PERIODIC MULTIPLICATIVE ALGORITHMS OF SELMER TYPE

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Abstract

The multiplicative acceleration of the 2-dimensional Selmer algorithm is considered. Its behavior is more or less unknown and no general result on convergence is known. In this paper we show that periodic expansions do in fact converge and the coordinates of the limit point are rational functions of the largest eigenvalue of the periodicity matrix. Some comparisons are made with the Jacobi-Perron algorithm (Remark1) and the triangle sequence (Section 3).

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1. Introduction

Recently a renewed interest on multidimensional continued fractions has been observed. A survey of recent literature is given in Schweiger ([5],[6]).

Special attention has been paid to 2-dimensional algorithms. The most well known example is the Jacobi-Perron algorithm. Choose integers b_0, b_1, b_0 such that $b_0 \geq b_1 > 0, b_0 \geq b_2 > 0$ and define

$$k_1 := [\frac{b_2}{b_1}], k_2 := [\frac{b_0}{b_1}]$$

and

$$\kappa(b_0, b_1, b_2) := (b_1, b_2 - k_1 b_1, b_0 - k_2 b_1).$$

This can be transformed into a 2-dimensional map as follows. We use the projection

$$\pi(b_0, b_1, b_2) := \left(\frac{b_1}{b_0}, \frac{b_2}{b_0}\right).$$

Then there is a unique map K such that $\pi \circ \kappa = K \circ \pi$. The map K is given piecewise on a suitable subset of the unit square as

$$K(x_1, x_2) = (\frac{x_2}{x_1} - k_1, \frac{1}{x_1} - k_2)$$

and is therefore an obvious generalization of the map connected with regular continued fractions.

Selmer [7] proposed the following algorithm. Choose integers $b_0 \geq b_1 \geq b_2 > 0$ and define

$$\sigma(b_0, b_1, b_2) := (b_0 - b_2, b_1, b_2) =: (c_0, c_1, c_2).$$

Then there is a permutation π such that $c_{\pi 0} \geq c_{\pi 1} \geq c_{\pi 2}$. We use again the projection

$$\pi(b_0, b_1, b_2) := (\frac{b_1}{b_0}, \frac{b_2}{b_0}).$$

Then we arrive at the following map on $B^2 := \{(x_1, x_2) : 0 \leq x_2 \leq x_1 \leq 1\}$.

$$S(x_1, x_2) = (\frac{x_1}{1 - x_2}, \frac{x_2}{1 - x_2}), 0 < x_1 < 1, x_1 + x_2 < 1$$

$$S(x_1, x_2) = (\frac{1 - x_2}{x_1}, \frac{x_2}{x_1}), 1 \leq x_1 + x_2, 0 \leq x_2 \leq \frac{1}{2}$$

$$S(x_1, x_2) = (\frac{x_2}{x_1}, \frac{1 - x_2}{x_1}), 1 \leq x_1 + x_2, \frac{1}{2} \leq x_2 \leq 1.$$

A remarkable property about the Selmer algorithm is the fact that for almost all $w \in B^2$ there is an $n = n(w)$ such that for all $m \geq n$, $S^m w \in \{(x_1, x_2) \in B^2 : 1 \leq x_1 + x_2\}$ (see [5]).

It is natural to consider the multiplicative acceleration of the algorithm, i.e., we perform in the first step

$$\tau(b_0, b_1, b_2) := (b_0 - kb_2, b_1, b_2), k : [\frac{b_0}{b_1}], k \geq 1.$$

The second step is to reorder again. In a similar way as before this leads to the map

$$T(x_1, x_2) = (\frac{x_2}{x_1}, \frac{1 - kx_2}{x_1}), k(x) := [\frac{1}{x_2}].$$

The time-1-partition has the cells

$$B(k) := \{x \in B^2 : k(x) = k\} = \{x \in B^2 : \frac{1}{k+1} < x_2 \leq \frac{1}{k}\}.$$

Then we see that

$$TB(k) = \{x \in B^2 : kx_1 + x_2 \geq 1\}.$$

Therefore no cylinder is full and $TB(k)$ is not a union of cylinders of rank 1.

