# KING AND ROOK VS. KING
# ON A QUARTER-INFINITE BOARD

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## Abstract

In [3], the following open problem is posed: Played on a quarter-infinite board, with initial position WKa1, WRb2 and BKc3, can White win? In this paper, we show that the answer is yes. Indeed, White has a winning strategy which can be implemented within an $11 \times 9$ region.

## 1. Introduction

Having its humble beginnings in the context of recreational mathematics, combinatorial game theory has matured into an active area of research. Along with its natural appeal, the subject has applications to complexity theory, logic, graph theory and biology. For these reasons, combinatorial games have caught the attention of many people and the large body of research literature on the subject continues to increase. The interested reader is directed to [1, 2, 4, 5] and to A. Fraenkel's excellent bibliography, found in [4].

In [3], the following open problem is posed: Played on a quarter-infinite board, with initial position WKa1, WRb2 and BKc3, can White win?

We show that White has a winning strategy which can be implemented within an $11 \times 9$ region. Since the original problem does not specify who moves first, we assume that White moves first. Throughout this paper, we will make use of "standard" chess

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notation and terminology. That is, we label the columns and rows of the quarter-infinite board with a, b, c, ... and $1, 2, 3, \ldots$, respectively. A column is called a *file* and a row is called a *rank*. Also, let (WK$s_i$, WR$s_j$, BK$s_k$) denote the game position where the White King is on square $s_i$, the White Rook is on square $s_j$, and the Black King is on square $s_k$.

## 2. White's Strategy

**Definition.** Suppose that Black's King is trapped inside a rectangular region of finite area, and that White's King is protecting its Rook. We say that this is a *winning position for White*.
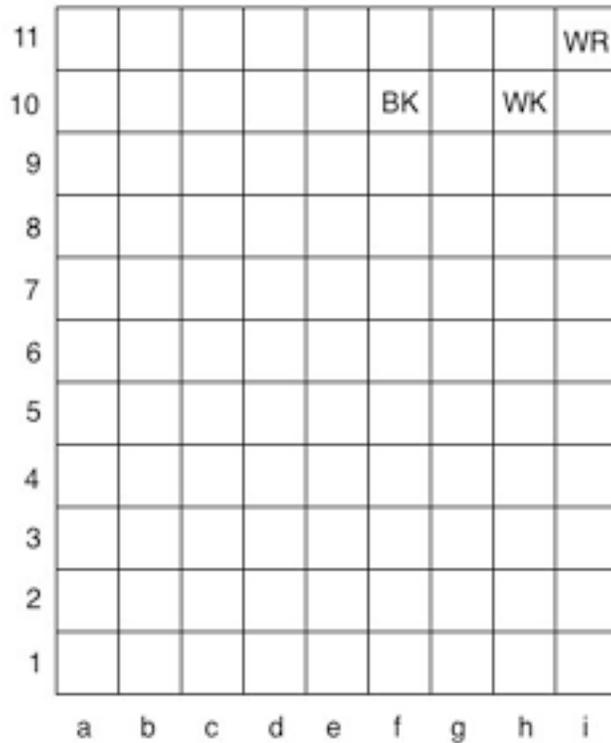


Figure 1: This is a winning position for White.

White's strategy is simple. His goal is to obtain a winning position. Once this has been achieved, White proceeds in the following way: Using his King and Rook, he systematically decreases the size of the rectangular region and eventually checkmates the Black King in an area close to the a1 square. For example, suppose that White has trapped Black in the following position: (WKe3, WRd4, BKc3) with White to move. Then, **1. Ke4 Kb3; 2. Kd3 Kb2; 3. Rc4 Kb3; 4. Kd4 Kb2; 5. Rc3 Kb1; 6. Kd3 Kb2;**

**7. Kd2 Kb1; 8. Rc2 Ka1; 9. Kc3 Kb1; 10. Kb3 Ka1; 11. Rc1,** and Black's King is checkmated.

Can White ever hope to obtain a winning position? At first glance, it seems as if this might not be possible.

