INFINITE FAMILIES OF DIVISIBILITY PROPERTIES MODULO 4 FOR NON–SQUASHING PARTITIONS INTO DISTINCT PARTS

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Abstract

In 2005, Sloane and Sellers defined a function $b(n)$ which denotes the number of non-squashing partitions of n into distinct parts. In their 2005 paper, Sloane and Sellers also proved various congruence properties modulo 2 satisfied by $b(n)$. In this note, we extend their results by proving two infinite families of congruence properties modulo 4 for $b(n)$. In particular, we prove that for all $k \geq 3$ and all $n \geq 0$,

$$\begin{aligned} b(2^{2k+1}n + 2^{2k-2}) &\equiv 0 \pmod 4 \quad \text{and} \\ b(2^{2k+1}n + 3 \cdot 2^{2k-2} + 1) &\equiv 0 \pmod 4. \end{aligned}$$

1. Introduction and Statement of Results

In 2005, Sloane and Sellers [5] defined the function $b(n)$ which counts the number of non–squashing partitions of n into distinct parts (as part of their work on enumerating non–squashing stacks of boxes under a particular set of constraints). More precisely, let us say that a partition of a natural number n is *non-squashing* if, when the parts are arranged in nondecreasing order, say $n = p_1 + p_2 + \cdots + p_k$ with $1 \leq p_1 \leq p_2 \leq \cdots \leq p_k$, we have

$$p_1 + \cdots + p_j \leq p_{j+1}$$

for all $1 \leq j \leq k-1$. Then $b(n)$ is the number of non–squashing partitions of n into *distinct* parts. So, for example, $b(10) = 9$ with the following partitions being allowed:

				1		1	1	2
	1	2	3	2	4	3	4	3
10	9	8	7	7	6	6	5	5

The values of $b(n)$ for relatively small n can be found in Sloane's Online Encyclopedia of Integer Sequences [4, A088567].

[1]This work was initiated while the third author was a visiting fellow at the Isaac Newton Institute, University of Cambridge.

In [5], Sloane and Sellers proved a number of facts related to $b(n)$ including the following:

Theorem 1 ([5], Theorem 2) *The numbers $b(n)$ satisfy the recurrence*

$$\begin{aligned} b(0) &= b(1) = 1, \\ b(2m) &= b(2m-1) + b(m) - 1 \quad \text{for } m \geq 1, \\ b(2m+1) &= b(2m) + 1 \quad \text{for } m \geq 1. \end{aligned}$$

Theorem 2 ([5], Theorem 2) *The generating function for $b(n)$ is given by*

$$B(q) = \sum_{n=0}^{\infty} b(n) q^n = \prod_{i=0}^{\infty} \frac{1}{1-q^{2^i}} - \sum_{i=1}^{\infty} \frac{q^{2^i}(1-q^{2^{i-1}})}{\prod_{j=0}^{i}(1-q^{2^j})}.$$

Theorem 3 ([5], Corollary 4) *The value of $b(n) \mod 2$ is as follows (all congruences are $\mod 2$):*

$$\begin{aligned} &b(0) \equiv 1, \\ &\text{if } n \text{ is odd}, b(n) \equiv b(n-1) + 1, \\ &b(8m+2) \equiv 1, \;\; b(8m+6) \equiv 0, \\ &b(16m+4) \equiv 0, \;\; b(16m+12) \equiv 1, \\ &\text{for } m > 0, b(16m) \equiv b(8m), \;\; b(32m+8) \equiv 0, \;\; b(32m+24) \equiv 1. \end{aligned}$$

At this point, several comments are in order. First, considering congruences satisfied by $b(n)$ was only a secondary, if not tertiary, goal in [5]. Hence, Theorem 3 is the only result in [5] which deals with congruences satisfied by $b(n)$. In contrast, the goal of this note is to focus attention on divisibility properties satisfied by $b(n)$ in arithmetic progression. This is a worthy goal given the clear relationship that $b(n)$ has with the unrestricted binary partition function whose generating function is given by

$$\prod_{i=0}^{\infty} \frac{1}{1-q^{2^i}}.$$

This is easily seen to be the first term in the generating function for $b(n)$ as noted in Theorem 2 above. It should be noted that Rødseth, Sellers, and Courtright [3] further solidified the relationship between $b(n)$ and the unrestricted binary partition function when they proved the following:

Theorem 4 *For all $n \geq 0$ and all $r \geq 2$,*

$$b(2^{r+1}n) \equiv b(2^{r-1}n) \pmod{2^{r-2}}.$$

As noted in [3], this family of "internal congruences" is extremely reminiscent of the family of congruences which Churchhouse [1] observed for the unrestricted binary partition function. (A similar family of congruences for a restricted binary partition function was also proved by Rødseth and Sellers [2].)