2. Periodic Expansions for Multiplicative Selmer Algorithm

Let $k_1, k_2, ...$ be a sequence of natural numbers. Then we define

$$B_0^{(-2)} = B_1^{(-2)} := 0, B_2^{(-2)} := 1$$

$$B_0^{(-1)} := 1, B_1^{(-1)} = B_2^{(-1)} := 0$$

$$B_0^{(0)} = B_2^{(0)} := 0, B_1^{(0)} := 1$$

and by recursion

$$B_i^{(s+1)} := k_{s+1} B_i^{(s-1)} + B_i^{(s-2)}, i = 0, 1, 2,\ s \geq 0.$$

If we introduce matrices

$$\beta(k) := \begin{pmatrix} 0 & k & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

then we see that the relations

$$\beta^{(s)}(k_1, ..., k_s) := \beta(k_1)...\beta(k_s) = \begin{pmatrix} B_0^{(s-1)} & B_0^{(s)} & B_0^{(s-2)} \\ B_1^{(s-1)} & B_1^{(s)} & B_1^{(s-2)} \\ B_2^{(s-1)} & B_2^{(s)} & B_2^{(s-2)} \end{pmatrix}$$

hold. If $y = T^s x$ and $k_i = k(T^{i-1}x)$, $1 \leq i \leq s$, then we find that

$$x_i = \frac{B_i^{(s-1)} + y_1 B_i^{(s)} + y_2 B_i^{(s-2)}}{B_0^{(s-1)} + y_1 B_0^{(s)} + y_2 B_0^{(s-2)}}, i = 1, 2.$$

Theorem 1 Assume that the algorithm of $x = (x_1, x_2)$ eventually becomes periodic with period length p. Then x_1 and x_2 are rational functions in ρ, where ρ denotes the largest eigenvalue of the characteristic polynomial of the periodicity matrix $\beta^{(p)}$. Therefore x_1 and x_2 belong to a number field of degree ≤ 3.

Proof. Clearly, we can assume that the expansion is purely periodic with period length p. Let

$$\Gamma := \beta^{(p)}(k_1, ..., k_p) =: \begin{pmatrix} B_0^{(p-1)} & B_0^{(p)} & B_0^{(p-2)} \\ B_1^{(p-1)} & B_1^{(p)} & B_1^{(p-2)} \\ B_2^{(p-1)} & B_2^{(p)} & B_2^{(p-2)} \end{pmatrix}.$$

Note that

$$\Gamma^k = \begin{pmatrix} B_0^{(kp-1)} & B_0^{(kp)} & B_0^{(kp-2)} \\ B_1^{(kp-1)} & B_1^{(kp)} & B_1^{(kp-2)} \\ B_2^{(kp-1)} & B_2^{(kp)} & B_2^{(kp-2)} \end{pmatrix}.$$

Then the characteristic polynomial of Γ is given as

$$\chi_\Gamma(t) := \det(t\mathbf{1} - \Gamma) = t^3 - At^2 + Bt - 1.$$

If α, β, ρ are the eigenvalues we order them in a way such that $|\alpha| \leq |\beta| < \rho$. The Frobenius-Perron theorem says that $1 < \rho$. Therefore $\min(|\alpha|, |\beta|) < 1$. Since $\alpha\beta\rho = 1$ the case $\alpha = \beta$ cannot occur. If the polynomial $\chi_\Gamma(t)$ has a multiple root then it would be reducible. Therefore if $\chi_\Gamma(t) = (x-\beta)^2(x-\rho)$ then ρ would be a rational number which satisfies $|\rho| = 1$, a contradiction. □

The difficulty for the present algorithm has two reasons. There is no convergence result known and $\max(B_0^{(s)}, B_0^{(s)-1})$ can be $B_0^{(s)}$ or $B_0^{(s-1)}$.

