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SOME NEW VAN DER WAERDEN NUMBERS AND SOME VAN DER WAERDEN-TYPE NUMBERS

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Abstract

The van der Waerden number $w(r; k_1, k_2, \cdots, k_r)$ is the least m such that given any partition $\{1, 2, \ldots, m\} = P_1 \cup P_2 \cup \cdots \cup P_r$, there is an index $j \in \{1, 2, \ldots, r\}$ such that P_j contains an arithmetic progression of length k_j. We have computed exact values of some (30) previously unkown van der Waerden numbers and also computed lower bounds of others. Let $w_d(r; k_1, k_2, \ldots, k_r)$ be the least m such that given any partition $\{1, 2, \ldots, m\} = P_1 \cup P_2 \cup \cdots \cup P_r$, there is an index $j \in \{1, 2, \ldots, r-1\}$ such that P_j contains an arithmetic progression of length k_j, or P_r contains an arithmetic progression of length k_r with common difference at most d. A table of observed values of $w_d(r; k_1, k_2, \ldots, k_r)$ for $d = 1, 2, \ldots, \left\lfloor \frac{m-1}{k_r-1} \right\rfloor$ is given.

1. Introduction

Van der Waerden's theorem [14] can be formulated (as in Chvátal [5]) as follows: Given any positive integer r and positive integers $k_1, k_2, \cdots, k_r$, there is an integer m such that given any partition

$$\{1, 2, \cdots, m\} = P_1 \cup P_2 \cup \cdots \cup P_r \tag{1}$$

there is always a class P_j containing an arithmetic progression of length k_j. Let us denote the least m with this property by $w(r; k_1, k_2, \cdots, k_r)$.

So far we know only six van der Waerden numbers such that $k_1 = k_2 = \cdots = k_r$. They are $w(2; 3, 3) = 9$ [5], $w(2; 4, 4) = 35$ [5], $w(3; 3, 3, 3) = 27$ [5], $w(2; 5, 5) = 178$ [13], $w(4; 3, 3, 3, 3) = 76$ [2] and $w(2; 6, 6) = 1132$ [11]. Landman et al. [12] gave an updated list of van der Waerden numbers. In Table 1, we present some previously unknown van der Waerden numbers. We have computed them using the cluster machines of ConCoCO Research Laboratory at Concordia University.

2. SAT and Van der Waerden Numbers

2.1. The Satisfiability Problem

A *truth assignment* is a mapping f that assigns 0 (interpreted as FALSE) or 1 (interpreted as TRUE) to each variable in its domain; we shall enumerate all the variables in the domain as $x_1, \ldots, x_n$. The complement $\bar{x}_i$ of each variable x_i is defined by

$$f(\bar{x}_i) = 1 - f(x_i) \text{ for all truth assignments } f.$$

Both x_i and $\bar{x}_i$ are called *literals*; a *clause* is a set of (distinct) literals, and a *formula* is a family of (not necessarily distinct) clauses.

A truth assignment *satisfies* a clause if it maps at least one of its literals to 1; the assignment *satisfies* a formula if and only if it satisfies each of its clauses. A formula is called *satisfiable* if it is satisfied by at least one truth assignment; otherwise it is called *unsatisfiable*. The problem of recognizing satisfiable formulas is known as the *satisfiability problem*, or SAT for short. The above definition is taken from Chvátal and Reed [6].

2.2. The DPLL Algorithm

Given a formula F and a literal u in F, we let $F|u$ denote the *residual formula* arising from F when $f(u)$ is set to 1: explicitly, this formula is obtained from F by (i) removing all the clauses that contain u, (ii) deleting $\bar{u}$ from all the clauses that contain $\bar{u}$, (iii) removing both u and $\bar{u}$ from the list of literals.

