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NEITHER $\prod_{k=1}^{n}(4k^2+1)$ NOR $\prod_{k=1}^{n}(2k(k-1)+1)$ IS A PERFECT SQUARE

Jin-Hui Fang [1]
Department of Mathematics, Nanjing Normal University, Nanjing 210097, P. R. China
fangjinhui1114@163.com

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Abstract

In this paper, by employing Cilleruelo's method, we prove that neither $\prod_{k=1}^{n}(4k^2+1)$ nor $\prod_{k=1}^{n}(2k(k-1)+1)$ is a perfect square for all $n > 1$, which confirms a conjecture of Amdeberhan, Medina, and Moll.

1. Introduction

Recently, there has been a renewed interest in investigating whether or not certain product sequences contain perfect squares. Amdeberhan, Medina and Moll [1] proposed several conjectures in this direction. Soon after, J. Cilleruelo [2] proved that the number

$$\prod_{k=1}^{n}(k^2+1)$$

is not a perfect square provided $n > 3$, which settles Conjecture 5.1 in [1]. Amdeberhan, Medina and Moll [1] also proposed the following conjecture.

Conjecture 1 ([1, Conjecture 7.1]). *The even and odd parts of* $\prod_{k=1}^{n}(k^2+1)$ *are defined by*

$$t_n := \prod_{k=1}^{n}(1+2k(k-1)), \text{ and } s_n := \prod_{k=1}^{n}(1+4k^2).$$

These products involve the triangular and square numbers respectively. Neither of them is a perfect square.

In this paper, by employing Cilleruelo's method, we confirm this conjecture.

Theorem 2. *Neither* $\prod_{k=1}^{n}(4k^2+1)$ *nor* $\prod_{k=1}^{n}(2k(k-1)+1)$ *is a perfect square for all* $n > 1$.

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2. Proof of Theorem 2

Proof. In this paper, p always denotes a rational prime.

Let $P_n = \prod_{k=1}^{n}(4k^2+1)$. Assume that P_n is a perfect square for some $n > 1$. Let p be a prime with $p|P_n$. Then $p^2|P_n$ and $p \equiv 1 \pmod 4$. If there exists a positive integer $k \le n$ with $p^2|4k^2+1$, then $p \le \sqrt{4n^2+1} < 2n+1$. Thus $p < 2n$. If there exist i, j, $1 \le i < j \le n$ with $p|4i^2+1$ and $p|4j^2+1$, then $p|4(j-i)(j+i)$. Thus either $p|j-i$ or $p|j+i$. So $p \le j+i < 2n$.
Hence

$$P_n = \prod_{\substack{p<2n \\ p\equiv 1 \pmod 4}} p^{\alpha_p}.$$

Let $n! = \prod_{p \le n} p^{\beta_p}$. Since $4^n n!^2 < P_n$, we have

$$\sum_{p\le n} \beta_p \log p < \frac{1}{2} \sum_{\substack{p<2n \\ p\equiv 1 \pmod 4}} \alpha_p \log p - n\log 2. \tag{1}$$

Since each interval of length p^j contains at most two solutions of $4x^2+1 \equiv 0 \pmod{p^j}$, we have

$$\alpha_p = \sum_{j \le \log(4n^2+1)/\log p} \#\{k \le n : p^j | 4k^2+1\} \le \sum_{j\le \log(4n^2+1)/\log p} 2\lceil n/p^j \rceil. \tag{2}$$

On the other hand

$$\beta_p = \sum_{j\le \log n/\log p} \#\{k \le n : p^j | k\} = \sum_{j \le \log n/\log p} \lfloor n/p^j \rfloor. \tag{3}$$

Thus we have

$$\begin{aligned} \alpha_p/2 - \beta_p &\le \sum_{j\le \log(4n^2+1)/\log p} \lceil n/p^j \rceil - \sum_{j\le \log n/\log p} \lfloor n/p^j \rfloor \\ &= \sum_{j \le \log n/\log p} (\lceil n/p^j \rceil - \lfloor n/p^j \rfloor) + \sum_{\log n/\log p < j \le \log(4n^2+1)/\log p} \lceil n/p^j \rceil \\ &\le \frac{\log(4n^2+1)}{\log p}. \end{aligned} \tag{4}$$

By (1) and (4) we have

$$\begin{aligned} \sum_{\substack{p\le n \\ p\not\equiv 1 \pmod 4}} \beta_p \log p &= \sum_{p \le n} \beta_p \log p - \sum_{\substack{p\le n \\ p\equiv 1 \pmod 4}} \beta_p \log p \\ &\le \frac{1}{2} \sum_{n<p<2n} \alpha_p \log p - n\log 2 + \log(4n^2+1)\pi(n;1,4), \end{aligned} \tag{5}$$

where $\pi(n; 1, 4)$ denotes the number of primes which are less than or equal to n and congruent to 1 modulo 4.

