

ON CONGRUENCE PROPERTIES OF CONSECUTIVE VALUES OF $P(N, M)$

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Received: 3/27/06, Revised: 1/24/07, Accepted: 2/28/07, Published: 3/27/07

Abstract

In the late 19th century, Sylvester and Cayley investigated the properties of the partition function $p(n, m)$. This function enumerates the partitions of a non-negative integer n into exactly m parts. The author here extends techniques developed in a previous publication to investigate the congruence properties of consecutive values of the function $p(n, m)$.

1. Introduction and Statement of Results

The study of congruence properties of the general partition function $p(n)$ has been a well-spring of mathematical research from the moment Ramanujan [9] first made his groundbreaking studies nearly eighty years ago. Inspired by Ramanujan's work, the results of Watson,[10] Atkin,[3] Dyson,[4] and Andrews and Garvan [2] have been fertile ground for the accomplishments of Ono [8] and most recently Mahlburg [7]. The results of this paper deal with an intermediate partition function to $p(n)$, namely the restricted partition function $p(n, t)$, which enumerates the partitions of a non-negative integer n into exactly t parts. This function was studied extensively by Sylvester and Cayley [1] at the end of the 19th century as it relates to the theory of invariants.

The relationship between $p(n, t)$ and $p(n)$ is clear. The total number of partitions of n is equal to the sum of the number of partitions of n into exactly one part, exactly two parts, and so forth until we conclude with the number of partitions of n into exactly n parts:

$$p(n) = p(n, 1) + p(n, 2) + p(n, 3) + \cdots + p(n, n - 1) + p(n, n).$$

For our work we require the following.

Definition 1 *For any natural number β , we define $\text{lcm}(\beta)$ to be the least common multiple of the numbers from 1 through β .*

For example, $\text{lcm}(4)$ equals the least common multiple of 1, 2, 3, and 4 and so we have $\text{lcm}(4) = 12$.

In a previous publication [6], the author established the following two Ramanujan like congruences of $p(n, \ell)$ for every odd prime ℓ .

Theorem 1 For ℓ an odd prime and $m \geq \text{lcm}(\ell - 1) - \frac{\ell-3}{2}$,

$$p(m\ell, \ell) - p(m\ell - \text{lcm}(\ell), \ell) \equiv 0 \pmod{\ell}.$$

Corollary 1 For ℓ an odd prime, $m \geq 0$, and $0 \leq k \leq \frac{\ell-3}{2}$, $p(m \cdot \text{lcm}(\ell) - k\ell, \ell) \equiv 0 \pmod{\ell}$.

The first goal of this paper is to state and prove Theorem 2 and Corollary 2. These results establish Ramanujan like congruences for consecutive values of $p(n, t)$ modulo every prime ℓ . So that we may cleanly state Theorem 2, we include a second definition here.

Definition 2 Given any natural number t with $t > 1$, let d be any of the natural numbers that are multiples of $\text{lcm}(t)$ and let ℓ be a prime divisor of d . We set

$$S(\ell, t, d) = \sum_{\delta \geq 0} \ell^\delta \left(\left\lfloor \frac{t}{\ell^\delta} \right\rfloor - \left\lfloor \frac{t}{\ell^{\delta+1}} \right\rfloor \right).$$

For a given δ , the expression $\left\lfloor \frac{t}{\ell^\delta} \right\rfloor - \left\lfloor \frac{t}{\ell^{\delta+1}} \right\rfloor$ enumerates the numbers between 1 and t that are divisible by ℓ^δ but not by any larger power of ℓ , namely $\ell^{\delta+1}$. Furthermore, $S(\ell, t, d)$ is a finite sum. For any t and ℓ there exists some Δ so that for every $\delta \geq \Delta$, we have $\left\lfloor \frac{t}{\ell^\delta} \right\rfloor = 0$.

Theorem 2 Suppose ℓ^α is a primary factor of d . If $S(\ell, t, d) \leq \ell^\alpha$, then for $n \geq d - \sum_{j=2}^{t-1} j$,
 $p(n, t) - p(n - d, t) \equiv 0 \pmod{\ell}$.

Moreover, by the definition of $\text{lcm}(t)$, if Theorem 2 holds for particular ℓ and t , then for any other \mathbf{t} such that $2 \leq \mathbf{t} \leq t$ we have that $p(n, \mathbf{t}) - p(n - d, \mathbf{t}) \equiv 0 \pmod{\ell}$.

Corollary 2 Given $S(\ell, t, d) \leq \ell^\alpha$ as in Theorem 2, $m \geq 0$, and σ an integer such that $-\sum_{j=2}^{t-1} j \leq \sigma \leq t - 1$, we have $p(m \cdot d + \sigma, t) \equiv 0 \pmod{\ell}$.