Algorithm 1 RECURSIVE DPLL ALGORITHM [7, 8]

function DPLL(F)
  while F includes a clause C such that $|C| \leq 1$ **do**
    if $C = \emptyset$ **then return** UNSATISFIABLE
    else if $C = \{v\}$ **then** $F = F|v$
  end while
  if $F = \emptyset$ **then return** SATISFIABLE
  Choose a literal x_i using a branching rule
  if DPLL($F|x_i$) = SATISFIABLE **then return** SATISFIABLE
  if DPLL($F|\overline{x}_i$) = SATISFIABLE **then return** SATISFIABLE
  return UNSATISFIABLE
end function

2.3. SAT Encoding of Van der Waerden Numbers

We want to construct an SAT formula (an instance of the satisfiability problem) which is satisfiable if and only if $w(r; k_1, k_2, \cdots, k_r) > n$. We consider the following two cases:

For $r = 2$, we have

(i) variables: x_i for $1 \leqslant i \leqslant n$.

(ii) clauses:

(a) $\{\bar{x}_a, \bar{x}_{a+d}, \cdots, \bar{x}_{a+d(k_1-1)}\}$ with $1 \leqslant a \leqslant n - k_1 + 1$, $1 \leqslant d \leqslant \lfloor (n - a)/(k_1 - 1) \rfloor$.

(b) $\{x_a, x_{a+d}, \cdots, x_{a+d(k_2-1)}\}$ with $1 \leqslant a \leqslant n - k_2 + 1$, $1 \leqslant d \leqslant \lfloor (n - a)/(k_2 - 1) \rfloor$.

Here, $x_i = $ TRUE implies $i \in P_1$, $x_i = $ FALSE implies $i \in P_2$ and if x_i is not assigned but the formula is satisfied, then i can be arbitrarily placed in any of the partitions. Clauses (a) prohibit the existence of an arithmetic progression of length k_1 in P_1 and clauses (b) prohibit the existence of an arithmetic progression of length k_2 in P_2.

If $r > 2$, we have the following

(i) variables: we assign one boolean variable for each integer and block-of-partition pair. Thus the variable $x_{i,j}$ with $1 \leqslant i \leqslant n, 1 \leqslant j \leqslant r$, is true if and only if the integer i belongs to a block P_j of a partition. This generates nr variables.

(ii) clauses:

(a) INTEGER i IS IN AT LEAST ONE BLOCK: We want at least one block-of-partition to contain integer i. Thus out of all partitions available, at least one should be true. This generates one clause for each integer, a total of n clauses. The clause for the integer i looks like

$$\{x_{i,1}, x_{i,2}, \cdots, x_{i,r}\}.$$

(b) INTEGER i IS CONTAINED IN AT MOST ONE BLOCK: We want an integer not to be contained in more than one block-of-partition. To do so, we add the following clauses:

$$\{\bar{x}_{i,s}, \bar{x}_{i,t}\} \text{ for } 1 \leqslant i \leqslant n, 1 \leqslant s < t \leqslant r.$$

(c) NO ARITHMETIC PROGRESSION OF LENGTH k_j IN BLOCK P_j: This is the most important constraint. For $1 \leqslant j \leqslant r$, $1 \leqslant a \leqslant n - k_j + 1$ and $1 \leqslant d \leqslant \lfloor (n-a)/(k_j - 1) \rfloor$, we add the following clauses:

$$\{\bar{x}_{a,j}, \bar{x}_{a+d,j}, \cdots, \bar{x}_{a+d(k_j-1),j}\}.$$

The above double-variable subscripts can easily be converted equivalently to single variable subscripts for convenience.

We have developed an efficient implementation of the DPLL algorithm which was used to solve several formulas corresponding to different values of n so that $w(r; k_1, k_2, \cdots, k_r) > n$ until we reached the van der Waerden number.

