n+1)+n+1, \quad \overline{h}_2(2^n-1) = \frac{1}{2}(3^n+1).$$

Proof. We prove this theorem by induction on n. By Lemma 2.4, we need only to prove the first formula. First, the formula is true for $n=0$. Now, we assume the formula is true for $n=k$. Then we have

$$\begin{aligned}
\overline{h}_2(2^{k+1}) &= \overline{h}_2(2^k) + 2\overline{h}_2(2^k-1) \\
&= 3\overline{h}_2(2^k) - 2(\overline{h}_2(2^k) - \overline{h}_2(2^k-1)) \\
&= \frac{3}{2}(3^k+1) + 3(k+1) - 2(k+1) \\
&\qquad\qquad \text{(by induction hypothesis and Lemma 4)} \\
&= \frac{1}{2}(3^{k+1}+1) + k + 2.
\end{aligned}$$

So the theorem is true for the case $n=k+1$ and the proof is completed. □

From Theorem 5 and Lemma 4 we can easily obtain

Corollary 6 *For all $n \geq 1$, we have*

$$\overline{h}_2(2^n-2) = \frac{1}{2}(3^n-1), \quad \overline{h}_2(2^n-3) = \frac{1}{2}(3^n-1)-n.$$

Now, we consider the function $\overline{h}_3(n)$.

Theorem 7 *For all $n \geq 0$, we have*

$$n \equiv 1,2,3 \pmod 4 \textit{ implies } \overline{h}_3(n) \equiv 0 \pmod 2.$$

Proof. We prove this theorem by induction on n. First the theorem is true for the case $n=0$ since

$$\overline{h}_3(1) = 2, \ \ \overline{h}_3(2) = 2, \ \ \overline{h}_3(3) = 4.$$

Now we assume the lemma is true for all $n \leq k$ and we consider the case $n = k+1$.

`Case 1.` $k = 3j$ for some integer $j < k$. Then from (2), (3), (4) and the induction hypothesis we have

$$\begin{aligned}\overline{h}_3(4(k+1)+1) &= \overline{h}_3(3(4j+1)+2)\\ &= 2\overline{h}_3(4j+1) \equiv 0 \pmod{2}\\ \overline{h}_3(4(k+1)+2) &= \overline{h}_3(3(4j+2))\\ &= \overline{h}_3(4j+2) + 2\overline{h}_3(4j+1) \equiv 0 \pmod{2},\\ \overline{h}_3(4(k+1)+3) &= \overline{h}_3(3(4j+2)+1)\\ &= 2\overline{h}_3(4j+2) + \overline{h}_3(4j+1) \equiv 0 \pmod{2}.\end{aligned}$$

`Case 2.` $k = 3j+1$ for some integer $j < k$. We also have

$$\begin{aligned}\overline{h}_3(4(k+1)+1) &= \overline{h}_3(3(4j+3))\\ &= \overline{h}_3(4j+3) + 2\overline{h}_3(4j+2) \equiv 0 \pmod{2},\\ \overline{h}_3(4(k+1)+2) &= \overline{h}_3(3(4j+3)+1)\\ &= 2\overline{h}_3(4j+3) + \overline{h}_3(4j+2) \equiv 0 \pmod{2},\\ \overline{h}_3(4(k+1)+3) &= \overline{h}_3(3(4j+3)+2)\\ &= 2\overline{h}_3(4j+3) \equiv 0 \pmod{2}.\end{aligned}$$

`Case 3.` $k = 3j+2$ for some integer $j < k$. We also have

$$\begin{aligned}\overline{h}_3(4(k+1)+1) &= \overline{h}_3(3(4j+4)+1)\\ &= 2\overline{h}_3(4j+4) + \overline{h}_3(4j+3) \equiv 0 \pmod{2},\\ \overline{h}_3(4(k+1)+2) &= \overline{h}_3(3(4j+4)+2)\\ &= 2\overline{h}_3(4j+4) \equiv 0 \pmod{2},\\ \overline{h}_3(4(k+1)+3) &= \overline{h}_3(3(4j+5))\\ &= \overline{h}_3(4j+5) + 2\overline{h}_3(4j+4) \equiv 0 \pmod{2}.\end{aligned}$$

So the theorem is true for the case $n = k+1$ and the proof is completed. □

From Theorem 7 and Recurrence (4) we can prove

Theorem 8 *For all $n \geq 0$, $k \geq 0$, we have*

$$\overline{h}_3(4n\cdot 3^k + 2\cdot 3^k - 1) \equiv 0 \pmod{2^{k+1}},$$
$$\overline{h}_3(4n\cdot 3^k + 3\cdot 3^k - 1) \equiv 0 \pmod{2^{k+1}},$$
$$\overline{h}_3(4n\cdot 3^k + 4\cdot 3^k - 1) \equiv 0 \pmod{2^{k+1}}.$$

