#A27 INTEGERS 9 (2009), 333-342

CHARACTERIZATIONS OF WORDS WITH MANY PERIODS

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Received: 9/30/08, Accepted: 2/9/09

Abstract

In 2001, Droubay, Justin, and Pirillo studied the so-called standard episturmian word. In 2006 and 2007, Fischler defined another type of word in the framework of simultaneous diophantine approximation. Both classes of words were described in terms of palindromic prefixes of infinite words. In 2003, the authors introduced words which they called extremal FW words after they had met so-called FW words later. Words from both classes appeared as unique words (up to word isomorphism) satisfying certain properties related to periods. In the present paper the connections between all these words are displayed in the form of four comparable characterizations of the four mentioned word classes.

1. Introduction

Let A be a finite set with at least two elements, the so-called alphabet. The free monoid A^* generated by A is the set of the finite words on A. The empty word is denoted by ϵ. Put $A^+ = A^* \setminus \epsilon$. For a word $u = u_1 u_2 \cdots u_m$ with $u_i \in A$ for $i = 1, \dots, m$ we denote the length m of u by $|u|$ and the number of distinct letters occurring in u by $\sharp u$. A word is called constant if it is a power of one letter.

A factor of a word $u = \{u_i\}_{i=1}^m$ is a word $u_h u_{h+1} \cdots u_j$ with $1 \leq h \leq j \leq m$. It is called a prefix of u if $h = 1$ and a suffix if $j = m$. For $p \leq m$ the word u is called periodic with period p if $u_i = u_{i+p}$ for $1 \leq i \leq m - p$. The reversal of $u = u_1 u_2 \cdots u_m$ is $\overline{u} := u_m u_{m-1} \cdots u_1$. The word u is called a palindrome if $u = \overline{u}$. Given u, its palindromic right-closure is the (unique) shortest palindrome $u^{(+)}$ which has prefix u. In this paper we study the palindromic prefixes of words with many periods.

A word is called standard episturmian whenever for every prefix v of u, $v^{(+)}$ is also a prefix of u (cf. [1], [7]). Justin and Pirillo (cf.[7, Theorem 2.10]) proved that a word u is standard episturmian if and only if there exists a word $\Delta(u) = x_1 x_2 \cdots x_K \in A^+$ such that if $u[0] = \epsilon$ and $u[k] = (u[k-1]x_k)^{(+)}$ for $k = 1, \dots, K$, then $u[K] = u$. For comparison with other characterizations we shall prove the following variant.

Theorem 1. *A word u is standard episturmian if and only if there exists a K such that u can be generated as follows: $u[0] := \epsilon$; for $k = 1, \dots, K$ either $u[k] = u[k-1]v[k]u[k-1]$, where $v[k]$ is a letter that does not occur in $u[k-1]$, or $u[k] =$*

$u[k-1]u'[k-1]$, where $u[k-1] = u[l-1]u'[k-1]$ and l is the largest integer less than k such that if $u[l] = (u[l-1]x)^{(+)}$, then $u[k] = (u[k-1]x)^{(+)}$.

The latter condition implies that if $\Delta(u) = x_1 x_2 \cdots x_K$, then l is the largest integer $< k$ such that $x_l = x_k$. The 'only if' part of Theorem 1 was already observed by Justin and Pirillo, see [7, p. 287].

We call a word u an extremal FW word (i.e. extremal Fine and Wilf word) if there exist positive integers $p_1 < p_2 < \cdots < p_r = |u|$ such that u has periods $p_1, p_2, \ldots, p_r$, but not period $\gcd(p_1, p_2, \ldots, p_r)$, u has the maximal length for such a word, and $\sharp u \geq \sharp v$ for every word v with these properties. We say that u is an extremal FW word for period set $P := \{p_1, p_2, \ldots, p_r\}$. We proved in [8] that if u is an extremal FW word for the set P, then u is unique apart from word isomorphism and it is a palindrome. We denote the length of the word u which is extremal for the period set P by $L(P) - 1 = L(p_1, p_2, \ldots, p_r) - 1$. It follows from the following theorem that every standard episturmian word is an extremal FW word. The converse is false as the example *aabaaaabaa* shows.

