#G02 INTEGERS 9 (2009), 411-422

ON PEBBLING GRAPHS BY THEIR BLOCKS

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Received: 11/19/08, Revised: 4/8/09, Accepted: 4/18/09

Abstract

Graph pebbling is a game played on a connected graph G. A player purchases pebbles at a dollar a piece and hands them to an adversary who distributes them among the vertices of G (called a configuration) and chooses a target vertex r. The player may make a pebbling move by taking two pebbles off of one vertex and moving one of them to a neighboring vertex. The player wins the game if he can move k pebbles to r. The value of the game (G, k), called the k-pebbling number of G and denoted $\pi_k(G)$, is the minimum cost to the player to guarantee a win. That is, it is the smallest positive integer m of pebbles so that, from every configuration of size m, one can move k pebbles to any target. In this paper, we use the block structure of graphs to investigate pebbling numbers, and we present the exact pebbling number of the graphs whose blocks are complete. We also provide an upper bound for the k-pebbling number of diameter-two graphs, which can be the basis for further investigation into the pebbling numbers of graphs with blocks that have diameter at most two.

1. Introduction

Graph pebbling is a game played on a connected graph $G = (V, E)$.[1] A player purchases pebbles at a dollar a piece, and hands them to an adversary who distributes them among the vertices of G (called a *configuration*) and chooses a target, or *root* vertex r. The player may make a *pebbling move* by taking two pebbles off of one vertex and moving one of them to a neighboring vertex. The player wins the game if he can move k pebbles to r, in which case we say that r is *k-pebbled*. Another common terminology calls the configuration *k-fold r-solvable*. The *value* of the game (G, k), called the *k-pebbling number* of G and denoted $\pi_k(G)$, is the minimum cost to the player to guarantee a win. That is, it is the smallest positive integer m of pebbles so that, from every configuration of size m, one can move k pebbles to any root. If k is not specified, it is assumed to be one.

For example, by the pigeonhole principle we have $\pi(K_n) = n$, where K_n is the complete graph on n vertices. From there, induction shows that $\pi_k(K_n) = n + 2(k-1)$. Induction also proves that $\pi_k(P_n) = k2^{n-1}$, where P_n is the path on n vertices. These two graphs illustrate the tightness of the two main lower bounds

[1] We assume the notation and terminology of [11] throughout.

$\pi(G) \geq \max\{n(G), 2^{\mathrm{diam}(G)}\}$, where $\mathrm{diam}(G)$ is the *diameter* of G, the number of edges in a maximum induced path. Another fundamental result uses the path fact and induction to calculate the k-pebbling number of trees (see [2]). The survey [7] contains a wealth of information regarding pebbling results and variations.

Complete graphs and paths are examples of *greedy* graphs. That is, the most efficient pebbling moves are directed towards the root. More formally, a pebbling move from u to v is greedy if $\mathrm{dist}(v, r) < \mathrm{dist}(u, r)$, where $\mathrm{dist}(x, y)$ denotes the distance between x and y. A greedy solution uses only greedy moves. A graph G is greedy if every configuration of size $\pi(G)$ can be greedily solved. If a graph is greedy, then we can assume every pebbling move is directed towards the root. The greedy property of trees follows from the No-Cycle Lemma of [9] (see also [4, 8]), which states that the digraph whose arcs represent the pebbling moves of a minimal solution contains no directed cycles. A *cut vertex* of a graph is a vertex that, if removed, disconnects the graph. The *connectivity* κ of a graph is the minimum number of vertices whose deletion disconnects the graph or reduces it to only one vertex. Two important results relate diameter and connectivity to pebbling numbers. Pachter, Snevily, and Voxman proved the first.

