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SIMPLIFICATION OF PARTIZAN GAMES IN MISÈRE PLAY

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Abstract

We demonstrate the similarities and differences between normal play games and misère play combinatorial games. In particular, we examine the question of why the analysis of misère play games is more difficult than their normal counterparts. Misère games form a partial order similar to that of normal play games and we show the form of a game can be simplified through domination and reversibility, defined appropriately.

1. Introduction

The games we consider are combinatorial games—two players, called Left and Right, who move alternately, with no chance devices and with perfect information. There are two *play* or *winning* conventions: in *normal play*, the first player who has no move loses, whereas in *misère play* the first player who has no move *wins*. Under both play conventions, games are partitioned into four outcome classes: $\mathcal{L}$ is the set of games that Left wins regardless of whether he or she plays first or second; $\mathcal{R}$ is the set of games that Right wins regardless of whether he or she plays first or second; $\mathcal{P}$ is the set of games that the second player wins; and $\mathcal{N}$ is the set of games that the first player wins.

Since the difference between the two conventions only involves a change in the winning condition it would be easy to think that games that were Left wins are now Right wins, and games that were first player wins are now second player wins and vice versa. This is not the case. A simple counterexample is NIM played with two heaps, both of size 2. This is

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a second player win regardless of the play convention. In [2, 3] it is mentioned that misère games (games played under the misère convention) are harder to analyze. A survey of the literature reveals only a few papers either devoted to the analysis of a single game [6, 9] or to a restricted class of games. See [1, 4, 5, 10] for examples. Recently, however, Plambeck [7] (see also [8]) has made a major step forward in analyzing impartial misère games. This paper was stimulated by this breakthrough and we first try to answer the question: "What makes misère games so hard?" Or more precisely, the question:

Question 1 *"What makes misère games so miserable?" from a mathematical point-of-view.*

Given games G and H, the *disjunctive sum* of G and H is the game in which on each turn a player chooses one of the two games and makes a move in it. This is denoted by $G + H$. Also, $-G$ denotes the game where the roles of Left and Right are reversed. The most basic game is $\{|\}$, which is denoted by 0. The structure of normal play games is given in [2, 3]. We give the salient features below. (For any undefined terms the reader should consult one of these books.)

Under the normal play convention, games with the disjunctive sum:

1. form a partial ordered abelian group;
2. are partitioned into equivalence classes; and
3. each equivalence class has a unique *canonical form* obtained by removing dominated options and bypassing reversible options.

Expanding briefly on these points, under the normal play convention, two games, G and H, are defined as being *equal*, denoted $G = H$, if for all games K, $G + K$ has the same outcome as $H + K$. This is an equivalence relation. All second player win games, i.e. those in $\mathcal{P}$, are equivalent to 0. This is the identity of the group. The partial order is given by $G \geq H$ if Left wins $G + K$ whenever he wins $H + K$, and if $G \in \mathcal{L}$ then $G > 0$ and $G \in \mathcal{R}$ then $G < 0$. Operationally, the important results are:

$$G = H \text{ iff } G - H = 0; \text{ and } G \geq H \text{ iff } G - H \geq 0.$$

Finally, if $G \geq X$ and $H \geq Y$ then $G + H \geq X + Y$.

A hint that the analysis of misère games will be more difficult than normal play is that there is no equivalence class that $G - G$ lies in for all G. Recall in impartial games that $-G = G$, and again, in misère NIM $2 + 2$ is in $\mathcal{P}$ but $1 + 1$ is in $\mathcal{N}$. The short answer to Question 1 and what we present in this paper is that

1. misère games are partitioned into equivalence classes, but they may be small in comparison to those induced by normal play—as we shall prove, 0 is in a class by itself (see Corollary 8);

2. there are concepts similar to dominated and reversible options that can be used to simplify games, thus making a start towards finding a canonical form; and

