POSITIONS OF VALUE *2 IN GENERALIZED DOMINEERING AND CHESS

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Received: 5/29/04, Revised: 7/9/05, Accepted: 7/31/05, Published: 8/1/05

Abstract

Richard Guy [5] asks whether the game-theoretic value *2, the value of a nim-heap of size 2, occurs in the games of Domineering or chess. We demonstrate positions of that value in Generalized Domineering and chess.

1. Overview

We follow the terminology of Winning Ways [1], considering two-player alternating-turn finite-termination complete-information chance-free games whose loss conditions are exactly the inability to move, and which always end in a loss. According to this definition, chess is not a game because it can be tied or drawn and the loss condition is not the inability to move. We will answer these objections.

A game position can be expressed recursively as $\{G^L \mid G^R\}$ where G^L, called the left-options of G, is a collection of positions to which one player, called Left, can move, if it is her turn, while G^R, the right-options of G, is a collection of positions to which the other player, called Right, can move if it is hers. We abuse terminology by also using the symbols G^L and G^R to refer to generic elements of these collections.

We can recursively define a binary operation $+$ on the collection of game positions by $G{+}H = \{G^L{+}H, G{+}H^L \mid G^R{+}H, G{+}H^R\}$ and an involution $-$ by $-G = \{-G^R \mid -G^L\}$. Then we can partially order game positions up to an equivalence relation by $G \geq H$ if Left can win $G + (-H)$ going second. This equivalence relation is usually taken to be equality; $G = H$ if $G - H$ is a win for whichever player goes second.

Employing modest restrictions on the number of options available from any position, the collection of all game positions becomes an Abelian group under $+$ with inverse $-$ and identity the equality class of $\{|\}$, called 0. It turns out that one can determine the outcome class of a game by comparing it to 0. If $G > 0$ then G is a win for Left regardless of who plays first, if $G < 0$ then it is a win for Right, if $G = 0$ it is a win for whichever player goes second, and if G is incomparable with 0 it is a win for whomever goes first.

A game consists of a collection of legal game positions. A game is called impartial if from each legal game position both players have exactly the same options, i.e., if $G^L = G^R$ (as sets of game positions) for every position G. A game that is not impartial is called partizan. It is well-known that every impartial game position is equivalent to a nim-heap of a particular size, called the Sprague-Grundy number, nim-value, or nimber of the position. This is certainly not true in general for a partizan game position, but it is possible for a position in a partizan game to be equivalent to a nim-heap. In particular, a 0-position in any game, where neither player has a canonical move, is always equivalent to an empty nim-heap. If a game has nim-value n, it is referred to as $*n$, with $*$ standing in for $*1$ and 0 for $*0$.

The simplest nimbers, 0 and $*$, occur in very simple guises in many partizan games. However, it is often much more difficult to find positions of other nimber values. The next nimber, $*2$, defined as $\{0, * \mid 0, *\}$ is known not to appear in some partizan games and has proven elusive in others. In this note, we demonstrate generalized Domineering positions and a chess position with value $*2$.

2. Domineering

Domineering, discussed in Winning Ways [1] and On Numbers and Games [2], is a game played with dominoes on a rectangular board tiled with squares. Players take turns placing dominoes on unoccupied squares on the board. Each domino covers two squares. Left must place all her dominoes with a "vertical" orientation while Right must place hers with "horizontal" orientation. This game can easily be seen to be partizan. The simplest game position with value 0 is the empty position; the simplest game position with value $*$ is

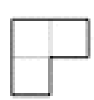


Figure 1: Domineering position of value $*$

Generalized Domineering, as a game, has the same rules to determine left and right options as domineering, but includes a larger class of legal positions. Any subset of a rectangular board is a legal position in generalized Domineering, whereas only subsets corresponding to a sequence of legal Domineering moves are legal positions in ordinary Domineering.

The following generalized domineering positions all have value $*2$ or $n + *2$ where n is an integer. The values were obtained with Aaron Siegel's cgsuite software [7].