We first note that

$$B_0^{(p-1)} + B_1^{(p)} + B_2^{(p-2)} = A = \rho + \beta + \alpha$$

and

$$\rho\beta\alpha = 1.$$

The Cayley-Hamilton Theorem (which was proved also by Perron in [4]) implies that

$$\Gamma^3 - A\Gamma^2 + B\Gamma - 1 = 0.$$

We multiply this equation with $\Gamma^k \beta(k_1, ..., k_j)$ and see that for the entries of the matrices the relations

$$B_i^{((3+k)p+j)} - AB_i^{((2+k)p+j)} + BB_i^{((1+k)p+j)} - B_i^{(kp+j)} = 0, 0 \leq i \leq 2, 0 \leq j < p$$

hold. The difference equation

$$a((3+k)p) - Aa((2+k)p) + Ba((1+k)p) - a(kp) = 0$$

has the general solution

$$a(kp) = d\rho^k + b\beta^k + a\alpha^k.$$

Remember that the case $\alpha = \beta$ does not occur in our situation. The numbers a, b, d are uniquely defined by the initial conditions

$$a(0) = d + b + a$$

$$a(p) = d\rho + b\beta + a\alpha$$

$$a(2p) = d\rho^2 + b\beta^2 + a\alpha^2.$$

Therefore we find

$$B_i^{(kp+j)} = d(i,j)\rho^k + b(i,j)\beta^k + a(i,j)\alpha^k.$$

Note further that from the recursion relations

$$B_i^{(s+2)} = k_{s+2}B_i^{(s)} + B_i^{(s-1)}$$

$$B_i^{(s+3)} = k_{s+3}B_i^{(s+1)} + B_i^{(s)}$$

follows that if $B_i^{(s)} \gg \rho^k$ then $B_i^{(s+j)} \gg \rho^k, j \geq 2$. Therefore, if $d(i,j) \neq 0$ for some j then $d(i,j) \neq 0$ for all $j, 0 \leq j < p$.

Now we know that

$$B_0^{(kp-1)} + B_1^{(pk)} + B_2^{(kp-2)} = \rho^k + \beta^k + \alpha^k.$$

As noted before if $y = T^s x$ then we find that

$$x_i = \frac{B_i^{(s-1)} + y_1 B_i^{(s)} + y_2 B_i^{(s-2)}}{B_0^{(s-1)} + y_1 B_0^{(s)} + y_2 B_0^{(s-2)}}, i = 1, 2.$$

But $y = T^s x \in \{x \in B^2 : \frac{1}{k_{s+1}+1} < x_2 \leq \frac{1}{k_{s+1}}\}$.

Therefore

$$\frac{1}{k_{s+1}+1} \leq \frac{B_i^{(s-1)} + B_i^{(s)} + B_i^{(s-2)}}{B_0^{(s-1)} + B_0^{(s)} + B_0^{(s-2)}} \leq k_{s+1} + 1, i = 1, 2.$$

If we take $M = \max(k_1, ..., k_p) + 1$ then we obtain

$$\frac{1}{M} \leq \frac{B_i^{(s-1)} + B_i^{(s)} + B_i^{(s-2)}}{B_0^{(s-1)} + B_0^{(s)} + B_0^{(s-2)}} \leq M, i = 1, 2.$$

This implies that $d(i,j) \neq 0$ for all $i = 1, 2$ and $j =, 1, ..., p-1$.