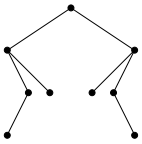
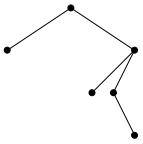
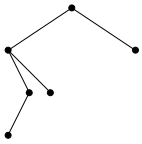
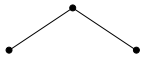
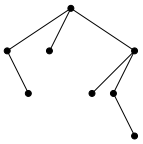
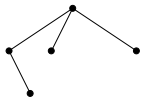
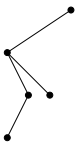
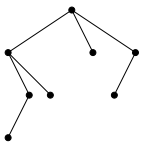
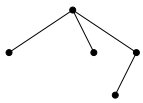
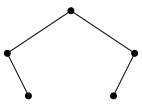
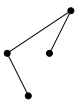
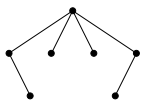
3. misère games form a partial order, but many games are incomparable which makes this less useful than in normal play games.

2. The Disjunctive Sum of Two Misère Games

Despite the earlier NIM example of a game being in two different classes under the two conventions, there still might be some relationship between normal play and misère play outcomes. The first result shows that this is false.

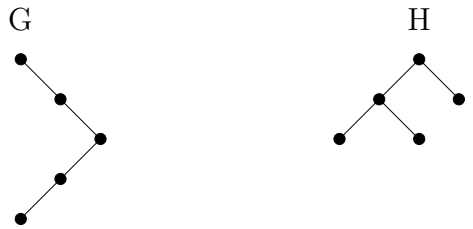
Theorem 2 *Let C_1 and C_2 be outcome classes. Then there exists a game G for which $G \in C_1$ in normal play and $G \in C_2$ in misère play.*

Proof. The following tables demonstrates an example of a game G for each of the 16 possible combinations for C_1 and C_2:

Normal → Misère ↓	P	L	R	N
P				
L				
R				
N				

□

Despite there not being a correlation between the outcome class in normal play and the outcome class in misère play, does misère play preserve some of the structure of the outcome classes present in normal play? For example, if $G, H \in \mathcal{L}$ does it follow that $G + H \in \mathcal{L}$? This is false and in the worst way. With G and H as given below,



both are in $\mathcal{L}$ but $G + H \in \mathcal{R}$. In fact, there is no correlation between the individual outcomes and the outcome of the sum, as can be seen from the next result.

Theorem 3 *Let C_1, C_2 and C_3 be outcome classes. There exists games G and H with $G \in C_1$, $H \in C_2$ and $G + H \in C_3$.*

Proof. At the end of this paper, in the Appendix, there is a dictionary of positions demonstrating that the outcome class of the disjunctive sum of G and H do not depend on the outcome classes of G and H played in isolation. □

3. Equivalence Classes of Misère Games

Without a partial order it is difficult to build a reduction scheme for misère games similar to that of normal play games but it is not impossible.

First, we need definitions of equivalence.

Definition 4 *Given games G and H, $G \cong H$ iff G and H have identical game trees.*

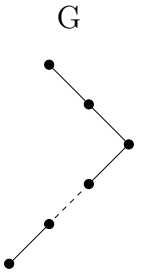
Definition 5 *Given games G and H, $o^-(G) = o^-(H)$ iff G and H are in the same outcome class.*

Definition 6 *Given games G and H, $G = H$ iff $o^-(G + K) = o^-(H + K)$ for all K. We say that G is* indistinguishable *from H.*

We begin by looking at the equivalence class [0].