3. Some New Van der Waerden Numbers

By a good partition, we shall mean a partition of the form (1) such that no P_j contains an arithmetic progression of k_j terms. Examples of good partitions corresponding to the numbers $w(r; k_1, k_2, \cdots, k_r) - 1$ are given. We denote partitions as strings; for example 11221122 means $P_1 = \{1, 2, 5, 6\}$ and $P_2 = \{3, 4, 7, 8\}$.

Table 1: Some previously unknown exact van der Waerden numbers and some lower bounds

$w(r; k_1, k_2, \cdots, k_k)$			Example of a good partition
$w(2; 4, 9)$	$>$	254	
$w(3; 2, 3, 9)$	$=$	90	33333332 33332333 33332322 32333332 33322333 33332333 33323333 33133223 23333323 33223333 33323233 3
$w(3; 2, 3, 10)$	$=$	108	33333233 33333332 33233323 33333323 33322333 32323333 32333233 13333333 33233333 22323333 33233322 33333323 33333333 233
$w(3; 2, 3, 11)$	$=$	129	33333322 33323333 33322333 22333333 33332323 33333233 33333332 33233323 23333333 33323233 32331333 33333223 33333233 33233333 33323333 23323333
$w(3; 2, 3, 12)$	$=$	150	33333333 33323233 23333333 33323223 33233333 33333323 33333223 23333333 33332333 32323333 33333332 23333333 33233333 32333332 12233333 33333322 33332333 23333333 33332
$w(3; 2, 3, 13)$	$=$	171	33333333 33332333 33323333 33332232 23333333 33233333 23333333 32332323 33333323 33333332 32333333 33333321 33333332 23233333 33323233 33223333 33332332 32333333 32333333 33233333 33233333 33
$w(3; 2, 3, 14)$	$>$	195	
$w(3; 2, 4, 8)$	$>$	150	
$w(3; 2, 5, 5)$	$=$	180	33232332 32223233 32222322 22322323 33232223 33323333 23323222 32333222 23222232 23233323 22233332 33332333 32223233 32322322 22322223 33232223 23323333 23333212 32333232 23222232 22233323 22232332 223
$w(3; 3, 3, 6)$	$>$	105	
$w(3; 3, 3, 7)$	$>$	142	
$w(3; 3, 3, 8)$	$>$	160	
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Table 1: Some previously unknown exact van der Waerden numbers and some lower bounds

$w(r; k_1, k_2, \cdots, k_k)$			Example of a good partition
$w(3; 3, 4, 5)$	$>$	137	
$w(3; 3, 5, 5)$	$>$	228	
$w(4; 2, 2, 3, 8)$	$=$	83	44444434 44433434 44434443 43444334 44444144 24344344 44434344 44344434 44443344 44434444 44
$w(4; 2, 2, 3, 9)$	$=$	99	43443444 44444343 44343444 44443433 43444344 44442444 44444144 44443444 34334344 44444343 44343444 44444344 34
$w(4; 2, 2, 3, 10)$	$=$	119	34434444 43443444 44444434 44443344 44444344 44334424 44344444 44433444 44444341 43444444 34344344 44444334 44434444 44434444 344444
$w(4; 2, 2, 4, 5)$	$=$	75	43434444 34434441 33343343 33444434 43444433 34334333 44443443 44443334 33443424 44
$w(4; 2, 2, 4, 6)$	$=$	93	33343344 44433434 34444343 33444434 44434444 43334444 34333423 43314444 43433344 44343344 44343444 3343
$w(4; 2, 2, 4, 7)$	$>$	129	
$w(4; 2, 2, 5, 5)$	$>$	181	
$w(4; 2, 3, 3, 5)$	$=$	86	43433444 34444224 33232444 43442424 32244232 43434444 14444343 42324422 34242443 44442324 34224
$w(4; 2, 3, 4, 4)$	$>$	93	
$w(4; 3, 3, 3, 4)$	$>$	100	
$w(4; 3, 3, 4, 4)$	$>$	131	
$w(5; 2, 2, 2, 3, 4)$	$=$	29	54554555 44143555 45544255 5445
$w(5; 2, 2, 2, 3, 5)$	$=$	44	55544545 55454425 55345555 45555144 54555454 455
$w(5; 2, 2, 2, 3, 6)$	$=$	56	45555545 55545455 54455555 45551423 44555554 55544555 5545555
$w(5; 2, 2, 2, 3, 7)$	$=$	72	55555544 54555455 55552445 44555545 55515555 45555445 44355555 54555454 4555555
$w(5; 2, 2, 2, 3, 8)$	$=$	88	55455455 55544555 45455554 55555535 55555454 41455555 55454455 55545555 55255544 55455555 4555545
$w(5; 2, 2, 2, 3, 9)$	$>$	105	
$w(5; 2, 2, 2, 4, 4)$	$=$	54	54554544 45544454 55255454 44554445 45315545 44455444 54554
$w(5; 2, 2, 2, 4, 5)$	$=$	79	55554554 55554445 44544455 55455455 55444544 54442555 35555155 44454454 44555545 445555