From this we imply that

$$\lim_{k\to\infty} \frac{B_1^{(kp)}}{B_0^{(kp)}} = \frac{d(1,0)}{d(0,0)} =: z_1, \ \lim_{k\to\infty} \frac{B_2^{(kp)}}{B_0^{(kp)}} = \frac{d(2,0)}{d(0,0)} =: z_2$$

exist. The equation

$$\Gamma^k \begin{pmatrix} 1 \\ x_1 \\ x_2 \end{pmatrix} = \lambda^k \begin{pmatrix} 1 \\ x_1 \\ x_2 \end{pmatrix}$$

shows that

$$B_0^{(kp-1)} + x_1 B_0^{(pk)} + x_2 B_0^{(kp-2)} = \lambda^k$$

for some positive eigenvalue of the periodicity matrix Γ. Since $B_0^{(kp-1)} \sim \rho^k$ we see that $\lambda = \rho$. Observe that $z = (z_1, z_2)$ is a periodic point with the same expansion. Hence $z = x$. We can calculate x_1 and x_2 as rational functions in ρ from the equation

$$\Gamma \begin{pmatrix} 1 \\ x_1 \\ x_2 \end{pmatrix} = \rho \begin{pmatrix} 1 \\ x_1 \\ x_2. \end{pmatrix}$$

Remark 1 It is well known that a similar theorem holds true for the Jacobi-Perron algorithm ([4]). But there are some important differences.

(1) The characteristic polynomial is irreducible for the Jacobi-Perron algorithm but in our case the characteristic polynomial can be reducible. Let $p = 1$ and $k_1 = k$. Then $\chi_\Gamma(t) = t^3 - kt - 1$. For $k = 2$ we see that $\beta = -1$. Note further that for $k \geq 3$ we find $\beta < -1$. Here $(x_1, x_2) = (\frac{1}{\rho}, \frac{1}{\rho^2})$.

(2) The Jacobi-Perron algorithm has periodic expansions of every (primitive) period $p \geq 1$. For the multiplicative Selmer algorithm period $p = 2$ does not occur. Suppose that $x = (x_1, x_2)$ has period 2. In this case we get $x = (\frac{1}{\lambda(\lambda - k_2)}, \frac{1}{\lambda})$ for some eigenvalue λ. Therefore we get $Tx = (\frac{1}{\lambda(\lambda - k_1)}, \frac{1}{\lambda})$. Then $k_1 = [\lambda] = k_2$. However, period $p = 3$ can occur. Let $k_1 = 2, k_2 = 3, k_3 = 5$. Then we obtain $\rho = 6.92167$. The condition $0 \leq x_2 \leq x_1 \leq 1$ is satisfied since $k_1 + 1 < \rho, k_2 + 1 < \rho, k_3 + 1 < \rho$.

Theorem 2 For periodic algorithms with $\alpha \neq \beta$ we obtain the estimates

$$|B_0^{(pg)} x_i - B_i^{(pg)}| \ll |\beta|^g, i = 1, 2$$

but there is at least one index i such that

$$\max(|B_0^{(pg)} x_i - B_i^{(pg)}|, |B_0^{(pg+2)} x_i - B_i^{(pg+2)}|) \gg |\beta|^g$$

holds for infinitely many values of g.

Proof. Since the quantities $B_0^{(pg)} x_i - B_i^{(pg)}$ obey the same recursion relation we find

$$B_0^{(pg)} x_i - B_i^{(pg)} = k(i,0)\rho^g + m(i,0)\beta^g + n(i,0)\alpha^g, i = 1, 2.$$

As we have just shown

$$\lim_{k \to \infty} \frac{B_1^{(kp)}}{B_0^{(kp)}} = x_1, \ \lim_{k \to \infty} \frac{B_2^{(kp)}}{B_0^{(kp)}} = x_2$$

and since $B_0^{(pg)} \sim \rho^g$ we get $k(1,0) = k(2,0) = 0$. Hence for $j = 1$ and $j = 2$ we find

$$B_0^{(pg)} x_i - B_i^{(pg)} = m(i,0)\beta^g + n(i,0)\alpha^g, i = 1, 2.$$

This shows that the first part of the theorem is true.