Theorem 7 *If $H \ncong 0$ then $H \neq 0$.*

Proof. We begin by letting H be some non-trivial game. Without loss of generality we can assume it has some left option. We then consider G:



We can now show that $H \neq 0$ by showing that $o^-H + G \neq o^-0 + G$. The string of consecutive Left moves in G is longer than the depth of the game tree of H. We can easily verify that $G \in \mathcal{L}$. Now, when we play $G + H$, Right can win playing first by playing in G. Since Left has no moves in G^R, he must play in H. Right can now play again to $G^{RR} + H^L$. Since G^{RR} is a string of left moves longer than any sequence of play in H^L we can deduce that Right will win this game. Since Right can win $G + H$ playing first we know $G + H \notin \mathcal{L}$ and hence $o^-(G + H) \neq o^-(G)$. □

Corollary 8 *The equivalence class under $=$ of 0 contains only one element.*

Proof. This follows directly from the previous theorem □

As we can see, this is a major difference from normal play games. Rather than having all games within an outcome class being equivalent to 0, we discover that in misère play there are no other games which are equivalent to 0. This naturally brings up the question of whether or not all equivalence classes are trivial. The following theorem demonstrates that this is fortunately not the case.

Theorem 9 *There exists games G and H with $G \ncong H$ such that $G = H$.*

Proof. Consider the following games G and H:

Let K be a game and assume Right can win $G + K$ playing first. Then Right can also win $H + K$ playing first by following the same strategy while ignoring the string of moves starting at x. Likewise, if Right can win $G + K$ playing second, he can win $H + K$ playing second.

Now assume Left can win $G + K$ playing first. We would like to show that Left can win $H + K$ playing first. This is equivalent to showing that Right cannot win $H + K$ playing second. We proceed by contradiction and assume that Right can win playing second. If Right never plays to x, then we can use the same strategy when playing $G + K$, giving Right a winning strategy playing second which gives a contradiction since we originally assumed Left would win playing first. So, Right must play to x at some point in her winning strategy. But then Right can win $G + K$ playing second by ignoring the option to y. This again gives a contradiction so we can conclude that Right cannot win $H + K$ playing second.

So, if Left can win $G + K$ playing first, Left can win $H + K$ playing first. Using a similar argument we can show that if Left can win $G + K$ playing second, then Left can win $H + K$ playing second.

Therefore, putting the results together, we can conclude that $H + K$ has the same outcome class as $G + K$. □

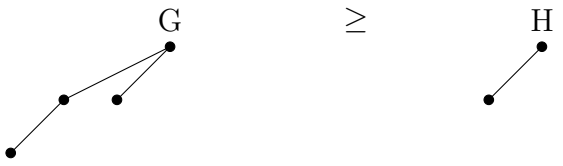
This result tells us that misère games have non-trivial equivalence classes under $=$. We can also see from the construction that there is something similar to domination for misère games. Is there also an operation analogous to reversible options? And if so, does this lead to a canonical form? We make precise a notion of a dominated and of a reversible option, but the question of a canonical form remains open.

Definition 10 *Given games G and H,*

$$G \geq H \text{ iff Left wins } H + K \text{ implies Left wins } G + K \ \forall K$$

Using this basic definition for $G \geq H$ (that applies to all games in both normal and misère play), we can find a couple of general constructions for misère games which are indistinguishable. Much like normal play games, we find rules similar to domination and reversibility.

We note that there are misère games G and H with $G \neq H$ which satisfy the above definition. As an example, we can choose the following two games:



The following three theorems are motivated by the process of simplifying games in normal play. In particular, applying Corollary 12 is similar to removing a dominated option while

applying Theorems 13 and 14 are similar to bypassing reversible moves.

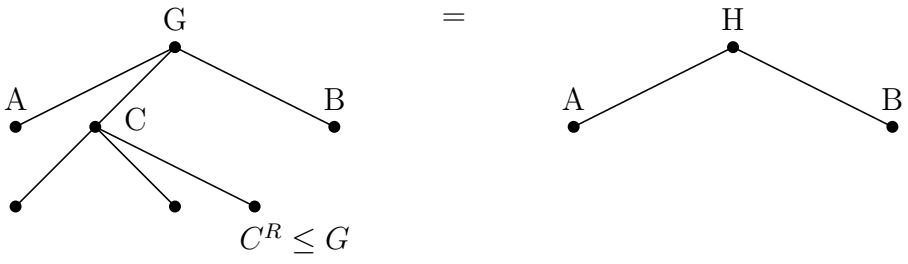
Theorem 11 *If* $(H^R \supseteq G^R \neq \emptyset$ *or* $H^R = G^R = \emptyset)$ *and* $(G^L \supseteq H^L \neq \emptyset$ *or* $G^L H^L = \emptyset)$ *then* $G \geq H$.