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Table 1: Some previously unknown exact van der Waerden numbers and some lower bounds

$w(r; k_1, k_2, \cdots, k_k)$			Example of a good partition
$w(5; 2, 2, 2, 4, 6)$	$>$	100	
$w(5; 2, 2, 3, 3, 4)$	$=$	63	55443453 53543545 55332335 45553455 54543144 55535335 35445553 544343
$w(5; 2, 2, 3, 3, 5)$	$>$	88	
$w(6; 2, 2, 2, 2, 3, 3)$	$=$	21	66556655 43216655 6655
$w(6; 2, 2, 2, 2, 3, 4)$	$=$	33	65656655 46566366 56215565 66655666
$w(6; 2, 2, 2, 2, 3, 5)$	$=$	50	56665655 65666525 66636666 46665565 56166565 56665666 6
$w(6; 2, 2, 2, 2, 3, 6)$	$=$	60	66665666 66556665 66666552 35466656 66665566 65656666 56616656 665
$w(6; 2, 2, 2, 2, 4, 4)$	$=$	56	56656555 66555656 63665655 56655565 66166465 25665556 5666555
$w(6; 2, 2, 2, 2, 4, 5)$	$>$	83	
$w(6; 2, 2, 2, 3, 3, 3)$	$=$	42	55446465 56655646 44531246 46556655 64644565 4
$w(7; 2, 2, 2, 2, 2, 3, 3)$	$=$	24	67766776 16345727 6677667
$w(7; 2, 2, 2, 2, 2, 3, 4)$	$=$	36	77676677 76646277 57773616 67776676 777
$w(8; 2, 2, 2, 2, 2, 2, 3, 3)$	$=$	25	87877883 78126578 77488787
$w(9; 2, 2, 2, 2, 2, 2, 2, 3, 3)$	$=$	28	89899828 99597148 86889399 898

We also provide an example of a good partition corresponding to each of several van der Waerden numbers that were recently found by Kouril [10].