With the above in mind, we can shed more direct light on our goal for this note by first proving the following two infinite families of divisibility properties modulo 2 satisfied by $b(n)$.

Theorem 5 *For all $n \geq 0$ and all $k \geq 4$,*

$$\begin{aligned} b(2^k n + 2^{k-2}) &\equiv 0 \pmod{2} \quad \textit{and} \\ b(2^k n + 3 \cdot 2^{k-2} + 1) &\equiv 0 \pmod{2}. \end{aligned}$$

Proof. The proof of this theorem is actually bundled up in the statement of Theorem 3 although this is not apparent at first reading. First, note that the $k = 4$ cases of this theorem are

$$\begin{aligned} b(16n + 4) &\equiv 0 \pmod{2} \quad \text{and} \\ b(16n + 13) &\equiv 0 \pmod{2}. \end{aligned}$$

The first of these is explicitly stated in Theorem 3. The second follows from the fact that

$$\begin{aligned} b(16n + 13) &= b(16n + 12) + 1 \quad \text{by Theorem 1} \\ &\equiv 1 + 1 \pmod{2} \quad \text{by Theorem 3} \\ &\equiv 0 \pmod{2}. \end{aligned}$$

Next, note that the $k = 5$ cases of this theorem are

$$\begin{aligned} b(32n + 8) &\equiv 0 \pmod{2} \qquad \text{and} \\ b(32n + 25) &\equiv 0 \pmod{2}. \end{aligned}$$

Again, these follow in the same manner from statements made in Theorem 3.

To complete the proof, we note that a proof by mathematical induction on the parameter k can now be performed (using these $k = 5$ results as the basis). To prove the first family of congruences in the theorem, assume

$$b(2^k n + 2^{k-2}) \equiv 0 \pmod{2}$$

for all $n \geq 0$ and some $k \geq 5$. Then we know

$$\begin{aligned} b(2^{k+1} n + 2^{k-1}) &= b(16(2^{k-3} n + 2^{k-5})) \\ &\equiv b(8(2^{k-3} n + 2^{k-5})) \pmod{2} \qquad \text{from Theorem 3} \\ &= b(2^k n + 2^{k-2}). \end{aligned}$$

The other family of congruences in this theorem follows in similar fashion. □

Our main goal in this work is to prove the following two theorems which provide divisibility properties of $b(n)$ modulo 4. (Note the striking resemblance between Theorems 5 and 7.)

Theorem 6 *For all* $n \geq 0$,

$$\begin{aligned} b(32n+20) &\equiv 0 \pmod{4} \quad \textit{and} \\ b(32n+29) &\equiv 0 \pmod{4}. \end{aligned}$$

Theorem 7 *For all* $n \geq 0$ *and all* $k \geq 3$,

$$\begin{aligned} b(2^{2k+1}n + 2^{2k-2}) &\equiv 0 \pmod{4} \quad \textit{and} \\ b(2^{2k+1}n + 3 \cdot 2^{2k-2} + 1) &\equiv 0 \pmod{4}. \end{aligned}$$

The proofs of these two theorems that we provide below are elementary, relying heavily on generating function manipulations as well as the $r = 4$ case of Theorem 4 above (which is the key ingredient in the induction step of the proof below as Theorem 3 was in the induction proof of Theorem 5.)

2. Infinite Families of Divisibility Properties Modulo 4

We begin this section by proving Theorem 6.

Proof of Theorem 6. Our proof begins by an elementary rewriting of the generating function in Theorem 2 in the following form:

$$\begin{aligned} \sum_{n=0}^{\infty} b(n)q^n &= \prod_{i=0}^{\infty} \frac{1}{1-q^{2^i}} - \sum_{i=1}^{\infty} \frac{q^{2^i}(1-q^{2^{i-1}})}{\prod_{j=0}^{i}(1-q^{2^j})} \\ &= \left(\frac{(1-q^{32})^5}{\prod_{i=0}^{4}(1-q^{2^i})} \right) \frac{1}{(1-q^{32})^5} \prod_{i=5}^{\infty} \frac{1}{1-q^{2^i}} \\ &\quad - \left(\sum_{i=1}^{5} \frac{q^{2^i}(1-q^{2^{i-1}})(1-q^{32})^6}{\prod_{j=0}^{i}(1-q^{2^j})} \right) \frac{1}{(1-q^{32})^6} \\ &\quad - \sum_{i=6}^{\infty} \frac{q^{2^i}(1-q^{2^{i-1}})}{\prod_{j=0}^{i}(1-q^{2^j})}. \end{aligned}$$