Proof. We prove this result by induction on k. We first consider the case $k = 0$. Note that the case $k = 0$ is Theorem 2.7. Now, we assume

$$\overline{h}_3(4n\cdot 3^k + 2\cdot 3^k - 1) \equiv 0 \pmod{2^{k+1}},$$
$$\overline{h}_3(4n\cdot 3^k + 3\cdot 3^k - 1) \equiv 0 \pmod{2^{k+1}},$$
$$\overline{h}_3(4n\cdot 3^k + 4\cdot 3^k - 1) \equiv 0 \pmod{2^{k+1}}.$$

Then by Recurrence (4) we have

$$\begin{aligned}
\overline{h}_3(4n\cdot 3^{k+1} + 2\cdot 3^{k+1} - 1) &= \overline{h}_3(3(4n\cdot 3^k + 2\cdot 3^k - 1) + 2)\\
&= 2\overline{h}_3(4n\cdot 3^k + 2\cdot 3^k - 1) \equiv 0 \pmod{2^{k+2}},\\
\overline{h}_3(4n\cdot 3^{k+1} + 3\cdot 3^{k+1} - 1) &= \overline{h}_3(3(4n\cdot 3^k + 3\cdot 3^k - 1) + 2)\\
&= 2\overline{h}_3(4n\cdot 3^k + 3\cdot 3^k - 1) \equiv 0 \pmod{2^{k+2}},\\
\overline{h}_3(4n\cdot 3^{k+1} + 4\cdot 3^{k+1} - 1) &= \overline{h}_3(3(4n\cdot 3^k + 4\cdot 3^k - 1) + 2)\\
&= 2\overline{h}_3(4n\cdot 3^k + 4\cdot 3^k - 1) \equiv 0 \pmod{2^{k+2}},
\end{aligned}$$

thereby completing the proof. □

Furthermore, we obtain

Theorem 9 *For all $n \geq 0$, we have $\overline{h}_3(3^n) = 2^{n+1}$, and $\overline{h}_3(3^n - 1) = 2^n$.*

Proof. We prove this result via induction on n using Recurrences (2) and (4). First, the theorem holds for the case $n = 0$ since $\overline{h}_3(1) = 2$, $\overline{h}_3(0) = 1$. Now we assume the theorem is true for the case $n = k$. Then we have

$$\overline{h}_3(3^{k+1}) = \overline{h}_3(3^k) + 2\overline{h}_3(3^k - 1) = 2^{k+1} + 2\cdot 2^k = 2^{k+2},$$
$$\overline{h}_3(3^{k+1} - 1) = 2\overline{h}_3(3^k - 1) = 2^{k+1}.$$

So the theorem is true for the case $n = k + 1$ and the proof is complete. □

By Theorem 9 and Recurrence (4) we can easily prove

Corollary 10 *For all* $n \geq 0$*, we have* $\overline{h}_3(3^{n+1}+2) = 2^{n+2}$.

3. General Conclusion

Lemma 11*For all* $n \geq 0, m \geq 3$ *and* $j \geq 0$*, we have*

$$\overline{h}_m(m^j n) = \overline{h}_m(n) + (2^{j+1}-2)\overline{h}_m(n-1).$$

Proof. We prove this result by induction on j. Note that the case $j = 0$ is clear and the case $j = 1$ is the recurrence (2). Now, we assume the result is true for some positive integer j. This means we assume that

$$\overline{h}_m(m^j n) = \overline{h}_m(n) + (2^{j+1}-2)\overline{h}_m(n-1),$$

or that

$$\sum_{n\geq 0} \overline{h}_m(m^j n)q^n = (1 + (2^{j+1}-2)q)\overline{H}_m(q).$$

Then, we have

$$\begin{aligned} \overline{h}_m(m^{j+1}n) &= [q^{mn}](1 + (2^{j+1}-2)q)\overline{H}_m(q) \\ &= [q^{mn}](1 + (2^{j+1}-2)q)(1+q)(1+q+\cdots+q^m)\overline{H}_m(q^m) \\ &= [q^{mn}](1 + 2q^m + (2^{j+1}-2)q \cdot 2q^{m-1})\overline{H}_m(q^m) \\ &= [q^n](1 + (2^{j+2}-2)q)\overline{H}_m(q) \\ &= \overline{h}_m(n) + (2^{j+2}-2)\overline{h}_m(n-1). \end{aligned}$$

□

From Lemma 11 we can easily prove the following result, which generalizes Theorem 9.

Corollary 12 *For all* $n \geq 0$ *and* $m \geq 3$*, we have*

$$\overline{h}_m(m^n) = 2^{n+1}, \quad \overline{h}_m(m^n - 1) = 2^n.$$

We now prove a family of congruences using similar elementary techniques.