Depending on t , Corollary 2 gives us exactly $\frac{(t+1)(t)}{2} - 1$ consecutive values of $p(n, t)$ all being divisible by ℓ . Moreover, by the definition of $\text{lcm}(t)$, if Corollary 2 holds for particular ℓ and t , then for any other \mathbf{t} such that $2 \leq \mathbf{t} \leq t$, then $p(m \cdot d + \sigma, \mathbf{t}) \equiv 0 \pmod{\ell}$ for exactly $\frac{(t+1)(\mathbf{t})}{2} - 1$ consecutive values of $p(n, \mathbf{t})$.

The second goal of this paper is to extend Corollary 1 in which congruence properties of $p(n, m)$ are revealed for any odd prime ℓ serving as both the modulus and the number of

parts of the partition. Corollary 2 extends this result for the same modulus ℓ , but for any number of parts t such that $2 \leq t < \ell$. We state this result in the following theorem.

Theorem 3 For ℓ an odd prime, $m \geq 0$, and $2 \leq t \leq \ell$, $p(m \cdot lcm(\ell), t) \equiv 0 \pmod{\ell}$.

1.1 Examples of Theorems 2 and 3, and Corollary 2.

What is surprising about Theorem 2 and Corollary 2 is that they establish congruence properties for not just a specific number of parts as in Theorem 1 and Corollary 1 for exactly ℓ parts, but for any number of parts t for $2 \leq t \leq \ell$. Theorem 3 produces Ramanujan like congruences for the partitions of n into exactly t parts for $2 \leq t \leq \ell$ for every odd prime ℓ .

To better understand these congruences, we consider some specific examples. In Theorem 2, let $d = 36k = 2^2 \cdot 3^2 \cdot k$. We note that for the number of parts $t \in \{2, 3, 4\}$ we may write $36k = lcm(2) \cdot 18 \cdot k = lcm(3) \cdot 6 \cdot k = lcm(4) \cdot 3 \cdot k$. Then for $k \geq 1$ and $n = 36k + r$ where r depends on t and $r = -\sum_{j=2}^{t-1} j$,

$$\begin{aligned}
 p(36k + r, 4) - p(36(k - 1) + r, 4) &\equiv 0 \pmod{3} \quad \text{for } r \geq -5; \\
 p(36k + r, 3) - p(36(k - 1) + r, 3) &\equiv 0 \pmod{3, 2} \quad \text{for } r \geq -2; \\
 p(36k + r, 2) - p(36(k - 1) + r, 2) &\equiv 0 \pmod{3, 2} \quad \text{for } r \geq 0.
 \end{aligned}$$

In Corollary 2, let $d = 300 = 2^2 \cdot 3 \cdot 5^2$. For the number of parts $t \in \{2, 3, 4, 5, 6\}$ we may write $300 = lcm(2) \cdot 150 = lcm(3) \cdot 50 = lcm(4) \cdot 25 = lcm(5) \cdot 5 = lcm(6) \cdot 5$. For σ an integer depending on t such that $-\sum_{j=2}^{t-1} j \leq \sigma \leq t - 1$ and $m \geq 0$, we have the following collection of partition congruences.

- For $t = 6$ there are $\frac{\binom{7}{2}}{2} - 1 = 20$ consecutive congruences, namely all of $p(300m - 14, 6)$ **through** $p(300m + 5, 6) \equiv 0 \pmod{5}$.
- For $t = 5$ there are $\frac{\binom{6}{2}}{2} - 1 = 14$ consecutive congruences, $p(300m - 9, 5)$ **through** $p(300m + 4, 5) \equiv 0 \pmod{5}$.
- For $t = 4$ there are 9 consecutive congruences, $p(300m - 5, 4)$ **through** $p(300m + 3, 4) \equiv 0 \pmod{5}$.
- For $t = 3$ there are 5 consecutive congruences, $p(300m - 2, 3)$ **through** $p(300m + 2, 3) \equiv 0 \pmod{5, 2}$.
- For $t = 2$ there are 2 consecutive congruences, $p(300m, 2)$ **through** $p(300m + 1, 2) \equiv 0 \pmod{5, 3, 2}$.

In Theorem 3, with $\ell = 5$ and the number of parts t for $2 \leq t \leq 5$ we have: $p(60m, 5) \equiv 0 \pmod{5}$, $p(60m, 4) \equiv 0 \pmod{5}$, $p(60m, 3) \equiv 0 \pmod{5}$, and $p(60m, 2) \equiv 0 \pmod{5}$.

2. Background

Definition 3 The finite rising q -factorial $(a; q)_t$ is defined by $(a; q)_t = \prod_{n=0}^{t-1} (1 - aq^n)$.

For our purposes, we set $a = q$ so that $(q; q)_t = \prod_{n=1}^t (1 - q^n)$.