Table 2: Recently proved van der Waerden numbers

$w(r; k_1, k_2, \cdots, k_r)$			Example of a good partition
$w(2; 3, 14)$	=	186	22121222 22222222 22112222 21222222 21222222 22121222 22222122 22222122 22222221 12221222 22222222 22122112 22222222 21212222 21222222 22122222 12122222 22222122 21222222 2222A211 22221222 22222212 22212222 2 (where A is arbitrary).
$w(2; 3, 15)$	=	218	22222222 21222122 22222122 21122222 12222222 22221222 21222222 22211222 22222212 22222212 22212221 22222222 22221122 22222212 22222212 22222222 21222222 22212222 12122222 12112222 22222221 22222122 22222221 12222222 22221122 22222212 22222222 2
$w(2; 3, 16)$	=	238	2222A221 22222222 22222212 1222B222 22212211 22222222 22222221 22222112 12222222 22221221 21222222 22222221 22222222 22221222 12222222 22122122 22222122 22222222 22212212 22221121 22222222 22222122 12222222 12222222 222221C2 21222221 12222212 22222222 21222222 22222 (where ABC is arbitrary).
$w(2; 4, 8)$	=	146	112221A2 12222112 22222122 12222211 12212222 12111222 21221121 21222222 21122222 12211222 21222212 11222112 22222211 21222222 21122212 11222221 12212222 122B1121 2 (AB is arbitrary)
$w(2; 5, 6)$	=	206	21112111 22122221 11222211 11211122 21222212 11222221 12122221 22211121 11121221 12112212 11112111 22212222 12112222 21121222 21222111 21111212 21121122 12111121 11222122 22121122 22211212 22212221 11211112 22211122 22122AB1 21112 (AB is arbitrary)
$w(3; 2, 4, 7)$	=	119	33333322 23223332 33233233 33332233 32333232 33233333 22232233 22233323 23313333 33232323 32233232 33333322 32223333 23323332 333333

4. Van der Waerden Numbers Known So Far

Table 3 contains a complete listing of known van der Waerden numbers. A reference indicated by ConCoCO means the corresponding number was previously unknown.

Table 3: van der Waerden numbers known so far

$w(r; k_1, k_2, \cdots, k_r)$		Reference
$w(2; 3, 3)$	9	Chvátal [5]
$w(2; 3, 4)$	18	Chvátal [5]
$w(2; 3, 5)$	22	Chvátal [5]
$w(2; 3, 6)$	32	Chvátal [5]
$w(2; 3, 7)$	46	Chvátal [5]
$w(2; 3, 8)$	58	Beeler and O'Neil [2]
$w(2; 3, 9)$	77	Beeler and O'Neil [2]
$w(2; 3, 10)$	97	Beeler and O'Neil [2]
$w(2; 3, 11)$	114	Landman, Robertson and Culver [12]
$w(2; 3, 12)$	135	Landman, Robertson and Culver [12]
$w(2; 3, 13)$	160	Landman, Robertson and Culver [12]
$w(2; 3, 14)$	186	Kouril [10]
$w(2; 3, 15)$	218	Kouril [10]
$w(2; 3, 16)$	238	Kouril [10]
$w(2; 4, 4)$	35	Chvátal [5]
$w(2; 4, 5)$	55	Chvátal [5]
$w(2; 4, 6)$	73	Beeler and O'Neil [2]
$w(2; 4, 7)$	109	Beeler [1]
$w(2; 4, 8)$	146	Kouril [10]
$w(2; 5, 5)$	178	Stevens and Shantaram [13]
$w(2; 5, 6)$	206	Kouril [10]
$w(2; 6, 6)$	1132	Kouril and Paul [11]
$w(3; 2, 3, 3)$	14	Brown [3]
$w(3; 2, 3, 4)$	21	Brown [3]
$w(3; 2, 3, 5)$	32	Brown [3]
$w(3; 2, 3, 6)$	40	Brown [3]
$w(3; 2, 3, 7)$	55	Landman, Robertson and Culver [12]
$w(3; 2, 3, 8)$	72	Kouril [10]
$w(3; 2, 3, 9)$	90	ConCoCO
$w(3; 2, 3, 10)$	108	ConCoCO
$w(3; 2, 3, 11)$	129	ConCoCO
$w(3; 2, 3, 12)$	150	ConCoCO
$w(3; 2, 3, 13)$	171	ConCoCO
$w(3; 2, 4, 4)$	40	Brown [3]
$w(3; 2, 4, 5)$	71	Brown [3]
$w(3; 2, 4, 6)$	83	Landman, Robertson and Culver [12]
$w(3; 2, 4, 7)$	119	Kouril [10]
$w(3; 2, 5, 5)$	180	ConCoCO