## 3. Various Black Strategies

Since Black does not wish to lose, there are several strategies which he might employ. For example, Black can attempt to run off towards infinity. Of course as Black's King is trying to run away, White's Rook can easily contain it in a rectangular region of finite area. In this case, Black's King moves toward the White Rook to chase it away. This eventually forces the White Rook to move away, which results in a larger enclosed region in which Black's King can move in. Conceivably, Black can proceed in like manner, ad infinitum. The following theorem shows that Black cannot do this in a naive fashion.

**Theorem 1.** *If Black's King attempts to run away towards infinity (by increasing its $x$ and $y$ coordinates whenever possible), White obtains a winning position.*

*Proof.* **1. Re2 Kd4; 2. Kb2 Kd5; 3. Kc3 Kd6; 4. Kd4 Kd7; 5. Re11 Kd8; 6. Ke5 Kd9; 7. Kf6 Kd10; 8. Ri11 Ke10; 9. Kg7.** This is an important intermediate position (see first diagram of Figure 2). **9. ... Kf10; 10. Kh8 Kg10; 11. Ki9.** At this point, White easily obtains a winning position. [For example, if 11. ... Kg9, then 12. Ri10 Kg8; 13. Rh10.] □

A related strategy that Black might try is to always aggressively move his King towards the White Rook. However, this is a losing strategy for Black.

**Theorem 2.** *If Black's King always moves toward the White Rook, White obtains a winning position.*

*Proof.* **1. Re2 Kd3; 2. Re11 Kd4; 3. Kb2 Kd5; 4. Kc3 Kd6; 5. Kd4 Kd7; 6. Ke5 Kd8; 7. Kf6 Kd9; 8. Kg7 Kd10; 9. Ri11 Ke10.** Here, we have a position (see first diagram of Figure 2) where White easily obtains a winning position. [For example, 10. Kh8 Kf10; 11. Kh9 Kf9; 12. Rg11 Kf10; 13. Kh10.] □

Yet another strategy that Black might try is to attempt to hem in the White King so as to prevent it from ever being able to protect the White Rook. Unfortunately (for Black), this is not possible.

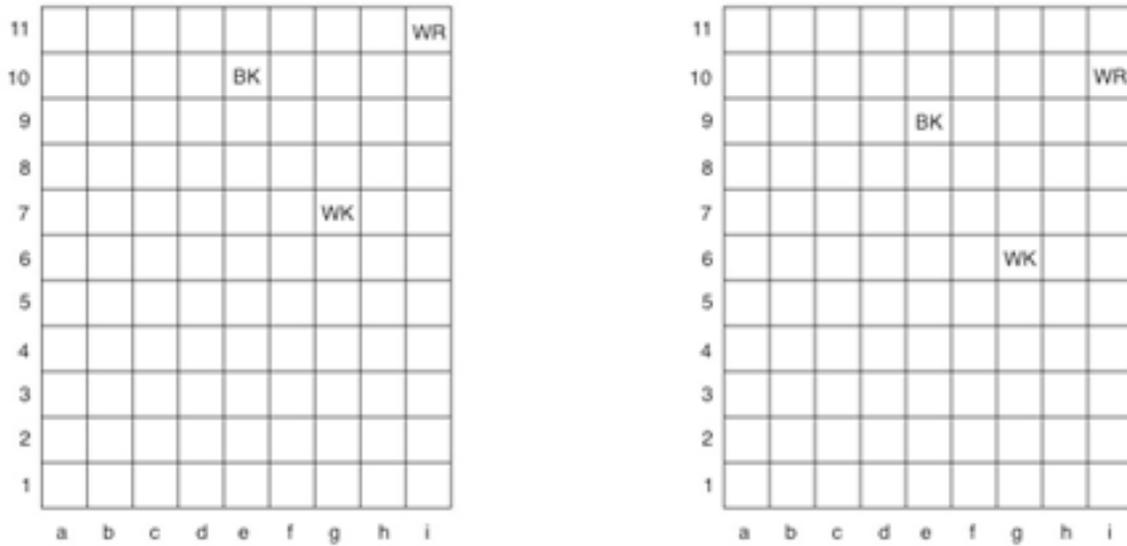**Theorem 3.** *The White King cannot be hemmed in and rendered harmless.*

Figure 2: In both of these cases, White can obtain a winning position.