The following lemma assists us in congruentially identifying certain rational functions with polynomials.

Lemma 1 Let ℓ be any prime and α a natural number. Then $(1 - q)^{\ell^\alpha} \equiv 1 - q^{\ell^\alpha} \pmod{\ell}$.

Proof. Let ℓ be any prime and b, α natural numbers. The lemma follows from the congruence $\binom{\ell^\alpha}{b} \equiv 0 \pmod{\ell}$ for $0 < b < \ell^\alpha$. □

We illustrate how we will use Lemma 1 by examining the way in which the following rational function is congruent to a polynomial modulo 5:

$$\begin{aligned} \frac{q^6(1-q^{300})}{(q; q)_6} &\equiv \frac{q^6(1-q^{12})^{25}}{(q; q)_6} \pmod{5} \\ &\equiv q^6 \cdot \frac{1-q^{12}}{1-q} \cdot \frac{1-q^{12}}{1-q^2} \cdot \frac{1-q^{12}}{1-q^3} \cdot \frac{1-q^{12}}{1-q^4} \cdot \frac{(1-q^{12})^5}{1-q^5} \cdot \frac{(1-q^{12})(1-q^{12})^{15}}{1-q^6} \pmod{5} \\ &\equiv q^6 \cdot (1 + q + q^2 + \dots + q^{11}) \cdot (1 + q^2 + \dots + q^{10}) \cdot (1 + q^3 + \dots + q^9) \\ &\quad \cdot (1 + q^4 + \dots + q^8) \cdot (1 + q^5 + \dots + q^{55}) \cdot (1 + q^6) \cdot (1 - q^{12})^{15} \pmod{5}. \end{aligned}$$

3. Proof of Theorem 2

Proof. It is well known that the generating function for the number of partitions of n into exactly t parts is given by $\sum_{n=0}^{\infty} p(n, t) q^n = \frac{q^t}{(q; q)_t}$. Turning to the generating formula for $p(n, t) - p(n - d, t)$ we have

$$\sum_{n=0}^{\infty} (p(n, t) - p(n - d, t)) q^n = \frac{q^t(1 - q^d)}{(q; q)_t}. \tag{1}$$

Since by hypothesis $\ell^\alpha | d$, where ℓ^α is a primary factor of d , we turn to Lemma 1 so that we may equate the right hand side of (1) to another rational function modulo ℓ :

$$\frac{q^t(1 - q^d)}{(q; q)_t} \equiv \frac{q^t(1 - q^{\frac{d}{\ell^\alpha}})^{\ell^\alpha}}{(q; q)_t} \pmod{\ell}. \tag{2}$$

Our purpose at this stage of the proof is to exploit certain properties of the right hand side of (2) to describe specific circumstances for which it will be equal to a polynomial modulo ℓ . Our focus will be on the factors of $1 - q^{\frac{d}{\ell^\alpha}}$ and the fact that there are ℓ^α of them.

First, for any $t_i \in \{1, 2, \dots, t\}$ such that $\gcd(t_i, \ell) = 1$, the expression $\frac{1 - q^{\frac{d}{\ell^\alpha}}}{1 - q^{t_i}}$ is a polynomial of degree $\frac{d}{\ell^\alpha} - t_i$. Such t_i are enumerated by $t - \lfloor \frac{t}{\ell} \rfloor$. Hence, we require at least this many factors of $(1 - q^{\frac{d}{\ell^\alpha}})$ in the numerator of the right hand side of (2) for our purpose.

Second, for δ a natural number with $1 \leq \delta \leq \alpha$, the remaining $t_i \in \{1, 2, \dots, t\}$ such that t_i is exactly divisible by ℓ^δ (and no higher power of ℓ), we have that the expression

$$\frac{(1 - q^{\frac{d}{\ell^\alpha}})^{\ell^\delta}}{1 - q^{t_i}} \pmod{\ell}$$

is a polynomial of degree $\frac{d}{\ell^{\alpha-\delta}} - t_i$ by Lemma 1. These remaining t_i 's are enumerated by $\lfloor \frac{t}{\ell^\delta} \rfloor - \lfloor \frac{t}{\ell^{\delta+1}} \rfloor$. It is clear that at least $\ell^\delta \left(\lfloor \frac{t}{\ell^\delta} \rfloor - \lfloor \frac{t}{\ell^{\delta+1}} \rfloor \right)$ factors of $(1 - q^{\frac{d}{\ell^\alpha}})$ are required for our purpose.