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Table 3: van der Waerden numbers known so far

$w(r; k_1, k_2, \cdots, k_r)$		Reference
$w(3; 3, 3, 3)$	27	Chvátal [5]
$w(3; 3, 3, 4)$	51	Beeler and O'Neil [2]
$w(3; 3, 3, 5)$	80	Landman, Robertson and Culver [12]
$w(3; 3, 4, 4)$	89	Landman, Robertson and Culver [12]
$w(4; 2, 2, 3, 3)$	17	Brown [3]
$w(4; 2, 2, 3, 4)$	25	Brown [3]
$w(4; 2, 2, 3, 5)$	43	Brown [3]
$w(4; 2, 2, 3, 6)$	48	Landman, Robertson and Culver [12]
$w(4; 2, 2, 3, 7)$	65	Landman, Robertson and Culver [12]
$w(4; 2, 2, 3, 8)$	83	ConCoCO
$w(4; 2, 2, 3, 9)$	99	ConCoCO
$w(4; 2, 2, 3, 10)$	119	ConCoCO
$w(4; 2, 2, 4, 4)$	53	Brown [3]
$w(4; 2, 2, 4, 5)$	75	ConCoCO
$w(4; 2, 2, 4, 6)$	93	ConCoCO
$w(4; 2, 3, 3, 3)$	40	Brown [3]
$w(4; 2, 3, 3, 4)$	60	Landman, Robertson and Culver [12]
$w(4; 2, 3, 3, 5)$	86	ConCoCO
$w(4; 3, 3, 3, 3)$	76	Beeler and O'Neil [2]
$w(5; 2, 2, 2, 3, 3)$	20	Landman, Robertson and Culver [12]
$w(5; 2, 2, 2, 3, 4)$	29	ConCoCO
$w(5; 2, 2, 2, 3, 5)$	44	ConCoCO
$w(5; 2, 2, 2, 3, 6)$	56	ConCoCO
$w(5; 2, 2, 2, 3, 7)$	72	ConCoCO
$w(5; 2, 2, 2, 3, 8)$	88	ConCoCO
$w(5; 2, 2, 2, 4, 4)$	54	ConCoCO
$w(5; 2, 2, 2, 4, 5)$	79	ConCoCO
$w(5; 2, 2, 3, 3, 3)$	41	Landman, Robertson and Culver [12]
$w(5; 2, 2, 3, 3, 4)$	63	ConCoCO
$w(6; 2, 2, 2, 2, 3, 3)$	21	ConCoCO
$w(6; 2, 2, 2, 2, 3, 4)$	33	ConCoCO
$w(6; 2, 2, 2, 2, 3, 5)$	50	ConCoCO
$w(6; 2, 2, 2, 2, 3, 6)$	60	ConCoCO
$w(6; 2, 2, 2, 2, 4, 4)$	56	ConCoCO
$w(6; 2, 2, 2, 3, 3, 3)$	42	ConCoCO
$w(7; 2, 2, 2, 2, 2, 3, 3)$	24	ConCoCO
$w(7; 2, 2, 2, 2, 2, 3, 4)$	36	ConCoCO
$w(8; 2, 2, 2, 2, 2, 2, 3, 3)$	25	ConCoCO
$w(9; 2, 2, 2, 2, 2, 2, 2, 3, 3)$	28	ConCoCO

5. Variations of Van der Waerden Numbers

5.1. On $w_1(2; k_1, k_2)$

Let $w_1(2; k_1, k_2)$ be the least m such that given any partition $\{1, 2, \cdots, m\} = P_1 \cup P_2$, there is a k_1 term arithmetic progression in P_1 or k_2 consecutive integers in P_2. The known values [4] are $w_1(2; 3, 2) = 9$, $w_1(2; 3, 3) = 23$, $w_1(2; 3, 4) = 34$, $w_1(2; 3, 5) = 73$, $w_1(2; 3, 6) = 113$ and $w_1(2; 3, 7) = 193$. Lower bounds of $w_1(2; 3, k)$ can be found in [9]. We have computed the following numbers:

