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ON $N \mid \varphi(N)D(N)+2$ AND $N \mid \varphi(N)\sigma(N)+1$

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Abstract

For any positive integer n, let $\varphi(n)$ denote the Euler totient function, $\sigma(n)$ denote the sum of the positive divisors of n and $d(n)$ denote the number of positive divisors of n. It is clear that if $n>4$ is an integer such that either $n \mid \varphi(n)d(n)+2$ or $n \mid \varphi(n)\sigma(n)+1$, then n is squarefree. The following results are proved: (1) Let t and n be two positive integers with $t \geq 2$ and $n|\varphi(n)d(n)+2$. If n has exactly t prime factors $p_1<p_2<\cdots<p_t$, then $p_i<(t\cdot 2^{t-1})^{2^{i-1}} (1\leq i\leq t)$. (2) If n is composite and $n|\varphi(n)\sigma(n)+1$, then n has at least three distinct prime factors.

1. Introduction

For any positive integer n, let $\varphi(n)$ denote the Euler totient function, $\sigma(n)$ denote the sum of the positive divisors of n and $d(n)$ denote the number of positive divisors of n. Obviously, if n is prime, then it divides $\varphi(n)d(n)+2$. Is this true for any composite n other than $n=4$? The question was posed in [1, B37]. Jud McCranie finds no others with $n<10^{10}$ (see [1, B37]). It is easy to see that if such n exists, then n is squarefree. In this paper, we prove the following result.

Theorem 1. *Let t and n be two positive integers with $t\geq 2$ and $n|\varphi(n)d(n)+2$. If n has exactly t prime factors $p_1<p_2<\cdots<p_t$, then $p_i<(t\cdot 2^{t-1})^{2^{i-1}} (1\leq i\leq t)$.*

Remark. In fact, similarly to the proof of Theorem 1, we can get a more precise inequality $p_i<2^{t-1}(t-i+1)(p_1-1)\cdots(p_{i-1}-1)$ for $2\leq i\leq t$. By using this we have proved that there are no integers n with $2\leq t\leq 4$ and $n \mid \varphi(n)d(n)+2$.

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Note that if n is prime, then it divides $\varphi(n)\sigma(n) + 1$. Now we consider the question: Does there exist any composite n with $n \mid \varphi(n)\sigma(n) + 1$? It is clear that if such an integer n exists, it must be squarefree. For this question, we prove that:

Theorem 2. *If n is composite and $n|\varphi(n)\sigma(n) + 1$, then n has at least three distinct prime factors.*

Remark. If $n \mid \varphi(n)d(n) + 2$ and $n \mid \varphi(n)\sigma(n) + 1$, then we have $n \mid \varphi(n)(2\sigma(n) - d(n))$. By $n \mid \varphi(n)\sigma(n) + 1$ we have $(n, \varphi(n)) = 1$ and n is squarefree. Thus $n \mid 2\sigma(n) - d(n)$. It is not difficult to prove that there are no squarefree composite n with $n \mid 2\sigma(n) - d(n)$ except for $n = 70$. But $70 \nmid \varphi(70)d(70) + 2$. So there is no composite n with $n \mid \varphi(n)d(n) + 2$ and $n \mid \varphi(n)\sigma(n) + 1$. We pose the following question.

Question. Determine all composite numbers such that $n \mid 2\sigma(n) - d(n)$.

Remark. There are only 14 such $n < 10^8$, namely 18, 70, 88, 132, 780, 11096, 17816, 518656, 1713592, 9928792, 11547352, 13499120, 17999992 and 89283592. It is easy to prove that if $2^{k+1} - k - 2$ is prime, then $2^k(2^{k+1} - k - 2)$ is such an integer. We have found that $2^{k+1} - k - 2$ is prime for $k = 3, 9, 13, 15, 25, 49, 55, 69, 115$. We pose the following conjectures.

Conjecture 1. There are infinitely many primes of the form $2^{k+1} - k - 2$, where k is a positive integer.