Lemma 13 *Let* $m \geq 4, j \geq 0$*, and* $3 \leq k \leq m-1$*. Then for all* $n \geq 0$*, we have*

$$\overline{h}_m(m^{j+1}n + m^j k) = (2^{j+2}-2)\overline{h}_m(n).$$

Proof. Using Lemma 11, we have

$$\begin{aligned}
\overline{h}_m(m^{j+1}n + m^j k) &= [q^{mn+k}](1 + (2^{j+1} - 2)q)\overline{H}_m(q) \\
&= [q^{mn+k}](1 + (2^{j+1} - 2)q) \\
&\quad \times(1 + 2q + 2q^2 + \cdots + 2q^m + q^{m+1})\overline{H}_m(q^m) \\
&= [q^{mn+k}](2q^k + (2^{j+1} - 2)q \cdot 2q^{k-1})\overline{H}_m(q^m) \\
&\qquad (\text{since } 3 \le k, \text{ so that } m + 2 < m + k) \\
&= [q^{mn}](2^{j+2} - 2)\overline{H}_m(q^m) \\
&= (2^{j+2} - 2)\overline{h}_m(n).
\end{aligned}$$

□

Remark 14 Lemma 13 implies that, for $m \ge 4$, $j \ge 0$ and $3 \le k \le m - 1$, we have

$$\overline{h}_m(m^{j+1}n + m^j k) \equiv 0 \pmod{2^{j+2} - 2}.$$

Theorem 15 *For all $n \ge 0$, $m \ge 4$, $j \ge 0$, $t \ge 1$ and k satisfying $3 \le k \le m - 1$, we have*

$$\overline{h}_m(m^{j+t}n + m^{j+t-1}k + \cdots + m^j k) = 2^t(2^{j+1} - 1)\overline{h}_m(n).$$

Proof. We prove this result by induction on t. The case $t = 1$ is proved in Lemma 13. Now we assume

$$\overline{h}_m(m^{j+t-1}n + m^{j+t-2}k + \cdots + m^j k) = 2^{t-1}(2^{j+1} - 1)\overline{h}_m(n),$$

or that

$$\overline{h}_m(m^{j+t-1}n + m^{j+t-2}k + \cdots + m^j k) = [q^n]2^{t-1}(2^{j+1} - 1)\overline{H}_m(q).$$

Then we have

$$\begin{aligned}
&\overline{h}_m(m^{j+t}n + m^{j+t-1}k + \cdots + m^j k) \\
&\quad = [q^{mn+k}]2^{t-1}(2^{j+1} - 1)\overline{H}_m(q) \\
&\quad = [q^{mn+k}]2^{t-1}(2^{j+1} - 1)(1 + q)(1 + q + \cdots + q^m)\overline{H}_m(q^m) \\
&\quad = [q^{mn+k}]2^{t-1}(2^{j+1} - 1) \cdot 2q^k\overline{H}_m(q^m) \\
&\quad = [q^n]2^t(2^{j+1} - 1)\overline{H}_m(q) \\
&\quad = 2^t(2^{j+1} - 1)\overline{h}_m(n).
\end{aligned}$$

□

Remark 16 Theorem 15 implies that, for all $n \geq 0$, $m \geq 4$, $j \geq 1$, $3 \leq k \leq m-1$, and $t \geq 1$,

$$\overline{h}_m(m^{j+t}n + m^{j+t-1}k + \cdots + m^j k) \equiv 0(\text{mod } 2^t(2^{j+1}-1)).$$

Corollary 17 *For all $t \geq 0$, $m \geq 4$, we have*

$$\overline{h}_m(m^{t+1} + m^t - 1) = 2^{t+2}. \tag{5}$$

Proof. A proof can be obtained by letting $n = m$, $j = 0$, $k = m-1$ in Theorem 15.

Remark 18 We can easily prove (5) is true for the case $m = 3$.

Acknowledgements The authors would like to thank the anonymous referee for his productive and useful comments and suggestions. Theorem 8 was suggested by the referee.

References

[1] G. E. Andrews, Congruence properties of the m-ary partition function, *J. Number Theory* **3** (1971) 104-110.

[2] R. Churchhouse, Congruence properties of the binary partition function, *Proc. Cambridge Philos. Soc.* **66** (1969) 371-376.

[3] S. Corteel, J. Lovejoy, Overpartitions, *Trans. Amer. Math. Soc.* **356** (2004) 1623-1635.

[4] K. M. Courtright, J. A. Sellers, Arithmetic properties for hyper m-ary partition functions, *Integers* **4** (2004).

[5] G. Dirdal, On restricted m-ary partitions,*Math. Scand.* **37** (1975) 51-60.

[6] G. Dirdal, Congruences for m-ary partitions, *Math. Scand.* **37** (1975) 76-82.

[7] L. L. Dolph, A. Reynolds, J. A. Sellers, Congruences for a restricted m-ary partition function, *Discrete Math* **219** (2000) 265 -269.

[8] H. Gupta, On m-ary partitions, *Proc. Cambridge Philos. Soc.* **71** (1972) 343-345.

[9] Q. L. Lu, On a restricted m-ary partition function, *Discrete Math* **275** (2004) 347-353.

[10] ∅. Rødseth, Some arithmetical properties of m-ary partitions, *Proc. Cambridge Philos. Soc.* **68** (1970), 447-453.

[11] Ø. Rødseth, J. A. Sellers, On m-ary partition function congruences : a fresh look at a past problem, *J. Number Theory* **87** (2001) 270-281.

[12] Ø. Rødseth, J. A. Sellers, On m-ary overpartitions, *Annals of Combinatorics* **9** (2005) 345-353.