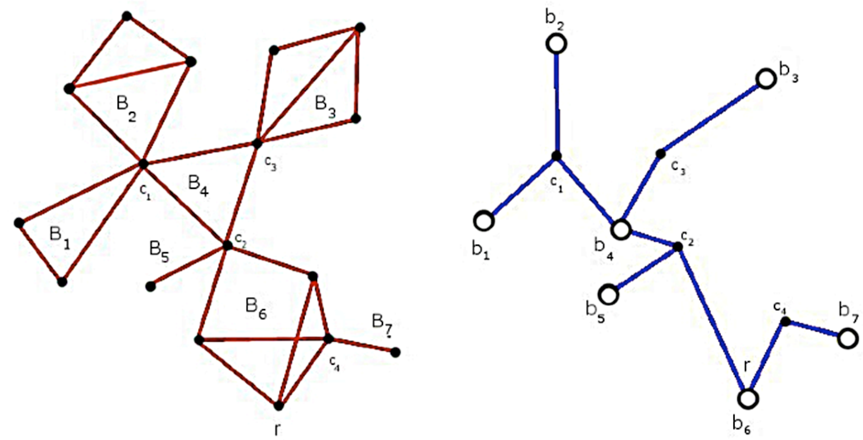
Result 1 ([10]). *If G is a connected graph on n vertices with $\mathrm{diam}(G) \leq 2$ then $\pi(G) \leq n + 1$.*

Clarke, Hochberg, and Hurlbert [3] characterized which diameter-two graphs have pebbling number n and which have pebbling number $n + 1$. We will use the graphs that describe that characterization in Section 3. Motivated by the characterization, Czygrinow, Hurlbert, Kierstead, and Trotter proved the second.

Result 2 ([5]). *If G is a connected graph on n vertices with $\mathrm{diam}(G) \leq d$ and $\kappa(G) \geq 2^{2d+3}$ then $\pi(G) = n$.*

This result states that high connectivity compensates for large diameter in keeping the pebbling number to a minimum. In this paper we exploit graph structures further to investigate pebbling numbers. A *block* of a graph G is a maximal subgraph of G with no cut vertex. Let $\mathcal{B}$ be the set of all blocks of G and $\mathcal{C}$ be the set of all cut vertices of G. Then the *block-cutpoint* graph of G, denoted $B(G)$, has vertices $\mathcal{B} \cup \mathcal{C}$, with edges (B, C) whenever $C \in V(B)$. Note that $B(G)$ is always a tree (see [11]). Figure 1 shows an example.

Here we instigate a line of research into using the k-pebbling numbers of $B(G)$ and of the blocks of G to give upper bounds on $\pi_k(G)$. To begin, we generalize Chung's tree result to weighted trees in Section 2. We then present the exact k-pebbling number of G when every block of G is complete in Section 3. Also in Section 3, we prove the following theorem, and show that there is a diameter-two graph G on $n \geq 6$ vertices with $\pi_k(G) = n + 4k - 3$ for all k (Theorem 11). Thus Theorem 3 is not known to be tight.



G $\qquad\qquad$ $B(G)$

Figure 1: A graph and its block-cutpoint graph

Theorem 3 *If G is a graph on n vertices with $\operatorname{diam}(G) \leq 2$ then $\pi_k(G) \leq n+7k-6$.*

Section 4 provides some further conjectures, questions, and possibilities for future research.

2. Trees and General Pebbling

A *tree* is a connected, acyclic graph, and a *forest* is a union of pairwise vertex-disjoint trees. A *leaf* of a tree is a vertex of degree one. An *r-path partition* of a particular tree T is a partition of the edges of T into paths, constructed by carrying out the following algorithm. Construct the sequence of pairs (T_i, F_i), where each T_i is a tree and each F_i is a forest, with $E(T_i) \cup E(F_i) = E(T)$, and $E(T_i) \cap E(F_i) = \emptyset$. Begin with $T_0 = r$, $F_0 = T$ and end with $T_t = T$, and $F_t = \emptyset$. At each stage, for some path P_i we have $P_i = T_i - T_{i-1} = F_{i-1} - F_i$, with the property that for each i, the intersection $V(P_i) \cap V(T_{i-1})$ is a leaf of P_i. The path partition is *r-maximal* if each P_i is the longest such path in F_{i-1}. An r-maximal path partition is *maximal* if r is one of the leaves of the longest path in T. An r-path partition of a tree is depicted in Figure 2, and a maximal path partition of a tree is depicted in Figure 3.

Define x_i to be the leaf of P_i in T_{i-1} and y_i to be the leaf of P_i not in T_{i-1}, and let $a_i = |E(P_i)|$.