Hence, so long as t is such that $\sum_{\delta \geq 0} \ell^\delta \left(\lfloor \frac{t}{\ell^\delta} \rfloor - \lfloor \frac{t}{\ell^{\delta+1}} \rfloor \right) \leq \ell^\alpha$, then, modulo ℓ , the right hand side of (2) can be expressed as a polynomial of degree $d - \sum_{j=1}^{t-1} j$. Thus $p(n, t) - p(n - d, t) \equiv 0 \pmod{\ell}$. □

4. Proof of Corollary 2

Proof. In Theorem 2, set $n = k \cdot d + \sigma$ for k an integer so that

$$p(k \cdot d + \sigma, t) - p(k \cdot d + \sigma - d, t) = p(k \cdot d + \sigma, t) - p((k - 1) \cdot d + \sigma, t) \equiv 0 \pmod{\ell}.$$

We proceed by induction on k . Let $k = 1$ so that

$$p(d + \sigma, t) = p(d + \sigma, t) - 0 = p(d + \sigma, t) - p((k - 1) \cdot d + \sigma, t) \equiv 0 \pmod{\ell}.$$

Note that for σ as in Corollary 2, $p(\sigma, t) = 0$.

Now suppose $p(m \cdot d + \sigma, t) \equiv 0 \pmod{\ell}$ is true for all $m < k$. Hence

$$p(k \cdot d + \sigma, t) - p((k - 1) \cdot d + \sigma, t) \equiv 0 \pmod{\ell},$$

which implies that $p(k \cdot d + \sigma, t) \equiv 0 \pmod{\ell}$ by the induction hypothesis. □

We note that Corollary 2 gives us exactly $\frac{(t-1)(t)}{2} - 1$ consecutive output values of $p(n, t) \equiv 0 \pmod{\ell}$.

5. Proof of Theorem 3

Our proof will reflect the fact that Theorem 3 is a combination of Corollaries 1 and 2. Corollary 1 treats the case for the number of parts $t = \ell$ and Corollary 2 covers the case for the number of parts t for $2 \leq t < \ell$.

Proof. Let ℓ be any odd prime, $m \geq 0$, and let t be such that $2 \leq t \leq \ell$.

Case 1. $t = \ell$. In Corollary 1 we set $k = 0$ so that $p(m \cdot \text{lcm}(\ell), \ell) \equiv 0 \pmod{\ell}$ and this part of the proof of Theorem 3 is complete.

Case 2. $2 \leq t < \ell$. Given any t such that $2 \leq t < \ell$, there exists a natural number b such that $\ell \cdot b \cdot \text{lcm}(t) = \text{lcm}(\ell)$. This part of the proof of Theorem 3 depends on Corollary 2 which depends on Theorem 2 so we set $d = \ell \cdot b \cdot \text{lcm}(t) = \text{lcm}(\ell)$ and the hypotheses of Theorem 2 are satisfied. Specifically, for $\alpha \geq 1$ where ℓ^α is primary factor of d , then $S(\ell, t, d) = t < \ell^\alpha$. We now set $\sigma = 0$ in Corollary 2 so that for any t such that $2 \leq t < \ell$,

$$p(m \cdot d, t) = p(m \cdot \ell \cdot b \cdot \text{lcm}(t), t) = p(m \cdot \text{lcm}(\ell), t) \equiv 0 \pmod{\ell}. \quad \square$$

There are many more results arising from the further extensions of the methods of this paper which are being explored.

6. Acknowledgements

The author would like to thank his advisor George Andrews for his continued guidance and James Sellers for the typesetting assistance. The author is especially grateful for the referee's time and effort.

References

- [1] G. E. Andrews, *Partitions: At the interface of q -series and modular forms*, Ramanujan Journal, **7** (2003), 385-400
- [2] G. E. Andrews and F. G. Garvan, *Dyson's crank of a Partition*, Bull. Amer. Math. Soc., **10** (1988), 167-171
- [3] A. O. L. Atkin, *Proof of a conjecture of Ramanujan*, Glasgow Math. J., **8** (1967), 14-32.
- [4] F. J. Dyson, *Some Guesses in the Theory of Partitions*, Eureka, **8** (1944), 10-15
- [5] H. Gupta, E. E. Gwyther and J. C. P. Miller, *Tables of Partitions*, Royal Soc. Math. Tables, Vol. 4, Cambridge University Press, Cambridge, 1958.
- [6] B. Kronholm, *On Congruence Properties of $p(n, m)$* , PAMS, **133** (2005), 2891-2895.
- [7] K. Mahlburg, *Partitions congruences and the Andrews-Garvan-Dyson crank*, Proc. Nat. Acad. Sci., **102** (2005), 15375-15376.
- [8] K. Ono, *Distribution of the partition function modulo m* , Annals of Math. **151** (2000), 293-307.
- [9] S. Ramanujan, *Collected Papers*, Cambridge University Press, London, 1927; reprinted: A. M. S. Chelsea, 2000 with new preface and extensive commentary by B. Berndt.
- [10] G. N. Watson, *Ramanujans Vermutung über Zerfallungszahlen*, J. reine und angew. Math **179** (1938), 97-128.