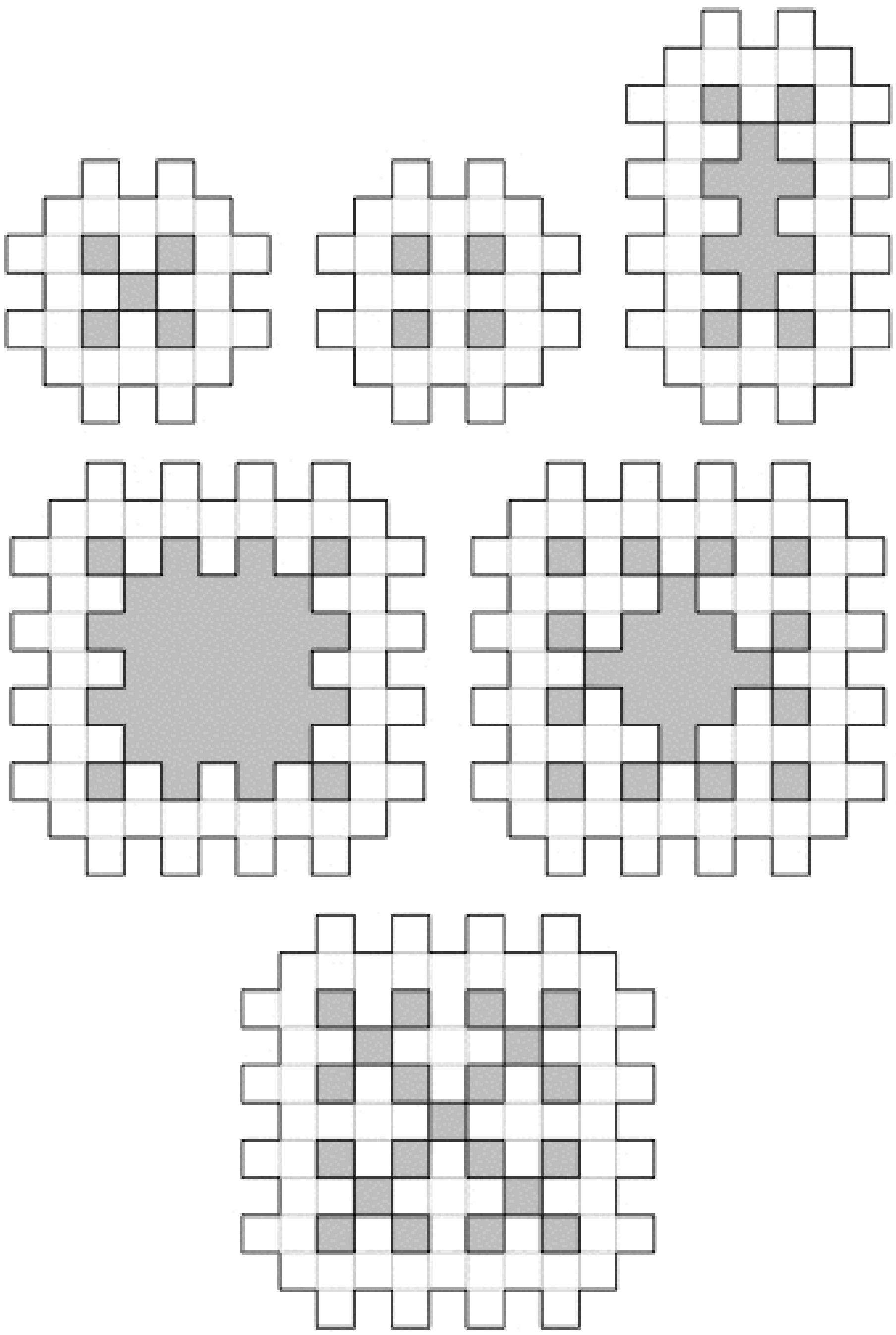


Figure 2: Generalized Domineering positions of value $n * 2$

Clearly, there is a pattern at work here. All simple rectangles of the above type (i.e., like the first, third, and fourth example) have value $*2$ modulo an integer. That is, rectangles with vertical side length $4n + 1$ and horizontal side length $4m + 1$ fitted with $2n$, $2m$ respective "outward-pointing bumps" alternating with $2n - 1$ or $2m - 1$ "inward-pointing bumps," have value $2(m - n) * 2$, for any $n, m \geq 1$. For two side lengths of $4n + 3, 4m + 3$, these rectangles have integral value, while for mixed side lengths of $4n + 1$ and $4m + 3$ they are hot. There is no obvious way to fit together the bumps on a rectangle with even side length on a planar Domineering board.