Lemma 4 *The configuration C on T defined by each $C(y_i) = 2^{a_i} - 1$ and $C(v) = 0$ for all other v is r-unsolvable.*

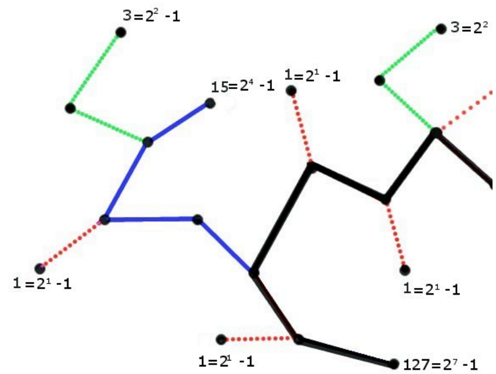


Figure 2: A non-maximal r-path partition of a tree, with its corresponding unsolvable configuration

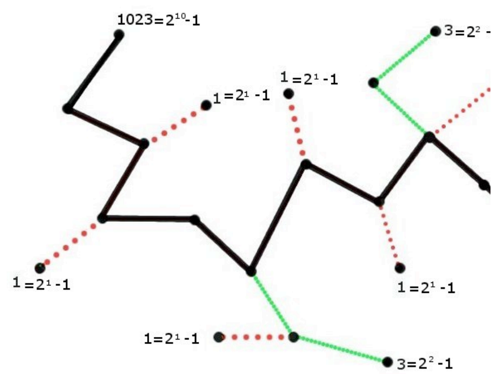


Figure 3: An r-maximal path partition of a tree, with its corresponding unsolvable configuration

Proof. We use induction. Let C_i be the restriction of C to T_i. The case in which $i = 0$ is trivial since the root has no pebbles. Now, assume that C_k is r-unsolvable on T_k. We know that the configuration on P_{k+1} is x_{k+1}-unsolvable because the pebbling number of a path of length l is 2^l. Thus, no pebbles can be moved to from P_{k+1} to T_k since $V(T_k) \cap V(T_{k+1}) = x_{k+1}$. Since we already know T_k is unsolvable, T_{k+1} is unsolvable also. Thus, by induction, the configuration C on T is r-unsolvable. $\square$

Chung's result generalizes this idea for k-pebbling.

Result 5 *[2]. If T is a tree and $a_1, a_2, \ldots, a_t$ is the sequence of the path size (i.e. the number of vertices in the path) in a maximum path partition of T, then $\pi_k(T) = k2^{a_1} + \sum_{i=2}^{t} 2^{a_i} - t + 1$.*

Chung's proof of this result uses induction performed on the vertices of T by fixing and then removing the root, thus dividing T into subtrees in order to use induction. We give a different proof of the more general Theorem 6, relying on the fact that trees are greedy.

First we consider a more general form of pebbling. For each edge e of a graph G we can assign a weight w_e. The weight is intended to signify that it takes w_e pebbles at one end of e to place 1 pebble at its other end. Hence the pebbling considered to this point has $w_e = 2$ for all e. We define the weighted pebbling number $\pi_k^w(G, r)$ to be the minimum number m so that every configuration of size m can k-pebble r by using w-weighted pebbling moves on G.

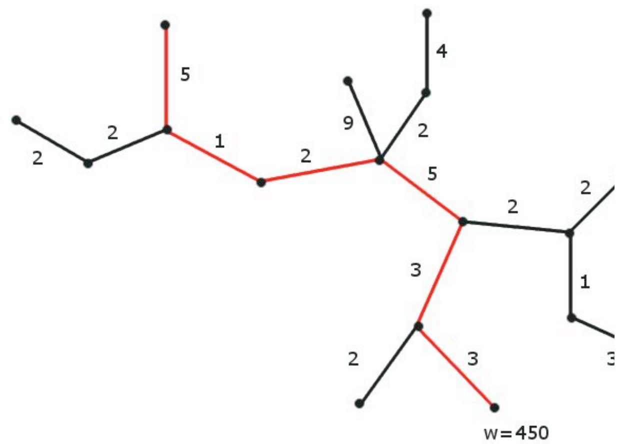


Figure 4: An edge-weighted tree

Given a weight function $w : E(G) \to \mathbb{N}$, we extrapolate to a weight function on the set of all paths of G, where $w(P)$ is the product of edge weights over all edges of the path P. Now when constructing maximal path partitions, we replace the condition "longest path" by "heaviest path" (greatest weight). This is equivalent for constant weight 2 pebbling. Nothing in the proof of Chung's theorem changes for weighted trees, but we introduce a new proof of the pebbling number of a weighted tree.