*Proof.* **1. Re2 Kd4; 2. Kb2 Kd3; 3. Re11 Kd2; 4. Re10 Kd3.** [Black cannot move 4. ... Kd1, since White easily obtains a winning position with 5. Kc3.] **5. Kc1 Kc3; 6. Kd1 Kd3; 7. Ke1** and now, White's King is ready to join in the fray (see the first diagram of Figure 3). Of course, there are other possible sequences of moves which can occur when Black tries to hem in the White King. For example, 1. Re2 Kd3; 2. Re11 Kc2; 3. Ka2 Kc3; 4. Kb1 Kd2; 5. Kb2 Kd3; 6. Kc1 Kc3; 7. Kd1 Kd3 8. Ke1 (see the second diagram of Figure 3). □

**Definition.** Let White's Rook be on the e-file and suppose that Black's King is trapped inside the rectangular region of finite area. If White's King is not on files a, b, or c, then the White King is *active* (and has been *activated*).

Note that Theorem 3 shows that White's King can always be made active.

## 4. Victory for White

White's first move should be **1. Re2.** In turn, Black can respond in a myriad of ways. The general strategy for White is to activate his King from the corner of the board. White's Rook will eventually end up on e11 (or e10), respectively. White's King then races to g7 (or g6), respectively. If Black plays passively, White has no difficulty in obtaining a winning position. Thus, Black must aggressively attack White's Rook. In the various scenarios which can occur, one of two key positions is reached (see Figure 2). In both of these instances, White easily obtains a winning position within an $11 \times 9$ region (even if it is Black's turn to move).
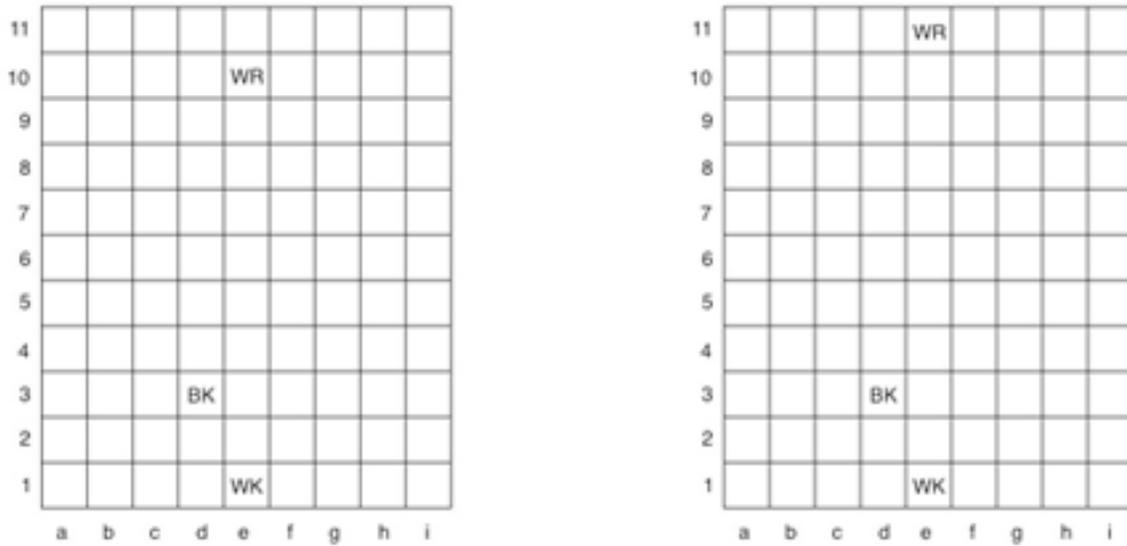
Figure 3: Two typical positions where White's King is active.