I have not been able to find any ordinary Domineering positions with value $*2$, and suspect that if such exist, they will be unlike the positions constructed here, which appear to depend strongly for their nimber value on the "holes" in the corners.

In order to prove that all rectangles of the appropriate type have value $n * 2$, let us note some peculiar characteristics of these rectangles. Ignoring the four corner squares for a moment, such positions consist of rows and columns of interlocking bumps. On a row of bumps, a left (vertical) move eliminates a bump and leaves zero, one, or two rows of bumps; a right move eliminates two adjacent bumps and leaves zero, one or two rows of bumps, along with two leftover squares of no value. The roles are reversed for a column.

Now considering the corners, Left can move on a corner and a vertically adjacent bump; Right can move on a corner and a horizontally adjacent one. This completes the list of possible moves on a rectangle as above or any subgame of one.

Thus motivated, we can define a new game called SQUID as a bookkeeping device. This game is played on a graph whose vertices are colored from the color set {red, green, blue}. A legitimate move for Left is to erase one blue or two adjacent non-blue vertices; a legitimate move for Right is to erase one red or two adjacent non-red vertices. Clearly, we can express any generalized Domineering rectangle as above as a position in this game as a rectangle of the same side length with green corners, blue horizontal sides, and red vertical sides.

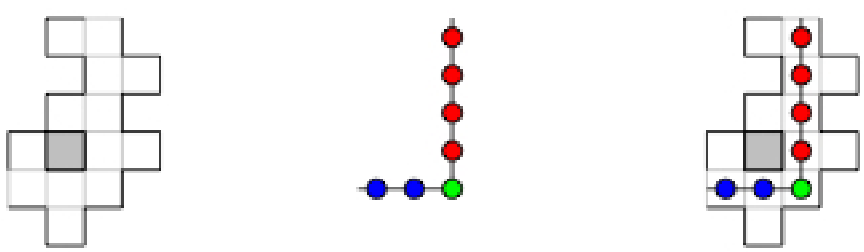


Figure 3: A position fragment in both games with an overlay to show the translation

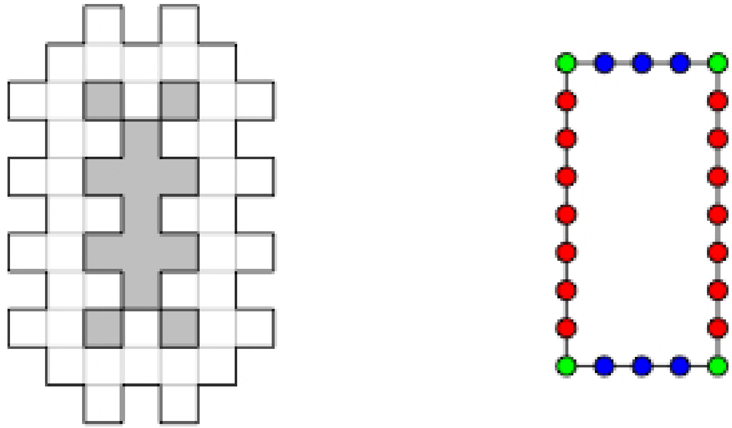


Figure 4: A $*2$ rectangle in Domineering and SQUID

Now that we have a framework, we can work locally with with positions in our new game. Our main aim is:

Proposition 1 *Any regular position in SQUID containing a chain of five blue vertices has value one less than the same position with the chain lengthened to nine blue vertices.*

Here a regular position is one in which the valence of any vertex is at most two, and a chain is a connected row of vertices. Henceforth we will assume that all our positions are regular. In fact, locally almost all of our vertices will be blue.

By symmetry the negative of this statement will be true for red chains. Since we have checked by computer the special case rectangle with dimensions 5×5 (three red/blue vertices to a side) and then the base cases of dimension 5×9 and 9×9, this will suffice to establish that all rectangles with side lengths $4n + 1$ and $4m + 1$ have the desired value, as any larger rectangle of the appropriate dimensions modulo four can be obtained from one of these two by repeated lengthening of subchains of length five by four.