Both of the quantities in parentheses are actually polynomials in q as is easily checked, while the denominators of the remaining portions of this representation of the generating function are clearly functions of q^{32}. These two facts are all that is needed now to dissect the generating function for $b(n)$ in order to obtain the generating functions for $b(32n+20)$ and $b(32n+29)$. Indeed, after expanding the polynomials above in MAPLE, extracting the relevant terms, dividing by the appropriate power of q (q^{20} respectively q^{29}), replacing q^{32} by q throughout, and simplifying we find

$$\begin{aligned}\sum_{n=0}^{\infty} b(32n+20)q^n \\ &= 4\left(\frac{15+135q+101q^2+5q^3}{(1-q)^5}\right)\prod_{i=0}^{\infty}\frac{1}{1-q^{2^i}} - 4\left(\frac{5+47q+67q^2+9q^3}{(1-q)^5}\right) \\ &\equiv 0 \pmod 4\end{aligned}$$

and

$$\begin{aligned}\sum_{n=0}^{\infty} b(32n+29)q^n \\ &= 4\left(\frac{35+155q+65q^2+q^3}{(1-q)^5}\right)\prod_{i=0}^{\infty}\frac{1}{1-q^{2^i}} - 4\left(\frac{11+61q+53q^2+3q^3}{(1-q)^5}\right) \\ &\equiv 0 \pmod 4.\end{aligned}$$

□

We move next to the proof of Theorem 7.

Proof of Theorem 7. As above, we rewrite the generating function for $b(n)$ in an advantageous form:

$$\begin{aligned}\sum_{n=0}^{\infty} b(n)q^n &= \prod_{i=0}^{\infty}\frac{1}{1-q^{2^i}} - \sum_{i=1}^{\infty}\frac{q^{2^i}(1-q^{2^{i-1}})}{\prod_{j=0}^{i}(1-q^{2^j})} \\ &= \left(\frac{(1-q^{128})^7}{\prod_{i=0}^{6}(1-q^{2^i})}\right)\frac{1}{(1-q^{128})^7}\prod_{i=7}^{\infty}\frac{1}{1-q^{2^i}} \\ &\quad -\left(\sum_{i=1}^{7}\frac{q^{2^i}(1-q^{2^{i-1}})(1-q^{128})^8}{\prod_{j=0}^{i}(1-q^{2^j})}\right)\frac{1}{(1-q^{128})^8} \\ &\quad -\sum_{i=8}^{\infty}\frac{q^{2^i}(1-q^{2^{i-1}})}{\prod_{j=0}^{i}(1-q^{2^j})}.\end{aligned}$$

Again, the quantities in parentheses are polynomials in q while the denominators of the remaining portions of this representation of the generating function are functions of q^{128}. These two facts are all that is needed now to dissect the generating function for $b(n)$ in order to obtain the generating functions for $b(128n+16)$ and

$b(128n+49)$. After expanding the polynomials above in MAPLE, extracting the relevant terms, dividing by the appropriate power of q (q^{16} respectively q^{49}), replacing q^{128} by q throughout, and simplifying we find

$$\sum_{n=0}^{\infty} b(128n+16)q^n$$

$$= 4\left(\frac{9+11319q+144618q^2+270494q^3+93949q^4+3899q^5}{(1-q)^7}\right)\prod_{i=0}^{\infty}\frac{1}{1-q^{2^i}}$$
$$-4\left(\frac{3+3592q+49505q^2+126048q^3+76445q^4+6632q^5-81q^6}{(1-q)^7}\right)$$
$$\equiv 0 \pmod 4$$

and

$$\sum_{n=0}^{\infty} b(128n+49)q^n$$

$$= 4\left(\frac{173+26803q+198354q^2+244118q^3+53809q^4+1031q^5}{(1-q)^7}\right)\prod_{i=0}^{\infty}\frac{1}{1-q^{2^i}}$$
$$-4\left(\frac{55+8530q+71571q^2+127940q^3+52053q^4+2010q^5-15q^6}{(1-q)^7}\right)$$
$$\equiv 0 \pmod 4.$$

The proof is completed by induction using the case $r=4$ of Theorem 4. □

3. Closing Thoughts

The authors have not found an arithmetic progression $2^t n+r$ such that, for all $n \geq 0$, $b(2^t n+r) \equiv 0 \pmod 8$. This is reminiscent of the behavior of the unrestricted binary partition function [1]. We doubt that such a progression exists.

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