Theorem 2. *A word u is an extremal FW word if and only if $\sharp u > 1$ and for some K it can be generated as follows: $u[0] := \epsilon$; for $k = 1, \ldots, K$ either $u[k] = u[k-1]v[k]u[k-1]$ where $v[k]$ is a letter which does not occur in $u[k-1]$, or $u[k] = u[k-1]u'[k-1]$ where $u[k-1] = u[l-1]u'[k-1]$ for some l with $1 \leq l < k$.*

The numbers $L(p_1, p_2, \ldots, p_r)$ also appear in a seemingly different context. In [10] we defined a quantity denoted by $L'(p_1, p_2, \ldots, p_r)$ which is the minimal value n for the following property: Let $X_1, X_2, \ldots, X_r$ be non-empty finite words in the alphabet A with $|X_i| = p_i$ for each i with $1 \leq i \leq r$. For each i with $1 \leq i \leq r$, let $W_i \in X_i\{X_1, \ldots, X_r\}^\infty$ (in other words W_i is an infinite concatenation of $X_1, \ldots, X_r$ beginning in X_i.) If $W_1, \ldots, W_r$ agree on a prefix of length n then $W_i = W_j$ for all i and j. We proved in [10] that $L'(p_1, p_2, \ldots, p_r) = L(p_1, p_2, \ldots, p_r)$ if $p_r \leq L(p_1, p_2, \ldots, p_{r-1})$ and $L'(p_1, p_2, \ldots, p_r) = p_r$ otherwise. Furthermore, examples of words realizing the maximal value $L' - 1$ for not satisfying the property are constructed by using extremal FW words. As was pointed out by T. Harju, the existence of L', but not the optimal bound, may be deduced from the details of the proof of the so-called *graph lemma* for infinite words (see Theorem 5.1 in Harju and Karhumäki [5]).

Extremal FW words are closely related to words introduced by S. Fischler. Let $\{n_i\}_{i=1}^\infty$denote the increasing sequence (assumed to be infinite) of all lengths of palindromic prefixes of a word u. Fischler [3] gave an explicit construction of all words u such that $n_{i+1} \leq 2n_i + 1$ for all i. He proved that among all such non-periodic words u the quantity limsup n_{i+1}/n_i is minimal for the Fibonacci word. We call the palindromic prefixes of the word u Fischler words. Later, Fischler [4] applied his study to simultaneous approximation to a fixed real number and its square by rational numbers with the same denominator. The following result shows that every extremal FW word is a Fischler word, but not conversely.

Theorem 3. *A word u is a Fischler word if and only if for some K it can be generated as follows: $u[0] := \epsilon$; for $k = 1, \ldots, K$ either $u[k] = u[k-1]v[k]u[k-1]$ where $v[k]$ is some letter, or $u[k] = u[k-1]u'[k-1]$ where $u[k-1] = u[l-1]u'[k-1]$ for some l with $1 \le l < k$.*

We call a word u a FW word (i.e. Fine and Wilf word) if there exist positive integers n and $p_1, p_2, \ldots, p_r$ such that u has length n and periods $p_1, p_2, \ldots, p_r$ and $\sharp u \ge \sharp v$ for any word v of length n and with periods $p_1, p_2, \ldots, p_r$. Note that for every positive integer n and periods $p_1, \ldots, p_r$ there exists a FW word. It is proved in [9] that the FW word for length n and periods $p_1, p_2, \ldots, p_r$ is unique apart from word isomorphism. We call u a FW word for period set $\{p_1, p_2, \ldots, p_r\}$ if it is the FW word for length $|u|$ and periods $p_1, p_2, \ldots, p_r$.

The word u is called a pseudo-palindrome if u is a fixed point of some involutary antimorphism θ of A^*; an involutary antimorphism is given by a map $\theta : A^* \to A^*$ such that $\theta \circ \theta = \mathrm{id}$ and $\theta(uv) = \theta(v)\theta(u)$ for all $u, v \in A^*$. The reversal operator $R : A^* \to A^*$ given by $R(u) = \overline{u}$ is a basic example, hence every palindrome is a pseudo-palindrome. It is proved in [9] that every FW word is a pseudo-palindrome.

Similar to the previous theorems we have the following characterization of FW words which implies that every extremal FW word is a FW word.