We need two short lemmas and then we will be ready to prove this. First, a tail of length n is defined as n adjacent vertices with one endpoint monovalent, i.e., wagging free.

Lemma 1 *Let A be a SQUID position.*

1. *Let A_2 be the same position with a blue tail of length two attached, either to a blue vertex or disjointly. Then $A_2 = A + \{1 \mid 0\}$.*

2. *Let A_3 be the same position with its tail lengthened to three. Then $A_3 \geq A$.*

Figure 5: Lemma 1

The proofs are straightforward, and involve only the usual tricks of combinatorial game theory analysis. In order to show, for example, that a game position G has value greater than or equal to zero, we must show that it can be won either by Left or by the second player to move. To show this it is both necessary and sufficient to demonstrate a Left response $(G^R)^L$ to every possible Right move which has value greater than or equal to zero; such positions are won by Left or by the second player to play, who in this case would be Left.

The other trick we use often is that if G and G' have exactly the same moves except some Left options of G and its subgames are not present in G', then $G \geq G'$. The idea is that if Left refuses to take certain moves it can only hurt her.

Proof.

1. We need to show that the difference game $A_2 - A + \{0 \mid -1\}$ is a second player win. We break it up into cases, and proceed by induction on the size of A. The base case where A is empty is trivially verified.

   Left moves first:

   - Left moves on the A part of A_2 or in $-A$; then Right responds with the corresponding move in the other of these components. By the inductive premise, Right has moved to a zero game.
   - Left moves on the tail of A_2. Right responds by moving to -1 in the $\{0 \mid -1\}$ component. Right can now refuse to move on the remainder of the tail, which can only increase the value of the position to $A + 1 - A - 1 = 0$.
   - Left moves on $\{0 \mid -1\}$ to 0. Right responds by moving on the tail, destroying it, leaving $A - A = 0$.

   Right moves first:

   - Right moves on the A part of A_2 or in $-A$; then Left responds with the corresponding move as above.
   - Right moves on both vertices of the tail, leaving $A - A + \{0 \mid -1\} = \{0 \mid -1\}$. Left moves to 0.
   - Right moves on the $\{0 \mid -1\}$ component to $A_2 - A - 1$. Left responds by moving on the square of the tail closer to A, leaving $A + 1 - A - 1 = 0$, where the $+1$ is the value of the isolated blue vertex.
   - Right possibly moves half-on, half-off the tail of A_2, leaving a position A' which looks like A missing a blue square along with a single vertex worth 1,

and $-A+\{0 \mid -1\}$. Left responds on $\{0 \mid -1\}$, leaving A', a single blue vertex, and $-A$. Now, A' along with a single blue vertex is the same position as A with some Right moves removed, so $A'+1 \geq A$. Then $A'+1-A \geq 0$, so that this is now a win for Left, as desired.

2. This is similar, but a little quicker. Since we only need to show $A_3 - A \geq 0$, all we care about is that Left can win going second. We proceed as before by induction, with trivial base case ($\emptyset_3 = \{2 \mid 1\}$).

   Now, as in the first case, if Right moves on the A part of A_3 or $-A$, then Left can respond with the corresponding move in the other component and we are done by the inductive premise. If Right moves on two of the three squares of the tail of A_3, then Left moves on the other square, leaving $A - A = 0$. Finally, if Right moves half-on, half-off the tail, leaving A' and a component which is two adjacent blue vertices, then Left can respond by taking one of these vertices, leaving $A'+1-A$, which as in the last case is at least 0, as desired.

Corollary 1 *A position A_4 with a length four blue tail attached disjointly or to a blue vertex or a position $A_{2,2}$ with two length two blue tails attached disjointly or to blue vertices satisfy $A_4 = A+1 = A_{2,2}$.*

Proof. This is easy, since we can write A_4 or $A_{2,2}$ as $(A_2)_2 = A+\{1 \mid 0\}+\{1 \mid 0\} = A+1$.