**Notation.** Suppose that the White King, White Rook and Black King are in a (legal) position on the quarter-infinite board. Let $D_W(s_i)$ denote the minimum number of consecutive (legal) moves the White King must make to reach square $s_i$, assuming that Black does not move. For the Black King, $D_B(s_j)$ is defined in a similar fashion.

**Lemma 1.** *Suppose that White's Rook is on square e11, with the Black King inside the rectangular region of finite area (ie. the Black King is on a square on the a, b, c, or d-file and also on the 1, 2, ..., or 10-rank). If $D_W(g7) \leq D_B(d10)$, then White can obtain a winning position.*

*Proof.* Without loss of generality, we can assume that White's King is active. Let us consider the worst case scenario. In this case, it is Black's turn to move. Now if Black plays passively and does not attempt to attack the Rook, then White easily obtains a winning position by bringing its King to f10. Thus, Black must march its King towards d10. As Black does this, White's King moves toward g7.

CASE (i). The White King arrives at g7 before the Black King arrives at d10. In this case, White easily obtains a winning position by moving his King to i10 and his Rook to i11.

CASE (ii). The White King is one move away from g7 and the Black King is on d10. In this case, **1. Ri11 Ke10; 2. Kg7** and White can obtain a winning position (see first diagram of Figure 2). □

**Lemma 2.** *Suppose that White's Rook is on square e10, with the Black King inside the rectangular region of finite area (ie. the Black King is on a square on the a, b, c, or d-file and also on the 1, 2, ..., or 9-rank). If $D_W(g6) \leq D_B(d9)$, then White can obtain a winning position.*

*Proof.* Without loss of generality, we can assume that White's King is active. Let us consider the worst case scenario. In this case, it is Black's turn to move. Now if Black plays passively and does not attempt to attack the Rook, then White easily obtains a winning position by bringing its King to f9. Thus, Black must march its King towards d9. As Black does this, White's King moves toward g6.

CASE (i). The White King arrives at g6 before the Black King arrives at d9. In this case, White easily obtains a winning position by moving his King to i9 and his Rook to i10.

CASE (ii). The White King is one move away from g6 and the Black King is on d9. In this case, **1. Ri10 Ke9; 2. Kg6** and White can obtain a winning position (see second diagram of Figure 2). □

**Theorem 4.** *White can obtain a winning position in an 11 × 9 region.*

*Proof.* White's first move is **1. Re2.** White then aggressively activates his King, with the goal of having his King and Rook on g7 (or g6) and e11 (or e10), respectively. Once White satisfies the hypothesis of Lemma 1 or 2, Black will lose.

- The Black King moves around passively. If Black allows the White Rook to move to e11 and the White King to move to f10 (without trying to hem in the White King or chase away the White Rook from e11), then White obtains a winning position.

- The Black King tries to run off to infinity. There are several ways that Black can do this. The worst case scenario (for White) is described in Theorem 1, where White eventually obtains a winning position. All other attempts by Black to run off to infinity yield positions which satisfy the hypothesis of Lemma 1, where White eventually obtains a winning position. For example, 1. Re2 Kc4; 2. Kb2 Kc5; 3. Kc3 Kc6; 4. Kd4 Kc7; 5. Re11 yields a position which satisfies the hypothesis of Lemma 1. This leads to a win for White.

- The Black King tries to hem in the White King. Theorem 3 shows one possible variation that can occur. At the end of this particular sequence of moves (see first diagram of Figure 3), the hypothesis of Lemma 2 are satisfied and thus White eventually wins. All other attempts by Black to hem in the White King yield positions which satisfy the hypothesis of Lemma 1 or 2, leading to a win for White. For example, some variations lead to positions such as (WKd2, WRe10, BKd4), (WKc2, WRe11, BKc4), (WKd3, WRe11, BKd5), etc., whence White easily obtains a winning position (see Figure 2).