Proof. Let K be a game so that Left wins $H + K$. If $G^L \supseteq H^L \neq \emptyset$ or $G^L = H^L = \emptyset$ then playing first, he can adopt the same winning strategy when playing $G + K$. If right plays first, he cannot win $H + K$. Since his moves in $G + K$ are a subset of the moves available in $H + K$, he cannot win $G + K$ either. Therefore, Left wins $G + K$ and we get that $G \geq H$. □

Corollary 12 *Given the game* $G = \{G^{L_1}, G^{L_2}, G^L | G^R\}$, *if we have that* $G^{L_1} \geq G^{L_2}$ *then* $G = \{G^{L_1}, G^L | G^R\}$.

Proof. This follows from the definition of $\geq$. In any line of play, Left can always do at least as well by playing to G^{L_1} as he would by playing to G^{L_2}. Therefore, the option to G^{L_2} will never be played and we get that $G = \{G^{L_1}, G^L | G^R\}$. □

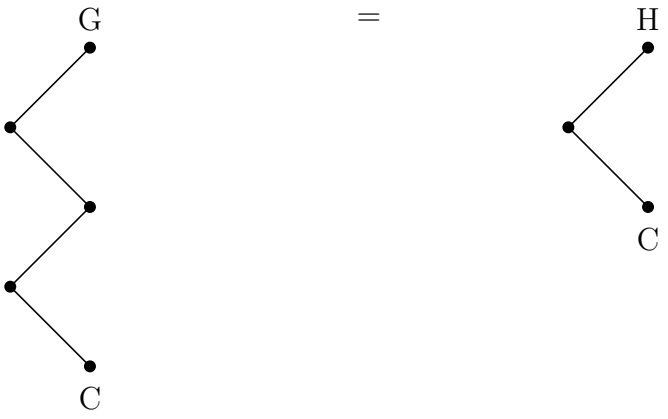
Theorem 13 *Let* $H = \{A|B\}$ *and* $G = \{A, C|B\}$ *where* A *is non-empty and* $A \supseteq C^{RL} \neq \emptyset$. *Furthermore, assume* $B \subseteq C^{RR}$ *if* B *non-empty and* $C^{RR} = \emptyset$ *if* $B = \emptyset$. *Then* $G = H$.



Proof. To begin, we can note that $G \geq H \geq C^R$ by applying theorem 12. Therefore, for any game K, if Left has a winning strategy in $H + K$, he also has a winning strategy in $G + K$.

Now we assume that Right has a winning strategy in $H + K$ for some K. When playing $G + K$, if Left never plays to C, we know he will lose since Right will follow their winning strategy from $H + K$. So, if Left can win $G + K$ he must at some point play to C. Right can immediately play to C^R. Since $C^R \leq H$ and we know Right can win $H + K$, Right can also win $C^R + K$. So, Left has no good move in $G + K$ and Right wins. Therefore, we can conclude that $G = H$. □

Theorem 14 *Let C be any set of games and let G and H be given as in the diagram below. Then $G = H$.*



Proof. Assume Left wins $H + K$ for some K. Since he has a move in H, it must at some point be a move he makes during his winning strategy. From this point, either Left will win before any further moves have been made in H, or Right will eventually move to $C + K'$ which we know is a winning position for Left. Moreover, any possible K' during Left's winning strategy makes $C + K'$ a winning position for Left.