Let $P_1, \ldots, P_t$ be an r-maximal path partition of T, with $w(P_1) \geq \cdots \geq w(P_t)$. Let $f_k^w(T, r) = kw(P_1) + \sum_{i=2}^{t} w(P_i) - t + 1$. For vertices x and y on a path P, denote by $P[x, y]$ the subpath of P from x to y.

Theorem 6 *Every weighted tree T satisfies $\pi_k^w(T, r) \leq f_k^w(T, r)$.*

Proof. The theorem is trivially true when $t = 1$ since T is a path.

For $t \geq 1$, define $T' = T - P_t$. Then $f_k(T, r) = f_k(T', r) + w(P_t) - 1$. Let P_j be a path containing the non-leaf endpoint x_t of P_t, and let vertex y_j be the leaf of T on P_j. Define $W = w(P_j[x_t, y_j])$. Thus we know from the maximal r-path construction that $W \geq w(P_t)$.

Let C be an unsolvable configuration on T with $|C| = f_k(T, r)$. Without loss of generality, we can assume that all the pebbles are on the leaves of a tree because the maximum sized unsolvable configuration sits on the leaves only. Let $s \geq 0$ be the number of pebbles P_t contributes to the vertex x_t, so we have $sw(P_t) \leq |C(P_t)| < (s+1)w(P_t)$.

Now define the configuration C' on T' by $C'(y_j) = C(y_j) + sW$ and $C'(v) = C(v)$ otherwise. Then,

$$\begin{aligned} |C'| &= |C| - [(s+1)w(P_t) - 1] + sW \\ &\geq f_k(T, r) - w(P_t) + 1 \\ &= f_k(T', r) \ . \end{aligned}$$

Hence C' is k-fold solvable on T'. Now define C^* on T' by $C^*(x_t) = C(x_t) + s$ and $C^*(v) = C(v)$ otherwise. In particular, because of greediness, C^* is k-fold r-solvable on T' because moving at most $sw(P_t)$ pebbles from y_j to x_t converts C' to a solvable subconfiguration of C^*. Now, since $C(P_t) \geq sw(P_t)$, the base case says we can move s pebbles from P_t to x_t, and in doing so we arrive again at C^* on T'. Hence C is k-fold r-solvable. □

We will use Theorem 6 to upper bound the pebbling number of graphs composed of blocks. The technique utilizes the block-cutpoint graph.

For a graph G and its block-cutpoint graph $B(G)$, let b_i denote the vertex of $B(G)$ that corresponds to the block B_i in G. For each block B_i, let x_i denote the cut vertex of G in B_i that is closest to the root (it is possible that some $x_i = x_j$). Let e_i denote the edge of $B(G)$ between b_i and x_i, and define its weight by $w(e_i) = \pi(B_i, x_i)$. Let all other edges have weight 1. For a root r of G, let B denote the block containing it, represented by the vertex b in $B(G)$. Let $B'(G)$ be the graph obtained from $B(G)$ by adjoining to b by an edge of weight 1 a new vertex r'. Then we arrive at the following theorem.

Theorem 7 *Every graph G satisfies $\pi_k(G, r) \leq \pi_k^w(B'(G), r')$.*

Proof. For a set U of vertices, denote by $C(U)$ the sum $\sum_{v \in U} C(v)$. Let $x(B_i)$ denote all the cut vertices of G in the block B_i. Given a configuration C on G, define C' on $B'(G)$ by

- $C'(x_i) = C(x_i)$ for all cut vertices x_i, and
- $C'(b_i) = C(B_i) - C(x(B_i))$ for all blocks B_i.