Now we will prove the second part of Theorem 2. In a similar way as before we find

$$B_0^{(pg+2)} x_i - B_i^{(pg+2)} = m(i,2)\beta^g + n(i,2)\alpha^g, i = 1, 2.$$

It is easy to verify that (see Schweiger 2000, p.5)

$$\det \begin{pmatrix} B_0^{(pg+2)} x_1 - B_1^{(pg+2)} & B_0^{(pg)} x_1 - B_1^{(pg)} \\ B_0^{(pg+2)} x_2 - B_2^{(pg+2)} & B_0^{(pg)} x_2 - B_2^{(pg)} \end{pmatrix} = \frac{1}{B_0^{(pg+1)} + B_0^{(pg+2)} y_1 + B_0^{(pg)} y_2}.$$

Here $T^p(x_1,x_2)=(y_1,y_2)$. Note that from $B_0^{(gp)}\sim\rho^g$ we obtain

$$\max_{i=1,2}(|B_0^{(pg)}x_i-B_i^{(pg)}|,|B_0^{(pg+2)}x_i-B_i^{(pg+2)}|)\gg\frac{1}{(\sqrt{\rho})^g}=\sqrt{|\alpha\beta|^g}.$$

For the last equality we used $\alpha\beta\rho=1$. If $|\alpha|=|\beta|$ then

$$\max_{i=1,2}(|B_0^{(pg)}x_i-B_i^{(pg)}|,|B_0^{(pg+2)}x_i-B_i^{(pg+2)}|)\gg|\beta|^g.$$

If $|\alpha|<|\beta|$ then at least one of the numbers $m(1,0),m(2,0),m(1,2),m(2,2)$ cannot vanish. If $m(1,0)=m(2,0)=m(1,2)=m(2,2)=0$ then we see that

$$\rho^{-g}\ll\det\begin{pmatrix}B_0^{(pg+2)}x_1-B_1^{(pg+2)} & B_0^{(pg)}x_1-B_1^{(pg)}\\ B_0^{(pg+2)}x_2-B_2^{(pg+2)} & B_0^{(pg)}x_2-B_2^{(pg)}\end{pmatrix}\ll|\alpha|^{2g}$$

which (by using $\alpha\beta\rho=1$) leads to $|\beta|\le|\alpha|$. Therefore we get again

$$\max_{i=1,2}(|B_0^{(pg)}x_i-B_i^{(pg)}|,|B_0^{(pg+2)}x_i-B_i^{(pg+2)}|)\gg|\beta|^g.$$

□

Remark 2 Since $|\beta|>1$ can occur we see that $\limsup_{g\to\infty}|B_0^{(pg)}x_i-B_i^{(pg)}|=\infty$ is possible. Therefore a generalization of Nakaishi's approach [3] seems to be out of reach.

3. Periodic Expansions for the Triangle Sequence

The shape of the time-1-matrices for Selmer's multiplicative algorithm suggest to investigate the algorithm with time-1-matrices

$$\beta(k):=\begin{pmatrix}1&k&1\\1&0&0\\0&1&0\end{pmatrix}.$$

These matrices belong to an algorithm which has been called the triangle sequence by Garrity [2] (see also [1]). The underlying map is given on B^2 by

$$T(x_1,x_2)=(\frac{x_2}{x_1},\frac{1-x_1-kx_2}{x_1}),k=[\frac{1-x_1}{x_2}].$$

There are some similarities but also some differences between these algorithms. The most important difference is the time-1-partition. The cylinders $B(k)$ are the triangles with vertices $(1,0),(\frac{1}{k+1},\frac{1}{k+1}),(\frac{1}{k+2},\frac{1}{k+2})$, $k\ge0$. Therefore all cylinders are full, i. e. $TB(k)=B^2,k\ge0$. If we set

$$\beta^{(s)}(k_1,...,k_s):=\beta(k_1)...\beta(k_s)=:\begin{pmatrix}B_{00}^{(s)}&B_{01}^{(s)}&B_{00}^{(s-1)}\\B_{10}^{(s)}&B_{11}^{(s)}&B_{10}^{(s-1)}\\B_{20}^{(s)}&B_{21}^{(s)}&B_{20}^{(s-1)}\end{pmatrix}$$

we find the recursion relations

$$B_{i0}^{(s+1)} = B_{i0}^{(s)} + B_{i1}^{(s)}, i = 0,1,2$$
$$B_{i1}^{(s+1)} = k_{s+1}B_{i0}^{(s)} + B_{i0}^{(s-1)}, i = 0,1,2.$$

Similar to the multiplicative Selmer algorithm we see easily the following properties. As before let α, β, ρ be the three eigenvalues of the characteristic polynomial ordered in a way such that $|\alpha| \leq |\beta| < \rho$.