Now we consider the game $G + K$. Left follows the same strategy he would as if he were playing $H + K$ moving in H when he normally would have moved in G. Now, if Right eventually plays in G, Left immediately responds in G as well. After Left's move, we are in a winning position for Left because it is identical to a position along his winning strategy in $H + K$. This shows that $G \geq H$.

Now we assume Right can win $H+K$ for some K and we note that $G^{LR} = H$. In the game $G + K$, Right follows the same strategy except that if Left plays in G, Right immediately responds in G to give the position $H + K'$. Since K' was arrived at by following a winning strategy as if playing $H + K$, we know it is a winning strategy for Right. Thus we get that $H \geq G$ and we can conclude that $G = H$. □

Naturally, we also get reversibility with Right's moves as well. This can be seen by taking the negatives of the games in the previous two theorems.

Theorem 15 *Repeated application of Theorems 13, 14 and Corollary 12 to a game A leaves a game B with $A = B$.*

Proof. Removing a dominated left option from a subgame G involves finding G^{L_1} and G^{L_2} with $G^{L_1} \geq G^{L_2}$. We can then simplify G by removing his option to G^{L_2}. Likewise, this can

be done with Right's options by finding G^{R_1} and G^{R_2} with $H^{R_1} \geq G^{R_2}$ in which case G^{R_1} is removed.

For some game K, consider $A + K$. By our definition of $\geq$, we know in each case that a winning strategy for either player will never use a dominated option in A since they can always do at least as well by playing to the other option. Moreover, a player cannot do any better by having removed the dominated option since it was never part of any winning strategy in the first place. Therefore, after removing a dominated option from A to get B, we know that $A + K$ and $B + K$ have the same outcome, i.e. $A = B$.

Bypassing a reversible move involves finding a subgame G of A with the form demonstrated in theorems 13 or 14. We then replace G by the game H from the appropriate theorem to get the new game B. Now, if playing B we end up at H, we know that $o^-(G+K) = o^-(H+K)$ for all K. Therefore, we can conclude that $o^-(A+K) = o^-(B+K)$ for all K, i.e. $A = B$.

Since $=$ is an equivalence relation, repeated use of these steps always leaves us within the same equivalence class. We also can see that each step leaves us with a simpler game tree with fewer vertices. □

This leaves us with the problem of determining if repeated use of this new interpretation of domination and reversibility leave us with a unique representative from each of the equivalence classes. The main difficulty is that we cannot apply the equivalent proof used for normal play games. The method used there is to consider two games G and H which are both arrived at by repeated use of domination and reversibility from a common game K. Now, we know that $G = H$ and therefore $G - H = H - H$. In normal play, this implies $G - H \in \mathcal{P}$ but this is not the case in misère play, as was mentioned in the introduction.

To resolve this issue, we must first determine if there are any other ways to simplify misère games while preserving equivalence. Then, we need a technique which says that if games G and H have no dominated or reversible options and $G = H$, then $G \cong H$. This is still an open problem.

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Co-authors

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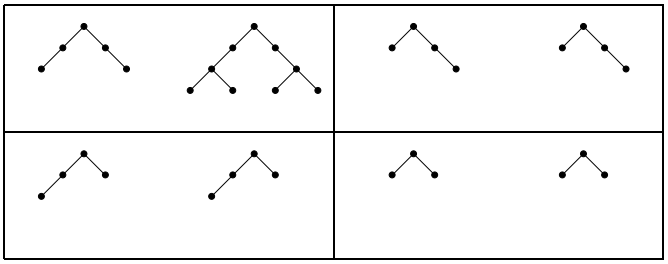
Appendix

The following table shows that the outcome class of the sum of two games does not depend on outcome classes of the games themselves. Each cell in the table shows two games G and H. The outcome class of G is C_1 and the outcome class of H is C_2. Its position shows the outcome class of $G + H$ according to the following diagram. Some of the 10 possible combinations are not shown as they can be derived by taking the negatives of some other entry.