If $p > n$, then

$$\frac{\log(4n^2+1)}{\log p} < \frac{\log(n+1)^3}{\log(n+1)} = 3.$$

By (2) we have $\alpha_p \le 4$.

If $p \le n$, then by (3) we have

$$\begin{aligned}\beta_p &\ge \sum_{j \le \log n/\log p} \left(\frac{n}{p^j} - 1\right) = n\left(\frac{1 - p^{-1-\lfloor \log n/\log p \rfloor}}{1 - 1/p} - 1\right) - \lfloor \log n/\log p \rfloor \\ &\ge n\left(\frac{1 - 1/n}{1 - 1/p} - 1\right) - \lfloor \log n/\log p \rfloor = \frac{n-p}{p-1} - \lfloor \log n/\log p \rfloor \\ &\ge \frac{n-1}{p-1} - \frac{\log(4n^2+1)}{\log p},\end{aligned}$$

where the last inequality is based on the fact $p \le n$.
Thus, by (5) we have

$$(n-1) \sum_{\substack{p \le n \\ p \not\equiv 1 \pmod 4}} \frac{\log p}{p-1} < \log(4n^2+1)\pi(n) + 2 \sum_{n<p<2n} \log p - n \log 2,$$

where $\pi(n)$ denotes the number of primes which are less than or equal to n.

Now we use the Chebyshev's estimates

$$\sum_{p \le n} \log p \le 2n \log 2, \qquad \sum_{n<p<2n} \log p \le 2n \log 2$$

and (see [3])

$$\pi(x) \le \frac{x}{\log x}\left(1 + \frac{1.2762}{\log x}\right) \quad (x > 1)$$

to obtain

$$\sum_{\substack{p \le n \\ p \not\equiv 1 \pmod 4}} \frac{\log p}{p-1} < \frac{\log(4n^2+1)}{n-1}\left(\frac{n}{\log n} + \frac{1.2762n}{\log^2 n}\right) + \frac{3n}{n-1}\log 2.$$

We know that the right-hand side is monotonic decreasing. Actually, that quantity is less than 7.14 for $n \ge 702007$.

For $n \ge 702007$, we have

$$\sum_{\substack{p \le n \\ p \not\equiv 1 \pmod 4}} \frac{\log p}{p-1} \ge \sum_{\substack{p \le 702007 \\ p \not\equiv 1 \pmod 4}} \frac{\log p}{p-1} > 7.14,$$

which proves the theorem for $n \ge 702007$.

Finally we have to check that P_n is not a square for $2 \leq n < 702007$.

- $17 = 4 \times 2^2 + 1$. The next time that the prime 17 divides $4k^2 + 1$ is for $k = 17 - 2 = 15$. Hence P_n is not a square for $2 \leq n \leq 14$.
- $101 = 4 \times 5^2 + 1$. The next time that the prime 101 divides $4k^2 + 1$ is for $k = 101 - 5 = 96$. Hence P_n is not a square for $5 \leq n \leq 95$.
- $1297 = 4 \times 18^2 + 1$. The next time that the prime 1297 divides $4k^2 + 1$ is for $k = 1297 - 18 = 1279$. Hence P_n is not a square for $18 \leq n \leq 1278$.
- $739601 = 4 \times 430^2 + 1$. The next time that the prime 739601 divides $4k^2 + 1$ is for $k = 739601 - 430 = 739171$. Hence P_n is not a square for $430 \leq n \leq 739170$.

Therefore $\prod_{k=1}^{n}(4k^2 + 1)$ is not a perfect square. The proof that $\prod_{k=1}^{n}(2k(k-1) + 1)$ is not a perfect square is completely similar. This completes the proof of Theorem 2. □

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