Conjecture 2. There are no odd composite n such that

$$n \mid 2\sigma(n) - d(n).$$

2. Proof of the Theorems

Proof of Theorem 1. We use induction on i to prove

$$p_i < (t \cdot 2^{t-1})^{2^{i-1}} (1 \le i \le t). \tag{1}$$

By $n \mid \varphi(n)d(n)+2$ we have $p_1p_2\cdots p_t \mid 2^t(p_1-1)(p_2-1)\cdots(p_t-1)+2$. Now we consider the case $p_1 \ge 3$. Then $p_1p_2\cdots p_t \mid 2^{t-1}(p_1 - 1)(p_2 - 1)\cdots(p_t - 1) + 1$. So there exists a positive integer k such that

$$2^{t-1}(p_1 - 1)(p_2 - 1)\cdots(p_t - 1) + 1 = kp_1p_2\cdots p_t. \tag{2}$$

If $k \ge 2^{t-1}$, then by (2) we have $2^{t-1}(p_1 - 1)(p_2 - 1)\cdots(p_t - 1) + 1 \ge 2^{t-1}p_1p_2\cdots p_t$. Thus, $2^{t-1}(p_1p_2\cdots p_t - (p_1 - 1)(p_2 - 1)\cdots(p_t - 1)) \le 1$. Obviously, this is impossible for $t \ge 2$. Hence, $k \le 2^{t-1} - 1$. By (2) we have

$$2^{t-1}(1 - \frac{1}{p_1})(1 - \frac{1}{p_2})\cdots(1 - \frac{1}{p_t}) + \frac{1}{p_1p_2\cdots p_t} = k. \tag{3}$$

So

$$2^{t-1}-1 \geq k > 2^{t-1}(1-\frac{1}{p_1})^t \geq 2^{t-1}(1-\frac{t}{p_1}).$$

The last inequality is based on the fact that $(1-x)^\alpha \geq 1-\alpha x$ for $0<x<1$ and $\alpha \geq 1$. Hence $p_1 < t2^{t-1}$. Now suppose that (1) is true for $i < j \leq t$. Since

$$\begin{aligned} k &= 2^{t-1}(1-\frac{1}{p_1})\cdots(1-\frac{1}{p_t}) + \frac{1}{p_1p_2\cdots p_t} \\ &= 2^{t-1}(1-\frac{1}{p_1})\cdots(1-\frac{1}{p_{j-1}})(1-\frac{1}{p_j})\cdots(1-\frac{1}{p_t}) + \frac{1}{p_1p_2\cdots p_t} \\ &\leq 2^{t-1}(1-\frac{1}{p_1})\cdots(1-\frac{1}{p_{j-1}})(1-\frac{1}{p_j}) + \frac{1}{p_1p_2\cdots p_j} \\ &= 2^{t-1}(1-\frac{1}{p_1})\cdots(1-\frac{1}{p_{j-1}})\left(1-\frac{1}{p_j}+\frac{1}{2^{t-1}(p_1-1)\cdots(p_{j-1}-1)p_j}\right) \\ &< 2^{t-1}(1-\frac{1}{p_1})\cdots(1-\frac{1}{p_{j-1}}), \end{aligned}$$

we have

$$2^{t-1}(1-\frac{1}{p_1})\cdots(1-\frac{1}{p_{j-1}}) - k > 0.$$

Since the left-side of the above inequality is a positive rational number, it is at least as large as $1/(p_1p_2\cdots p_{j-1})$. Thus

$$2^{t-1}(1-\frac{1}{p_1})\cdots(1-\frac{1}{p_{j-1}}) - k \geq \frac{1}{p_1p_2\cdots p_{j-1}}.$$

Hence

$$k \leq 2^{t-1}(1-\frac{1}{p_1})\cdots(1-\frac{1}{p_{j-1}})\left(1-\frac{1}{2^{t-1}(p_1-1)(p_2-1)\cdots(p_{j-1}-1)}\right). \tag{4}$$