Theorem 4. *A word u is a FW word if and only if for some K it can be generated as follows: $u[0] := \epsilon$; for $k = 1, \ldots, K$ either $u[k] = u[k-1]v[k]u[k-1]$ where $v[k]$ is not ϵ and consists of distinct letters or $u[k] = u[k-1]u'[k-1]$ where $u[k-1] = u[l-1]u'[k-1]$ for some l with $1 \le l < k$.*

Observe that standard episturmian words and Fischler words are defined in terms of palindromes whereas extremal FW words and FW words are defined in terms of periods. The relation between periods and (pseudo-)palindromic prefixes of (pseudo-)palindromic words is exhibited in the following equivalence.

Lemma 5. *Let w be a (pseudo-)palindrome and u a prefix of w. Then u is a (pseudo-) palindrome if and only if $|w| - |u|$ is a period of w.*

2. Proof of Lemma 5

Proof of Lemma 5. Denote the involutary antimorphism by θ. Thus $\theta(w) = w$. In case of proper palindromes θ is the reversal operator.

Suppose $|w| - |u|$ is a period of w. Then there are words v and v' such that $w = uv = v'u$. Since w is a (pseudo-)palindrome,

$$\theta(v)\theta(u) = \theta(uv) = \theta(w) = w = v'u.$$

By $|\theta(u)| = |u|$, we have $\theta(u) = u$. Thus u is a (pseudo-)palindrome.

Suppose u is a (pseudo-)palindrome. Then

$$uv = w = \theta(w) = \theta(v)\theta(u) = \theta(v)u.$$

Hence $|w| - |u| = |v| = |\theta(v)|$ is a period of w. □

3. Proofs of Theorems 1 and 3

Both in Theorem 1 and in Theorem 3 we have either $u[k] = u[k-1]v[k]u[k-1]$ where $v[k]$ is some letter or $u[k] = u[k-1]u'[k-1]$ where $u[k-1] = u[l-1]u'[k-1]$ and $l < k$. Observe that in the former case $u[k] \le 2|u[k-1]| + 1$ and in the latter case $|u[k]| = 2|u[k-1]| - |u[l-1]| \le 2|u[k-1]|$.

Proof of Theorem 3. Let u be a Fischler word. Let $u[k] (k = 0, 1, \ldots, K)$ be the sequence of increasing palindromic prefixes of u. Let n_k denote the length of $u[k]$ for $k = 0, 1, \ldots, K$. Then $n_k \le 2n_{k-1} + 1$ for all k. If $n_k = 2n_{k-1} + 1$, then $u[k] = u[k-1]v[k]u[k-1]$ for a letter $v[k]$, because both $u[k]$ and $u[k-1]$ are palindromes. If $n_k \le 2n_{k-1}$, then $u[k] = u[k-1]u'[k-1]$ for a word $u'[k-1]$ of length $n_k - n_{k-1} \le n_{k-1}$. By Lemma 5 $|u'[k-1]|$ is a period of $u[k]$, hence of $u[k-1]$. Write $u[k-1] = wu'[k-1]$ for some word w. Since

$$\begin{aligned}
\overline{u'[k-1]}wu'[k-1] &= \overline{u'[k-1]}u[k-1] = \overline{u'[k-1]}\;\overline{u[k-1]} \\
&= \overline{u[k-1]u'[k-1]} = \overline{u[k]} = u[k] = u[k-1]u'[k-1] \\
&= \overline{u[k-1]}u'[k-1] = \overline{wu'[k-1]}u'[k-1] \\
&= \overline{u'[k-1]}\;\overline{w}u'[k-1]
\end{aligned}$$

we see that w is a palindrome. Furthermore w is a prefix of $u[k-1]$ and therefore of u. By definition of Fischler word $w = u[l-1]$ for some $l < k$.

Let u be constructed as in Theorem 3. Let $\{n_k\}_{k=1}^{K}$ be the increasing sequence of the lengths of the palindromic prefixes of u. Then $n_k \le 2n_{k-1} + 1$ by the remark at the beginning of this section. □

Proof of Theorem 1. Let u be a standard episturmian word. Then, by Theorem 2.10 of Justin and Pirillo [7], there exists a word $\Delta(u) = x_1x_2 \ldots x_K \in A^+$ such that if $u[0] = \epsilon$ and $u[k] = (u[k-1]x_k)^{(+)}$ for $k = 1, \ldots, K$ then $u[K] = u$. Since $u[k] = u[k-1]x_ku[k-1]$ is a palindrome, we have $|u[k]| \le 2|u[k-1]| + 1$

for $k = 1, \ldots, K$. Thus every standard episturmian word is a Fischler word. By Theorem 3 it suffices to show that in the former case of the characterization $v[k] = x_k$ does not occur in $u[k-1]$ and in the latter case l is the largest integer $< k$ such that $x_l = x_k$.