Table 4: Some new values for $w_1(2; k_1, k_2)$

$w_1(r; k_1, k_2)$			Example of a good partition
$w_1(2;\ 3,\ 8)$	=	238	22222221 22222122 22222122 22221222 21222222 21222222 12212222 21222222 21222222 12222222 12222122 22212222 22212222 22122222 22122222 12222222 12122222 22122222 12222222 12222221 22222221 22221222 22122222 22122222 21222222 21122222 22122222 21222212 22222122 22222
$w_1(2;\ 4,\ 2)$	=	18	21212112 12112121 2
$w_1(2;\ 4,\ 3)$	=	62	22122122 12121221 22112212 21212212 21122122 12122122 11221122 12122
$w_1(2;\ 4,\ 4)$	=	229	22211222 12221122 21221212 22122212 21221222 11222122 12212221 22212122 12221122 21221212 21222112 22122121 22212221 22122122 21122212 21221222 12221212 21222112 22122121 22122211 22212212 12221222 12212212 22112221 22122122 21222121 22122211 22212221 1222
$w_1(2;\ 5,\ 2)$	=	32	21121121 21211212 12112121 2112112
$w_1(2;\ 6,\ 2)$	=	60	21A12121 21121121 12121212 11212121 21121212 12112112 11212121 B12 (where AB is arbitrary).
$w_1(2;\ 7,\ 2)$	=	301	21121121 21212112 11212121 12112121 2121121A 12121121 21212112 121B1211 21121212 11211212 12121121 21212112 12121211 21212121 12112121 21121121 21212121 12121211 21212112 12121212 11211212 12112112 12121211 21212121 12121212 11212121 21121121 21211211 21C12121 12121212 11212121 21121212 12112112 12121121 12121212 11212121 21121211 2112 (ABC is arbitrary).

5.2. On $w_d(r; k_1, k_2, \cdots, k_r)$

Let $w_d(r; k_1, k_2, \cdots, k_r)$ be the least m such that given any partition $\{1, 2, \cdots, m\} = P_1 \cup P_2 \cup \cdots \cup P_r$, there is an index $j \in \{1, 2, \cdots, r-1\}$ such that P_j contains an arithmetic progression of length k_j, or P_r contains an arithmetic progression of length k_r with common difference at most d. It is clear that this is a generalization of $w_1(2; k_1, k_2)$ and that $w_d(r; k_1, k_2, \cdots, k_r) = w(r; k_1, k_2, \cdots, k_r) = m$ for $d = \lfloor \frac{m-1}{k_r-1} \rfloor$. We observe the following:

Table 5: $w_d(2; k_1, k_2, \cdots, k_r)$

$w_d(r; k_1, \cdots, k_r)/d$	1	2	3	4	5	6	7	8	9	10	11	12	13
$w_d(2;\ 3,\ 3)$	23	13	11	9									
$w_d(2;\ 3,\ 4)$	34	28	28	18	18								
$w_d(2;\ 3,\ 5)$	73	51	31	23	22								
$w_d(2;\ 3,\ 6)$	113	75	48	37	34	32							
$w_d(2;\ 3,\ 7)$	193	130	78	65	51	51	46						
$w_d(2;\ 3,\ 8)$	238	168	129	83	78	74	61	58					
$w_d(2;\ 3,\ 9)$	>346	242	191	116	99	90	77	77	77				
$w_d(2;\ 4,\ 4)$	229	94	69	45	38	37	37	37	37	37	35		
$w_d(2;\ 4,\ 5)$		>299	134	114	93	74	74	74	74	74	55	55	55
$w_d(3;\ 3,\ 3,\ 3)$	139	119	61	43	41	35	34	33	33	33	33	27	27

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