Given an r'-solution S' of C' on $B'(G)$, which exists because of the identities $|C'| = |C| = \pi_k^w(B'(G), r')$, define the r-solution S of C on G by the following: replace every pebbling step along e_i in S' by some x_i-solution of some $\pi(B_i)$ of the pebbles in B_i. Then S is an r-solution. □

3. Larger Blocks

In this section we consider the cases in which all blocks are cliques or all have bounded diameters. The following proposition is well known.

Proposition 8 *If H is a connected spanning subgraph of G then $\pi_k(G, r) \leq \pi_k(H, r)$ for every root r.*

Proposition 8 holds because r-solutions in H are r-solutions in G. In particular, this holds when H is a breadth-first search spanning tree of G that is rooted at r and thus preserves distances to r in G. This allows us to prove the following.

Result 9 *Let G be a connected graph in which every block is a clique. Let T be a breadth-first search spanning tree of G. Then $\pi_k(G) = \pi_k(T)$.*

Proof. The fact that $\pi_k(G) \leq \pi_k(T)$ follows from Proposition 8. The fact that $\pi_k(G) \geq \pi_k(T)$ follows from showing that every r-solvable configuration C on G is r-solvable on T. Indeed, let S be an r-solution in G, and for a block B of G, denote by $x = x(B)$ the cut vertex of B that is closest to r. If the sequence is greedy, then all its edges are in T. If the sequence is not greedy, then S contains an edge from some vertex a to some vertex $b \neq x$. Replace this edge by the edge from a to x. The resulting sequence is an r-solution on T. Thus $\pi_k(G) = \pi_k(T)$. □

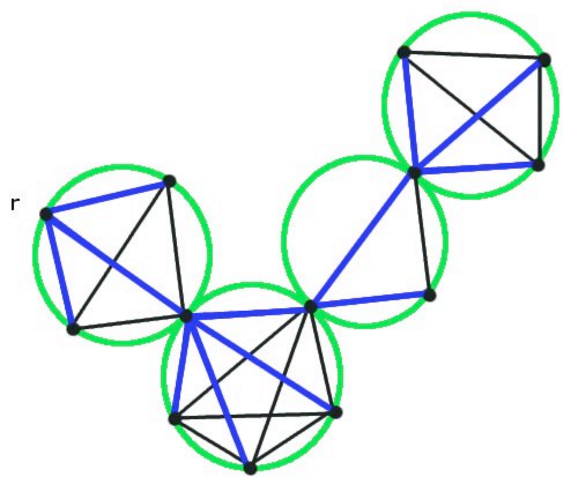


Figure 5: A clique block graph with its breadth-first search spanning tree

Corollary 10 *Let G be a connected graph in which every block is a clique. Let T be a breadth-first search spanning tree of G. Let $a_1, \ldots, a_t$ denote the path lengths* *in a maximal path partition of T rooted at r. Then* $\pi_k(G,r) = n + 2^{a_1}(k-1) + \sum_{i=1}^{t}(2^{a_i} - a_i - 1)$.

Note that the formula in Corollary 10 is of the form $n + c_1k + c_2$, which is also the form of the formula in Theorem 3. Also, the *fractional pebbling number*, defined as $\hat{\pi}(G) = \lim_{k\to\infty} \pi_k(G)/k$ is seen to be $\hat{\pi}(G) = 2^{\mathrm{diam}(G)}$ for such G. This is an instance of the Fractional Pebbling Conjecture of [7], recently proven in [6].

Now we provide the upper and lower bounds on diameter-two graphs. To show a lower bound, we will display an unsolvable configuration on an extremal graph $\mathcal{G}$. This is the graph that Clarke, et al. [3] used to characterize the diameter-two graphs with pebbling number $n+1$. The vertices of $\mathcal{G}$ are $\{a,b,c,p,q,r\} \cup_{z\in\{p,q,r,c\}} V(H_z)$, where H_p, H_q, H_r, and H_c are any graphs with the following properties:

- Every component of H_p, H_q, and H_r has some vertex adjacent to p, q, and r, respectively.
- Every vertex of H_p, H_q, and H_r is adjacent to a and c, b and c, a and b, respectively.
- Every vertex of H_c is adjacent to a, b, and c.