Suppose x_k occurs in $u[k-1]$. Then, by the definition of standard episturmian, $x_k = x_l$ for some $l < k$. Write $u[k-1] = u[l-1]w[l]$ for some word $w[l]$. Then x_k is the initial letter of $w[l]$ and

$$\begin{aligned}\overline{u[k-1]w[l]} &= \overline{w[l]}\ \overline{u[k-1]} = \overline{w[l]}\ u[k-1] = \overline{w[l]}u[l-1]w[l] \\ &= \overline{w[l]}\ \overline{u[l-1]}w[l] = \overline{u[l-1]w[l]}w[l] = \overline{u[k-1]}w[l] \\ &= u[k-1]w[l].\end{aligned}$$

Thus $u[k-1]w[l]$ is a palindrome. Hence

$$|(u[k-1]x_k)^{(+)}| \leq |u[k-1]w[l]| \leq 2|u[k-1]|.$$

So if the former case of the characterization for k holds, then $x_k = v[k]$ does not occur in $u[k-1]$.

Suppose $l < k$ is such that $x_l = x_k$. Then $u[k] = u[k-1]u'[k-1]$ with $u[k-1] = u[l-1]u'[k-1]$. Hence $|u[k]| - |u[k-1]| = |u'[k-1]| = |u[k-1]| - |u[l-1]|$. Thus $|u[k]| - |u[k-1]|$ is minimal if $|u[l-1]|$ is maximal. This implies that l is the largest integer $< k$ such that $x_l = x_k$.

Suppose a word u is constructed as described in Theorem 1. If $u[k] = u[k-1]v[k]u[k-1]$ where $v[k]$ does not occur in $u[k-1]$, then obviously $(u[k-1]v[k])^{(+)} = u[k]$. In the other case the inital letter x_k of $u'[k-1]$ occurs in $u[k-1]$ as the letter following the palindromic prefix $u[l-1]$ for some $l < k$. By choosing l as the largest integer $< k$ with this property, the criterion secures that $u[k] = (u[k-1]x_k)^{(+)}$. Thus u is standard episturmian by the characterization due to Justin and Pirillo. □

4. Some Lemmas

Lemma 6. (a) *If $p_1, p_2, \ldots, p_r$ are coprime integers and $p \in \{p_1, p_2, \ldots, p_r\}$, then $p, p_1 + p, p_2 + p, \ldots, p_r + p$ are coprime integers too.*
(b) *If $p_1 < p_2 < \cdots < p_r$ are coprime integers, then $p_1, p_2 - p_1, \ldots, p_r - p_1$ are coprime integers.*

Proof. a) Suppose $d \mid p, d \mid p + p_i$ for all i. Then $d \mid p_i$ for all i. The proof for (b) is similar. □

Lemma 7. ([9, Lemma 1]) *Let $u = u_1 \cdots u_m$ be a word with $\sharp u = s$ and periods $q_1 < \cdots < q_r$. Put $u' := u_1 \cdots u_{m-q_1}$. If $m \geq 2q_1 - y$, with $0 \leq y < q_1$, then u' is a word with $\sharp u' \geq s - y$ and periods $q_1, q_2 - q_1, \ldots, q_r - q_1$.*

Proof. Because of the period q_1, every letter of u occurs in u' with the possible exception of some of the y letters $u_{q_1-y+1}, u_{q_1-y+2}, \ldots, u_{q_1}$. Hence $\sharp u' \geq s - y$. For $t \leq m - q_j$ we have $u_t = u_{t+q_j} = u_{t+q_j-q_1}$. So u' has period $q_j - q_1$ for $j = 2, \ldots, r$. □

Lemma 8. ([9], Lemma 2) *Suppose $u = u_1 \cdots u_m$ has periods $q_1, \ldots, q_r$. Let $u' := u_1u_2 \cdots u_{m+q_1}$ have period q_1. Then the word u' has periods $q_1, q_2 + q_1, \ldots, q_r + q_1$.*