Proof of Proposition 1. Now let B_5 be a position with a chain of five blue vertices and B_9 the same position with a chain of nine blue vertices instead. We will show by induction that $B_9 - B_5 - 1 = 0$. The base case is again easy to verify, as a chain of length five has value 2 while a chain of length nine has value 3. As in the lemmas, we use the inductive step only to say that we can respond to a move far away from the chain in one component with the corresponding move in the other component, leaving a second player win by the inductive premise. The remaining cases are summarized below. It helps to organize the descriptions to have a labelling of the chains, as in Figure 6.

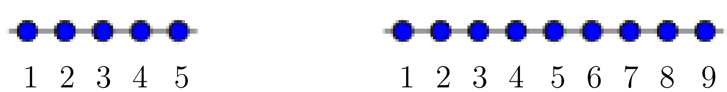


Figure 6: B_5 and B_9

Left moves first:

- Left moves on the n vertex of the chain of B_9. Right can respond by moving on the n vertex of $-B_5$ for $n \leq 4$, the 3 vertex of B_5 for $n = 5$, or the $n-4$ vertex of $-B_5$ for $n \geq 6$. The resulting positions differ by a length four or two length two blue tails, so by the corollary, this is a zero position.

- Left moves on the $n, n+1$ vertices of the chain of $-B_5$. For $n \neq 4$, Right responds with the $n, n+1$ vertices of the chain of B_9; for $n = 4$ Right responds with the $8, 9$ vertices of the chain of B_9. In either case the corollary again applies to say that the resulting position is 0.

- Similarly, if Left moves half-on, half-off, the chain of $-B_5$ then Right makes the corresponding move half-on and half-off the chain of B_9 and the corollary again applies

Right moves first:

- If Right moves on two adjacent vertices of the chain of B_9, or half-on and half-off the chain, Left responds on the chain of $-B_5$. If Right's move involves one of the endpoints, Left responds with the matching move involving the corresponding endpoint. If Right moves $(2, 3), (4, 5)$, or $(6, 7)$ then Left responds with $(2, 3)$; if Right moves $(3, 4), (5, 6)$, or $(7, 8)$ the Left responds with $(3, 4)$. In any case the corollary applies.

- If Right moves on a single vertex of $-B_5$, then Left responds on B_9; She responds to a move on 5 with one on nine, and otherwise responds to n with n.

- If Right moves on the -1, taking it to 0, leaving the position $B_9 - B_5$, then Left can move on any interior vertex of B_9, say, for specificity, on 2. This leaves a position $B_{1,7}$ with two trailing blue tails of lengths 1 and 7. Now by the second lemma above, this has value at least as high as the same position (call it $B_{1,4}$) with tails of lengths 1 and 4 (see Figure 7). Then this is at least as high in value as B_5, which is precisely the same position with an additional possible move for Right. So $B_{1,7} \geq B_{1,4} \geq B_5$ so that the position we are left with, $B_{1,7} - B_5$, is greater than or equal to zero, and thus a win for Left, as desired.

This concludes the proof of the proposition.

Figure 7: Application of the second lemma and comparison

3. Chess

Following Elkies [3],[4], we consider chess as a combinatorial game in certain very restricted settings. In order to avoid tied or drawn play, we require that every chess position end in a finite number of moves in a loss for either white or black. Also, so that we do not have to haggle about the exact value of a win in chess, we require our starting positions, when dominated options and reversible moves are removed, to pass through a zero-value position of mutual zugzwang, that is, a position where whoever moves, loses, assuming adequate play. The simplest example of a zero position is in the first board of Figure 8:

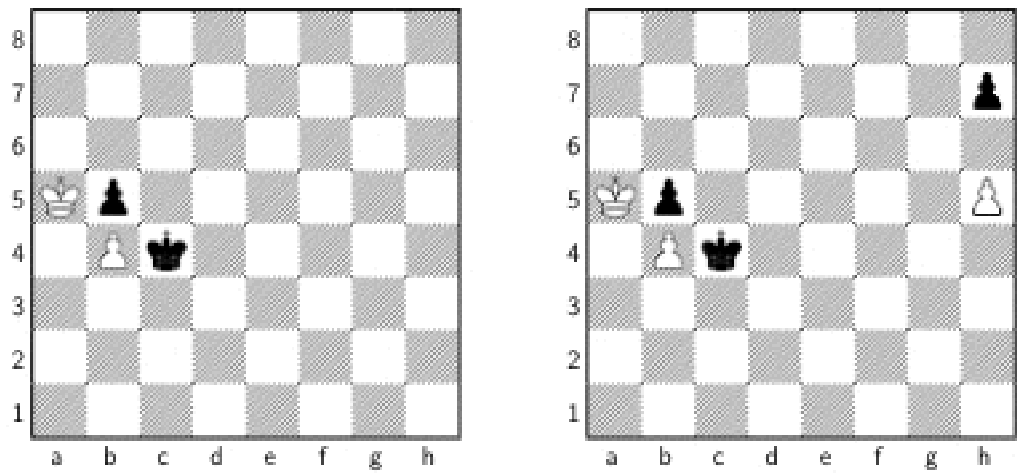


Figure 8: 0 and *

White can move only her king, but to do so leaves her pawn undefended so that black will immediately capture it and then shortly win the game. The same is true for black. So this is a true zero position in chess, where neither player has a viable move. We call starting positions that must pass through such a zero position with proper play combinatorial positions.

In the second board of Figure 8, we see a combinatorial position with value *, the same as a Nim-heap of size one. Whichever player has the turn advances the pawn on the h-file, leaving a position which is essentially equivalent to the previous one, i.e., where each player has only a losing move.

On the other hand, a position with value *2 is more difficult to come by. The pawns (or other pieces, but it is easier to manipulate pawns) must have multiple moves, but not “too many.” If pawns could not move forward, only capture, and the game ended with the player unable to move losing, then the board in Figure 9 would have the desired value.

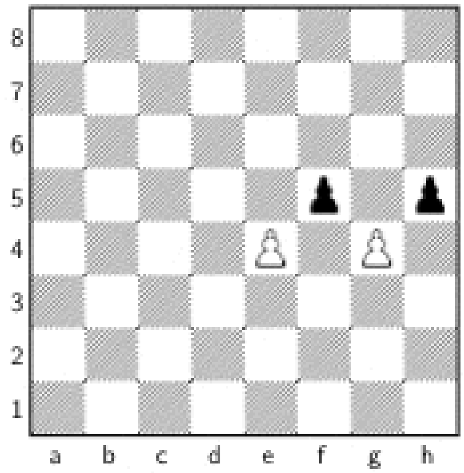


Figure 9: *2 position in capture-only chess

If it is white's turn, there are three options. The pawn on g4 can take at f5, winning the game (because there are no options for black). The same pawn can take at h5, leaving the pawns on e4 and f5 each with the possibility to capture the other. Finally, the e4 pawn can capture at f5, leaving the pawns on g4 and h5 with the ability to capture one another. Both of the latter two moves leave only one option for each player, namely the immediate win, so are equivalent to a Nim-heap of size 1. It is clear by symmetry that the options are the same for black. So each player can move to an empty game (that is, a Nim-heap of size 0) or a $*$ game, a Nim-heap of size 1. These are the options from a Nim-heap of size 2, and the value of the initial game is $\{0, * \mid 0, *\} = *2$.

Now, it is all well and good to find such a position on a chessboard with pieces that look like chess pieces, but this is not chess; there are no kings on the board, but even if the kings were locked in an apparent zugzwang as before, a pawn can advance as well as capture. As a chess position, such a game would not be combinatorial, because each player wants to advance a pawn first in order to promote first and thereby win, without ever passing through zugzwang. So how can we keep the pawns from advancing?