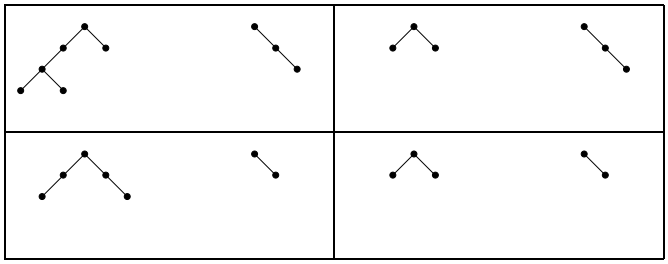
$C_1 + C_2$:

P	L
R	N

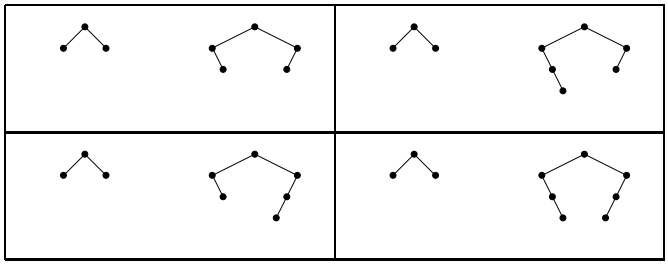
P+P:



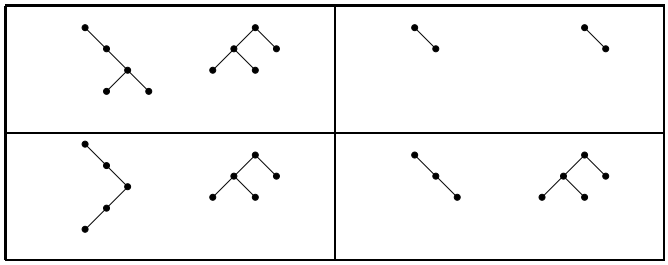
P+L:



P+N:



L+L:



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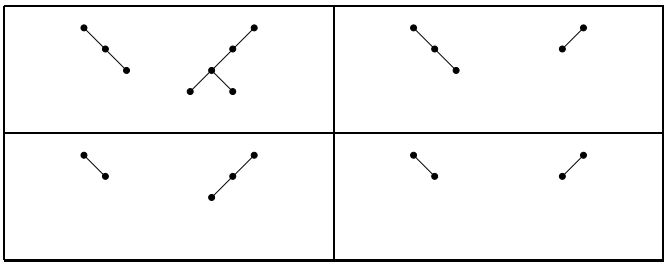

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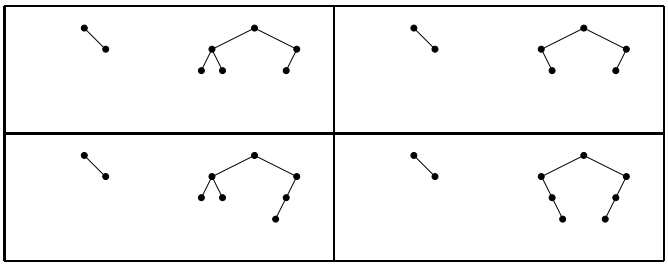
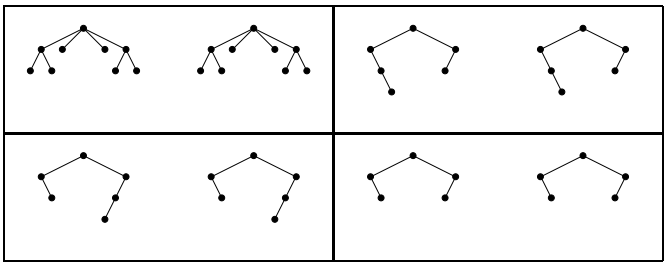
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