Proof. Note that if $q_1 \leq m$, then $u_{m+1} \cdots u_{m+q_1}$ is the suffix of u of length q_1. If $q_1 > m$, then $u_{q_1+1} \cdots u_{m+q_1}$ is a suffix of u. For $t \leq m - q_j$ we have $u_t = u_{t+q_j} = u_{t+q_j+q_1}$ for $j = 2, \ldots, r$. □

Lemma 9. *Every extremal FW word is a palindrome.*

Proof. [8, Theorem 4]. □

Lemma 10. *If w is a FW word for the period set $Q = \{q_1, \ldots, q_s\}$ and $u := wvw$ where $v \neq \epsilon$ and v consists of distinct letters none of which occurs in w, then u is a FW word for the period set $P := \{|w|+|v|, |w|+|v|+q_1, |w|+|v|+q_2, \ldots, |w|+|v|+q_s\}$.*

Proof. By Lemma 8, P is a set of periods of u. Suppose u is not a FW word for P. Then there exists a word u' with period set P such that $|u'| = |u|$ and $\sharp u' > \sharp u$. Let w' be the prefix of u' of length $|w|$. Then, by Lemma 7, w' has all periods from Q and $\sharp w' \geq \sharp u' - |v| > \sharp u - |v| = \sharp w$. This shows that w is not a FW word for Q, which contradicts our assumption. □

Lemma 11. *If w is a FW word for the period set $Q := \{q_1, \ldots, q_s\}$ and $u = ww'$ where $w = vw'$ for some v with $|v| \in Q$, then u is a FW word for the period set $P := \{|w'|, |w'| + q_1, |w'| + q_2, \ldots, |w'| + q_s\}$.*

Proof. By Lemma 8, P is a set of periods of u. Suppose u is not a FW word for period set P. Then there exists a word u' with period set P such that $|u'| = |u|$ and $\sharp u' > \sharp u$. Let v be the prefix of u' of length $|w|$. Then, by Lemma 7, v has all periods from Q and $\sharp v = \sharp u' > \sharp u = \sharp w$. This shows that w is not a FW word for Q, contradicting our assumption. □

Lemma 12. *If w is a FW word for the period set $Q := \{q_1, \ldots, q_s\}$ with $q_1 < q_2 < \cdots < q_s$ and $w = uv$ with $|v| = q_1$, then u is a FW word for the period set $Q' := \{q_1, q_2 - q_1, \ldots, q_s - q_1\}$ where we omit "q_1," if $|u| < q_1$ or $q_i = 2q_1$ for some i.*

Proof. Suppose u is not a FW word for Q'. Then there exists a word u' with $|u'| = |u|$, $\sharp u' > \sharp u$ and period set Q'.

If $|v| \leq |u|$, then consider $w' := u'v'$ where v' is the suffix of u' of length $|v|$. By Lemma 8, w' has all periods from Q. Moreover, $\sharp w' = \sharp u' > \sharp u = \sharp w$ and $|w'| = |w|$. This gives a contradiction with the assumption that w is a FW word.

If $|v| > |u|$, then v is of the form $v''u$. Consider $w' := u'v'u'$ where v' consists of $|v''|$ distinct letters, none of which appears in u'. Then w' has all periods from Q by Lemma 8. Moreover, $\sharp w' = \sharp u' + |v'| > \sharp u + \sharp v'' \geq \sharp w$, again in contradiction with the assumption. □

5. Proofs of Theorems 2 and 4

Proof of Theorem 4. We proved in Lemma 6 of [9] that every FW word can be generated in the way stated in the theorem. It remains to prove that every word u which can be generated by the given inductive procedure is a FW word. We use induction on k.

It is clear that $u[1] = v[1]$. Since $v[1]$ consists of $|v[1]|$ distinct letters, it is a FW word with period set $\{|v[1]|\} = \{|u[1]| - |u[0]|\}$. Suppose $u[k-1]$ is a FW word with period set

$$\{|u[k-1]| - |u[k-2]|, |u[k-1]| - |u[k-3]|, \ldots, |u[k-1]| - |u[0]|\}.$$

If $u[k] = u[k-1]v[k]u[k-1]$ with $v[k]$ as in the statement of the theorem, then, by Lemma 10, $u[k]$ is a FW word for periods

$$\begin{aligned}\{|u[k-1]| + |v[k]|, |u[k-1]| + |v[k]| + |u[k-1]| - |u[k-2]|, \ldots,\\ |u[k-1]| + |v[k]| + |u[k-1]| - |u[0]|\}\\ = \{|u[k]| - |u[k-1]|, |u[k]| - |u[k-2]|, \ldots, |u[k]| - |u[0]|\}.\end{aligned}$$