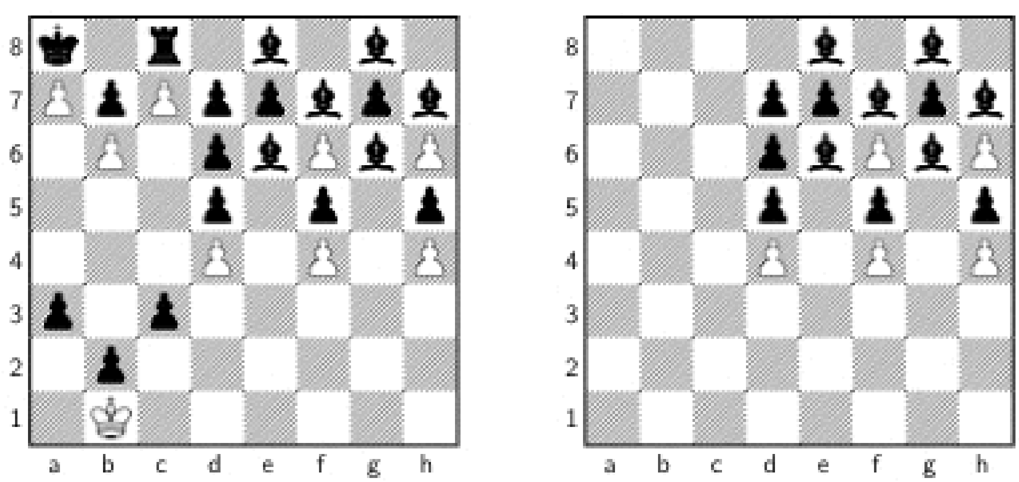


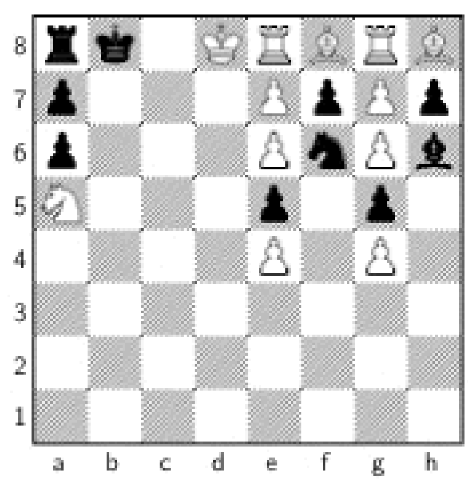
Figure 10: *2 position in 8x8 chess with arbitrary setup

The first board of Figure 10 is a combinatorial position (with an impossible arrangement of pieces) with the required properties. The interlocked bishops cannot themselves move, being blocked by immobile pawns, but their presence prevents the pawns from moving, except for the desired captures. So if we confine our attention to the files d through h, pictured on the second board, this is easily seen to be equivalent to the forced-capture game of Figure 9.

It may not be immediately obvious that the configurations in the upper and lower left amount to a mutual Zugzwang, but it is not particularly difficult to see. White can only move her king, which allows one of the black pawns to promote. Once black has a queen she can sacrifice her rook to free her king and then mate with queen and king.
On the other hand, if black moves a pawn, she loses all three and is forced to move the rook, which is either an immediate mate (c7xb8 or c7xd8) or allows white to take with the pawn on the b-file and then promote and win with queen and king. There is no danger of stalemate because white can always move her king and black can always move either the rook, or, if the rook is captured by b6xc7, her king or the pawn on b7.

This basic position, without the kings, gives an idea of how to create a capture-only situation in chess, but of course, just by the number of bishops and pawns, cannot be achieved through normal play.

Figure 11 is a position of value *2 and the movelist of an orthodox (i.e., fully legal) chess game, in algebraic notation, that results in it. The chess game was created with the help of retrograde analysis software [6]. No effort was made to find a shortest solution, or a solution that involved good chess play. It is obvious that an extra tempo move could result in the same position with black to play. For convenience, we will begin numbering possible variations at 68., the next move according to this particular chess game.



1. h3 Nc6	4. Qd2 Ne5	7. g4 Nc6	10. Nd4 Ne5
2. h4 Ne5	5. Kd1 Nc6	8. h5 Ne5	11. Nf5 Nc6
3. d3 Nc6	6. Qe1 Ne5	9. Nf3 Nc6	12. Ng3 Ne5