By (3) we have

$$\begin{aligned} k &= 2^{t-1}(1-\frac{1}{p_1})(1-\frac{1}{p_2})\cdots(1-\frac{1}{p_t}) + \frac{1}{p_1p_2\cdots p_t} \\ &> 2^{t-1}(1-\frac{1}{p_1})\cdots(1-\frac{1}{p_{j-1}})(1-\frac{1}{p_j})^t \\ &\geq 2^{t-1}(1-\frac{1}{p_1})\cdots(1-\frac{1}{p_{j-1}})(1-\frac{t}{p_j}). \end{aligned}$$

Combining the above inequality with (4), we have

$$1-\frac{t}{p_j} < 1-\frac{1}{2^{t-1}(p_1-1)(p_2-1)\cdots(p_{j-1}-1)}.$$

Thus, by the induction hypothesis, we have

$$p_j < 2^{t-1}t(p_1-1)\cdots(p_{j-1}-1) < (t2^{t-1})(t2^{t-1})^{1+2+\cdots+2^{j-2}} = (t2^{t-1})^{2^{j-1}}.$$

So when $p_1 \geq 3$ we have proved that $p_i < (t2^{t-1})^{2^{i-1}}$ for all $1 \leq i \leq t$. Now, we consider the case $p_1 = 2$. By $n \mid \varphi(n)d(n) + 2$, we have $2p_2 \cdots p_t \mid 2^t(p_2 - 1) \cdots (p_t - 1) + 2$. That is,

$$p_2 \cdots p_t \mid 2^{t-1}(p_2 - 1) \cdots (p_t - 1) + 1.$$

Similarly to the case $p_1 \geq 3$, we can prove that $p_i < (t2^{t-1})^{2^{i-1}} (1 \leq i \leq t)$ when $p_1 = 2$. □

Before the proof of Theorem 2, we first introduce a lemma.

Lemma. *There do not exist positive integers a, b with $a > 1$ and $b > 1$ such that $ab|a^2+b^2-2$.*

Proof of the lemma. Without loss of generality, we may assume that $a \leq b$. Now we use induction on b to prove the lemma.

It is easy to see that $ab \nmid a^2 + b^2 - 2$ when $b = 2$. Suppose that the lemma is true for $b < k$. Now we consider the case $b = k$. Suppose that there is an integer a with $k \geq a \geq 2$ and $ak \mid a^2 + k^2 - 2$. Then there exists a positive integer l with

$$a^2 + k^2 - 2 = lak. \tag{5}$$

By the Euclidean algorithm, there exist nonnegative integers q, r with $0 \leq r < a$ such that $k = aq + r$. By (5) we have

$$l = \frac{a^2 + k^2 - 2}{ak} = \frac{a^2 + (aq + r)^2 - 2}{a(aq + r)} = q + \frac{r(aq + r) + a^2 - 2}{a(aq + r)}.$$

Since

$$0 < \frac{r(aq + r) + a^2 - 2}{a(aq + r)} < 2$$

and by the above equation it is an integer, we have

$$\frac{r(aq + r) + a^2 - 2}{a(aq + r)} = 1 \tag{6}$$

and $l = q + 1$. By (6) and $l = q + 1$ we have

$$la(a - r) = a^2 + (a - r)^2 - 2. \tag{7}$$

If $r = 0$, then by (7) we have $a^2 \mid 2$, a contradiction with $a > 1$. So $r > 0$ and $k = aq+r > a$. By (7) and the induction hypothesis, we have $a - r = 1$. Thus by (7) we have $la = a^2 - 1$. Hence $a \mid 1$, which is impossible for $a > 1$. □

Proof of Theorem 2. Assume that $n = p_1p_2$, where p_1, p_2 are distinct primes. By $n \mid \varphi(n)\sigma(n) + 1$ we have $p_1p_2 \mid (p_1^2 - 1)(p_2^2 - 1) + 1$. Hence, $p_1p_2 \mid p_1^2 + p_2^2 - 2$. Now Theorem 2 follows from the lemma. □

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Reference

[1] Richard K.Guy, Unsolved Problems in Number Theory, First Edition, Springer-Verlag, 1981; Third Edition, Springer-Verlag, 2004.