If $u[k] = u[k-1]u'[k-1]$ with $u'[k-1]$ as in the statement of the theorem, then $|u'[k-1]| = |u[k-1]| - |u[l-1]|$ is in the period set of $u[k-1]$. From Lemma 7 it follows that $u[k]$ is a FW word for periods

$$\begin{aligned}\{|u'[k-1]|, |u'[k-1]| + |u[k-1]| - |u[k-2]|, \ldots,\\ |u'[k-1]| + |u[k-1]| - |u[0]|\}\\ = \{|u[k]| - |u[k-1]|, |u[k]| - |u[k-2]|, \ldots, |u[k]| - |u[0]|\}.\end{aligned}$$

This completes the induction step and shows that $u = u[K]$ is a FW word for period set $\{|u[K]| - |u[K-1]|, |u[K]| - |u[K-2]|, \ldots, |u[K]| - |u[0]|\}$. □

Proof of Theorem 2. Suppose u is an extremal FW word for some period set P. Then u is a FW word and can be constructed in the way indicated by Theorem 4. Suppose there is a k such that $u[k] = u[k-1]v[k]u[k-1]$ where $|v[k]| > 1$. Put $v[k] = x_1 \ldots x_m$. Then in u every x_1 is followed by x_2 and every x_2 is preceded by x_1.

Hence u is not a palindrome. This contradicts our assumption in view of Lemma 9. Thus u can be constructed by the given inductive procedure.

Let u be constructed according to the inductive procedure from the statement of Theorem 2. For $k = 0, 1, \ldots$ we define

$$P[k] := \{|u[k]| - |u[k-1]|, |u[k]| - |u[k-2]|, \ldots, |u[k]| - |u[0]|\}.$$

It is clear that $u[1] = v[1]$ is a FW word for the period set $\{1\}$. Since $\sharp u > 1$, there is some minimal $j > 1$ with $v[j] \neq v[1]$. Then $u[j]$ is of the form $u_1 u_2 \ldots u_n u_{n+1} \ldots u_{2n-1}$ with $u_n = v[j]$ and $u_i = v[1]$ for $i \neq n$. Observe that $u[j]$ has periods $n, n+1, \ldots, 2n-1$. Suppose $u[j]$ is not an extremal FW word for the period set $\{n, n+1, \ldots, 2n-1\}$. Then there exists an extremal FW word w with period set $\{n, n+1, \ldots, 2n-1\}$ and either $|w| > 2n-1$, $\sharp w > 1$ or $|w| = 2n-1$, $\sharp w > 2$. We consider the prefix w' of w of length $|w| - n$. By Lemma 12, w' is a FW word with period $(n+1) - n = 1$ and therefore constant. In the former case we have $|w| \geq 2n$ and w has period n and it follows that $\sharp w' = \sharp w > 1$. This yields a contradiction. In the latter case w has period n and $|w'| = n - 1$ implies $\sharp w' \geq \sharp w - 1 > 1$. Again this yields a contradiction. Thus $u[j]$ is an extremal FW word for some period set $P[j]$.

Next we apply induction on k, starting from $k = j + 1$. Let $u[k-1]$ be an extremal FW word with period set $P[k-1]$. As explained in the proof of Theorem 4, by Lemmas 10 and 11, $u[k]$ is a FW word for period set $P[k]$. Suppose it is not extremal. Let u' be an extremal FW word for period set $P[k]$.