13. a4 Nc6
14. a5 Ne5
15. c4 Nc6
16. f4 Ne5
17. Bg2 Nc6
18. b4 Ne5
19. f5 Nc6
20. a6 Ne5
21. d4 Nc6
22. b5 Nb8
23. Nf1 c5
24. Qb4 Qc7
25. Qxc5 Qd8
26. Qb4 bxa6
27. b6 Bb7
28. Nb1d2 Bc8
29. Bb7 Qc7
30. e4 d5
31. Qd6 Qd8
32. cxd5 e5
33. Nc4 Qxd6
34. Nf1d2 Qd8
35. Nb3 Be6
36. Bc8 Qd6
37. dxe6 Qd8
38. Bd7+ Qxd7
39. Kd2 Qd8
40. Nd6+ Qxd6
41. b7 Qd8
42. d5 Qf6
43. Kc3 Ne7
44. Kb4 Kd8
45. Bh6 g5
46. Bg7 Rg8
47. Bh8 Rg6
48. fxg6 Bh6
49. g7 Kc7
50. Rh1c1+ Kd6
51. Rc8 Qg6
52. Rg8 Kc7
53. hxg6 Nd7
54. Rc1+ Kb6
55. b8=B Kb7
56. Bd6 Nf5
57. Bf8 Nf6
58. Na5+ Kb8
59. d6 Ne7
60. Rc7 Nc8
61. Re7 Ne8
62. Kc5 Nc7
63. Re8 Nd5
64. Kc6 Nc7
65. Kd7 Nd5
66. Kd8 Nc8e7
67. dxe7 Nf6

Figure 11: Orthodox *2 chess position

First of all, the reader will verify that the system of pawns (e6,f7,g6,h7) is as before, assuming that none of the other pieces, particularly the black bishop or knight, moves. That is, there are three possible captures for each side. One of the three results in a configuration of pawns with no moves for either side (a zero game); the other two result in configurations with a capture option for each side (a $*$ game).

Now, other than these pawn moves, the only white piece able to move at all is the knight on a5. If this moves, then 68. ... Kb7+ is an immediate mate; white can only throw the knight in the way for one turn. So other than the pawn configuration, white has only a losing move, so that she is in Zugzwang if no such move is available.

Now let us turn our attention to black, who has moves by the bishop and knight. The pawn configuration is as before. Now, If the black knight is moved to any square but e8, white's Kd7 is an immediate mate; Therefore the only possible moves are 68. ... Bxg7 and 68. ... Nxe8.

White responds to 68. ... Bxg7 by capturing back with either bishop; this essentially frees up the white pieces to apply more pressure. Black can move either the knight or the pawn on h7. As before, any knight move but 69. ... Nxe8 is an immediate mate; with a white bishop on g7, 69. ... Nxe8 70. Bg7xe5+ forces either 70. ... Nc7 71. Bxc7# or 70 ... Nd6 71. Bxd6# The final possibility in this line, if the pawn on h7 is there, is 69. ... h5 or h6 70. Bg7xf6 which leaves black with only a pawn move before 71. Bxc7#.

So the only possible defense is 68. ... Nxe8. Here the response depends on the configuration of the pawns. If there is a white pawn on f7 then 69. fxe8=Q Bxg7 70. Kd7# is forced, so the analysis is easy. If there is a black pawn or no pawn there, then the analysis is better left to chess experts or computers, but 69. Kd7 leads in every case to a mate in 10. The main idea is to keep the black king on the 8th rank with the white king, force e8=Q Nxe8, and then to mobilize the rest of the trapped pieces with something like Bd6.

It can be verified that the original position and the positions obtained after one capture of the appropriate type (i.e., leaving a position of value $*$) are all forced mates for either side (in three for black and four for white, except when there is no white pawn on f7, in which case mate is in 11 for white). The unique first move in each case is the pawn move that eliminates all further capture possibilities in the pawn configuration. That is, the combinatorial moves are {exf7, gxf7, gxh7| fxe6, fxg6, hxg6} which translates to $\{*, 0, * \mid *, 0, *\} = *2$, as desired.

Acknowledgments

I would not have written this paper without the consistent prodding of Elwyn Berlekamp. I am indebted to Noam Elkies for a significant improvement in the zugzwang used in the final position, which greatly streamlined the analysis. I used the latex chess packages of Torben Hoffman and Piet Tutelaers. I would also like to thank the referee, whose comments induced me to expand the Domineering section.

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