Furthermore, (a,r,b,q,c,p) forms a 6-cycle, (a,b,c) forms a triangle, as shown in Figure 6, and no other edges than previously mentioned are included. Note that the diameter of $\mathcal{G}$ is 2.

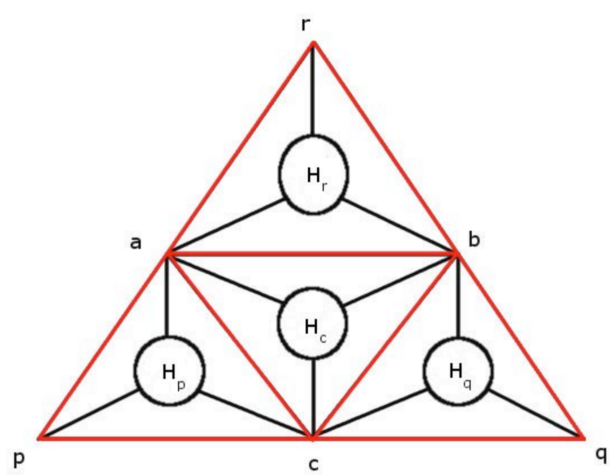


Figure 6: The extremal graph $\mathcal{G}$

Theorem 11 *For all $n \geq 6$, there is a graph G on n vertices with $\pi_k(G) \geq n+4k-3$ for all k.*

Proof. As suggested above, we show that $\mathcal{G}$ is such a graph. Distribute the following configuration of size $n + 4k - 4$ on the $\mathcal{G}$:

- Place $4k - 1$ pebbles on p.
- Place 3 pebbles on q.
- Place 1 pebble on every vertex in $\cup_{z\in\{p,q,r,c\}} H_z$ and 0 elsewhere.

The configuration is r-unsolvable since every solution costs at least 4 pebbles (because every pebble is at distance 2 from r), and so after $k - 1$ solutions at most n pebbles remain. In fact, the remaining configuration is a subconfiguration of the one defined above for $k = 1$, which was shown to be r-unsolvable in [10]. Hence $\pi_k(\mathcal{G}) > n + 4k - 4$. □

To prove Theorem 3 we consider the eight *cheap* configurations shown in Figure 7. We call them cheap because they lose a small number (at most 7) of pebbles in the process of moving one pebble to the root. In particular, their names indicate their *cost* (number of pebbles used). For example, in C7, C6, and C5, one moves an extra pebble onto where 3 sits to create C4A. Then one can reach C2 from C4B, C4A, and C3. Of course, C2 results in C1. There are more cheap solutions than these, but we do not need them in our argument.

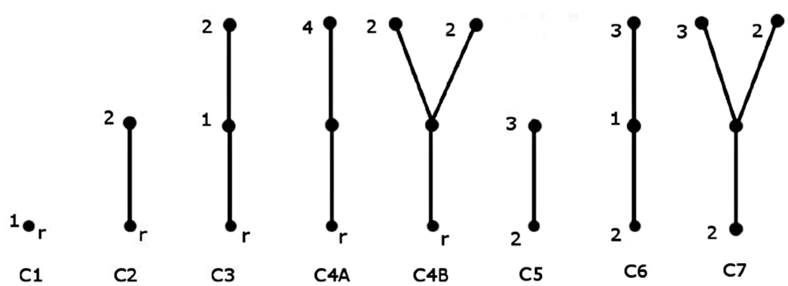


Figure 7: Cheap solutions of cost 7 or less

We show by contradiction that a cheap solution must exist, and thus a pebble can be moved to the root with the loss of at most 7 pebbles. The remaining $k - 1$ solutions will be found by induction.

Proof of Theorem 3. Assume that the configuration C of pebbles on G is of size $n+7k-6$ and has no cheap solutions of cost 7 or less. We will derive a contradiction to show that a cheap solution exists. Then after using a cheap solution we apply induction to get the remaining $k - 1$ solutions. The theorem is already true for $k = 1$ by Result 1. Define the following notation.