If $u[k] = u[k-1]v[k]u[k-1]$, then $|u'| \geq |u[k]| = 2|u[k-1]| + 1$. Let v' be the suffix of u' of length $|u[k]| - |u[k-1]|$. Then $u' = u''v'$ with $|u''| = |u'| - |u[k]| + |u[k-1]| \geq |u[k-1]$. By Lemma 12, u'' has all periods occurring in $P[k-1]$. By our assumption that $u[k-1]$ is an extremal FW word for $P[k-1]$ we have that either $|u''| \leq |u[k-1]|$ or $|u''| > |u[k-1]|$ and u'' has

$$\gcd(|u[k-1]| - |u[k-2]|, |u[k-1]| - |u[k-3]|, \ldots, |u[k-1]| - |u[0]|)$$

as a period. In the former case we obtain $|u''| = |u[k-1]|, \sharp u'' \leq \sharp u[k-1]$, hence $|u'| = |u[k-1]| + |v'| = |u[k]|$ and $u' = u''v''u''$ where $|v''| = 1$. Then $\sharp u[k] - 1 = \sharp u[k-1] \geq \sharp u'' \geq \sharp u' - 1$. Thus $u[k]$ is an extremal word for $P[k]$ itself. In the latter case u'' is constant, since the gcd of the period set is a period of u'' and it equals 1 by Lemma 6(a). Moreover, $|u''| \geq |u[k-1]| + 1 = |u[k]| - |u[k-1]| = |v'|$. Since $u' = u''v'$ and u' has period $|v'|$, we have $\sharp u' = \sharp u'' = 1$. This contradicts that u' is an extremal FW word for $P[k]$.

If $u[k] = u[k-1]u'[k-1]$ where $u[k-1] = u[l-1]u'[k-1]$ for some l with $1 \leq l < k$, then $|u'| \geq |u[k]| = 2|u[k-1]| - |u[l-1]|$. Let u'' be the prefix of u' of length $|u'| - (|u[k]| - |u[k-1]|) = |u'| - |u'[k-1]|$. By Lemma 12, u'' has all periods occurring in $P[k-1]$. As above, it follows that $|u''| \leq |u[k-1]|$ or u'' is a constant word with $|u''| > |u[k-1]|$. In the former case we have $|u'| =$

$|u''| + |u[k]| - |u[k-1]| \le |u[k]|$ and $|u'| \ge |u[k]|$ so that $|u'| = |u[k]|$. Furthermore u' has period $|u[k]| - |u[k-1]$ from $P[k]$ and u'' is a prefix of u' of length

$$|u'| - |u'[k-1]| \ge |u[k]| - |u[k]| + |u[k-1]| \ge |u'[k-1] = |u[k]| - |u[k-1]|.$$

Hence $\sharp u'' = \sharp u'$. Since u' is an extremal FW word for period set $P[k]$, and $u[k]$ with $|u[k]| = |u'|$ has also period set $P[k]$ but is not an extremal FW word for $P[k]$, we have $\sharp u' > \sharp u[k]$. Thus $|u''| = |u'| - |u'[k-1]| = |u[k]| - |u'[k-1]| = |u[k-1]|$ and $\sharp u'' = \sharp u' > \sharp u[k] = \sharp u[k-1]$ which contradicts that $u[k-1]$ is a FW word for $P[k-1]$. In the latter case $\sharp u' = \sharp u'' = 1$ contradicting that u' is an extremal FW word. This completes the induction step. We conclude that $u = u[K]$ is an extremal FW word for period set $P[K]$. □

6. Further Properties of FW Words

Corollary 13. *If w is an extremal FW word, then every palindromic prefix u of w with $\sharp u > 1$ is an extremal FW word.*

Corollary 14. *If w is a FW word, then every pseudo-palindromic prefix u of w is a FW word.*

The truth of the analogous statements for standard episturmian words and Fischler words follows immediately from their definitions.

Proof of Corollary 13. It suffices to prove it for the largest palindromic prefix of w, since thereafter we can apply induction. Suppose w is an extremal FW word for period set $q_1, q_2, \ldots, q_r$. Let u be the largest palindromic prefix of w. Then, by Lemma 5, $|w| - |u|$ is the shortest period of w. Hence, by Lemma 7, u has periods $q_1, q_2 - q_1, \ldots, q_r - q_1$. Suppose u is not an extremal FW word. Then there exists a non-constant word v with periods $q_1, q_2 - q_1, \ldots, q_r - q_1$ and $|v| > |u|$. Let v' be the suffix of v of length q_1 and consider the word vv'. By Lemma 8 the word has periods $q_1, q_2, \ldots, q_r$. Furthermore it is non-constant and has length $> |u| + q_1 = |w|$. This contradicts that w is an extremal FW word. □

Proof of Corollary 14. This proof is similar to the proof of Corollary 13. □

Ackowledgement. We thank the referee for his or her valuable comments.

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