□

## 5. White's Strategy for WKa1, WRb2 and BK$s_k$

Suppose that the initial position on a quarter-infinite board is WKa1, WRb2 and BK$s_k$, where $s_k$ is a square not on the a-file, b-file, 1-rank, or 2-rank. Can White win? The answer is yes.

**Theorem 5.** *Let the initial position on a quarter-infinite board be WKa1, WRb2 and BK$s_k$, where $s_k$ is a square not on the a-file, b-file, 1-rank, or 2-rank. Then, White has a winning strategy.*

*Proof.* Let the Black King be of distance $d$ from the White King. First, the White Rook moves horizontally (to the right) at least $2d$ squares. Whatever the Black King does, White's first goal is to move the White King to the right of the Black King, yet staying as close as possible to it. This is possible since the Black King is now to the left of the White Rook and the White King can eventually move to the right side of the White Rook. Now, once the Black King attacks the White Rook, the White Rook should move vertically at least $4d + 4$ squares. After approximately $2d + 2$ moves, we may assume the following:

- The White King is within $d + 2$ squares of the Black King.

- The White King is to the right of the Black King.

At this point, White's strategy is to move the White King diagonally up and to the right for the next $2d$ moves, no matter what the Black King does. At the end of this second stage, the White King is $d$ columns to the right of the Black King and within distance $d + 2$. Finally, the White Rook moves horizontally (to the right) $4d$ squares.

After this, the Black King is effectively boxed in. White easily establishes a winning position and it is straight-forward to checkmate the Black King. $\qquad\square$

## 6. An 11 × 9 Chessboard
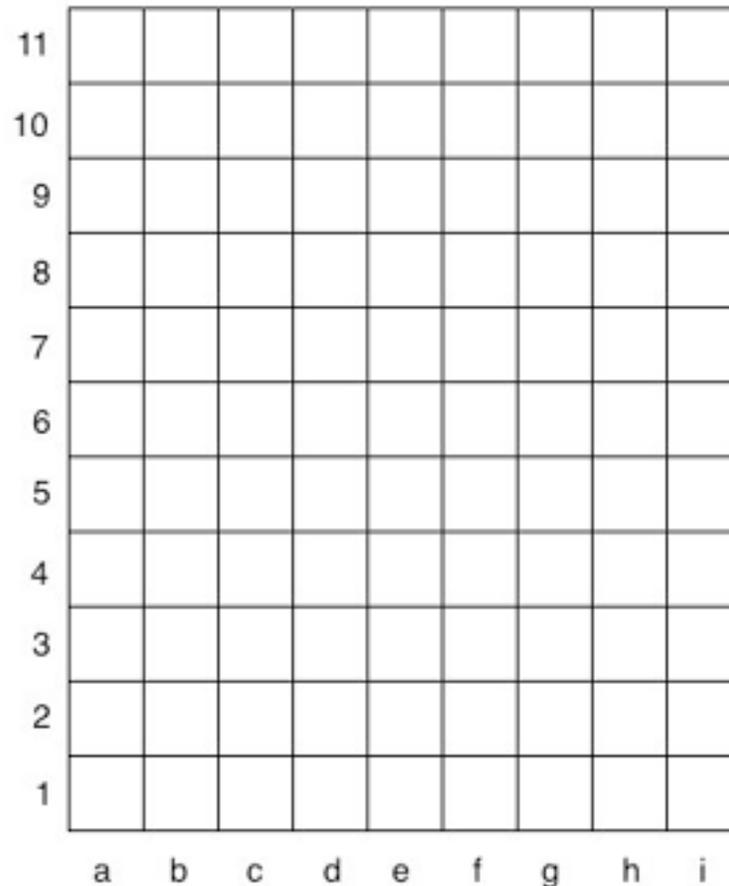


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