- N_i is the set of vertices with i pebbles.
- $N_{i,r}$ is the set of common neighbors of N_i and root r.
- $N_{i,j}$ is the set of common neighbors of pairs of vertices from N_i and N_j.
- $n_i = |N_i|$, $n_{i,j} = |N_{i,j}|$, $n_{i,r} = |N_{i,r}|$, and $n'_0 = |N'_0|$.
- $N'_0 = N_0 - N_{3,r} - N_{3,3} - N_{2,r}$.

Claim 12 *If C is a configuration on a diameter-two graph G with no cheap solutions, then*

S1. $N_{i,r} \subseteq N_0$ *for* $i \in \{2, 3\}$,

S2. $N_{3,3} \subseteq N_0$,

S3. $n_{i,r} \geq n_i$ *for* $i \in \{2, 3\}$,

S4. $|C| = 3n_3 + 2n_2 + n_1$,

S5. $n = n_3 + n_2 + n_1 + (n_{3,r} + n_{3,3} + n_{2,r} + n'_0)$, *and*

S6. $n_{3,3} \geq \binom{n_3}{2}$.

Proof of Claim 12. We refer to Figure 7. Statement S1 follows from the nonexistence of C3 because a pebble adjacent to the root and a vertex with at least two pebbles is a C3 configuration. Likewise, S2, S3, and S4 follow from the nonexistence of C6, C4B, and C4A respectively. Next, S5 simply partitions the vertices according to their number of pebbles, then uses the definition of N'_0. Finally, since C has no C5, no two vertices of N_3 are adjacent. However, because G has diameter two, every such x and y have a common neighbor. Now the nonexistence of C7 implies that such common neighbors are distinct, which implies S6. $\diamondsuit$

Next we use S4 and S5 to count $|C|$ in two ways:

$$3n_3 + 2n_2 + n_1 = n_3 + n_2 + n_1 + (n_{3,r} + n_{3,3} + n_{2,r} + n'_0) + 7k - 6.$$

Then S3 and S6 imply

$$\begin{aligned} 0 &= -2n_3 - n_2 + n_{3,r} + n_{3,3} + n_{2,r} + n'_0 + 7k - 6 \\ &\geq -n_3 + \binom{n_3}{2} + n'_0 + 7k - 6. \end{aligned}$$

Finally, by completing the square and using $n_0' \geq 1$ (since $r \in N_0'$) and $k \geq 1$, we have

$$\begin{aligned} 0 &< (n_3 - 3/2)^2 + (4 - 9/4) \\ &= 2\left[\binom{n_3}{2} - n_3 + 2\right] \\ &\leq 2\left[\binom{n_3}{2} - n_3 + n_0' + 7k - 6\right] \\ &\leq 0\ , \end{aligned}$$

which is a contradiction. Hence, C must contain a solution of cost at most 7, afterwhich at least $n + 7(k-1) - 6$ pebbles remain, from which we obtain $k - 1$ more solutions. □

4. Remarks

We believe that the upper bound of Theorem 3 can be tightened by reducing the coefficient of k. Doing this requires restricting cheap solutions to lesser cost, which necessitates considering more of them. For example, there are one cost-4, one cost-5, and four cost-6 solutions that were not used in our argument. Our lower bound has inspired the next conjecture.

Conjecture 13 *If G is a graph on n vertices with* $\mathrm{diam}(G) \leq 2$ *then* $\pi_k(G) \leq n + 4k - 3$.

Of course, the Fractional Pebbling Theorem implies that the coefficient of k is 4 in the limit; in fact, its proof is based on the pigeonhole principle — for large enough k, C4A exists. Also, Theorem 3 suggests the following problem.

Problem 14 Find upper bounds for the k-pebbling numbers of graphs of diameter d.

Along these lines, only the following result is known, proved by Bukh [1].

Theorem 15 *If* $\mathrm{diam}(G) = 3$, *then* $\pi(G) \leq (3/2)n + O(1)$.

In addition, the following question is still open.

Question 16 Is it possible to lower the connectivity requirement in Result 2?

The construction in [7] shows that $\kappa \geq 2^d/d$ is necessary.

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