(1) The characteristic polynomial can be reducible. Let $p = 1$ and $k_1 = k$. Then $\chi_\Gamma(t) = t^3 - t^2 - kt - 1$. For $k = 3$ we see that $\beta = -1$.

(2) It is possible to have $0 < \alpha < 1 < \beta < \rho$. Let $p = 2$ and $k_1 = 4, k_2 = 3$. Then $\chi_\Gamma(t) = t^3 - 5t^2 + 6t - 1$. Then $\rho = 6.480786620527056, \beta = 1.4097605688346069, \alpha = 0.109452810638337$.

In contrast to the multiplicative Selmer algorithm much more is known about convergence (see [1]). In fact, it is known that the triangle sequence is convergent for periodic expansions. This makes it easier to prove the following theorem, which is a strengthening of Theorem 9 of [2].

Theorem 3 Assume that the algorithm of $x = (x_1, x_2)$ eventually becomes periodic with period length p. Then x_1 and x_2 are rational functions in ρ, where ρ denotes the greatest eigenvalue of the characteristic polynomial of the periodicity matrix $\beta^{(p)}$. Therefore x_1 and x_2 belong to a number field of degree ≤ 3.

Proof. Since the proof closely follows Perron's proof ([4],[5]), we will only sketch it. Let (x_1, x_2) have a purely periodic expansion with periodicity matrix

$$\Gamma = \beta^{(p)} = \begin{pmatrix} B_{00}^{(p)} & B_{01}^{(p)} & B_{00}^{(p-1)} \\ B_{10}^{(p)} & B_{11}^{(p)} & B_{10}^{(p-1)} \\ B_{20}^{(p)} & B_{21}^{(p)} & B_{20}^{(p-1)} \end{pmatrix}.$$

Then

$$\begin{pmatrix} B_{00}^{(p)} & B_{01}^{(p)} & B_{00}^{(p-1)} \\ B_{10}^{(p)} & B_{11}^{(p)} & B_{10}^{(p-1)} \\ B_{20}^{(p)} & B_{21}^{(p)} & B_{20}^{(p-1)} \end{pmatrix} \begin{pmatrix} 1 \\ x_1 \\ x_2 \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ x_1 \\ x_2 \end{pmatrix}$$

for some eigenvalue $\lambda > 1$. Clearly, x_1 and x_2 are rational functions in λ. From $\Gamma^{p+1} = \Gamma^p\Gamma$ we obtain the relation

$$\frac{B_{00}^{((g+1)p)}}{B_{00}^{(gp)}} = B_{00}^{(p)} + B_{01}^{(p)}\frac{B_{10}^{(gp)}}{B_{00}^{(gp)}} + B_{00}^{(p-1)}\frac{B_{20}^{(gp)}}{B_{00}^{(gp)}}.$$

From the convergence of the algorithm we obtain

$$\lim_{g\to\infty} \frac{B_{00}^{((g+1)p)}}{B_{00}^{(gp)}} = B_{00}^{(p)} + B_{01}^{(p)}x_1 + B_{00}^{(p-1)}x_2 = \lambda.$$

Using this result and the recursion relations we obtain $B_{00}^{(gp)} \sim \lambda^g$ and $B_{01}^{(gp)} \sim \lambda^g$. However from

$$B_{00}^{(gp)} + B_{11}^{(gp)} + B_{20}^{(gp-1)} = \rho^g + \alpha^g + \beta^g$$

we see that $\lambda = \rho$. □

Remark 3 Clearly Theorem 2 can also be extended to the triangle sequence.

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